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Nonlinear Diffusion Waveshapes Generated by Possibly Finite Initial Disturbances*

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Submitted by Alex McNabb

INTRODUCTION

In this paper solutions of the semilinear diffusion equation

$$\partial u/\partial t = (\partial^2 u/\partial x^2) + f(u) \tag{1}$$

on the domain $-\infty < x < \infty$, $t > 0$ are investigated for large values of the time t. It is always assumed that $f(0) = f(1) = 0$, $f(u)$ is continuous on $[0, 1]$ and has right and left derivatives at $u = 0$ and $u = 1$, respectively, and also that for some number u_+: $0 < u_+ < 1$, $f(u) > 0$ for $u_+ < u < 1$, and if $0 \leqslant u < 1$, $\int_u^1 f(v)\, dv > 0$.

The purpose of the investigation is to show the development of an advancing wave-front solution from a variety of initial conditions. Such waves were first investigated in the classic papers of Fisher [1] and Kolmogorov, *et al.* [2], motivated by the application of (1) to the theory of an advancing wave of favorable genes. Later the waves became of interest in the theory of combustion [3–8]; this aspect is reviewed by Zel'dovich and Barenblatt [9]. Recently, Sattinger [10] has demonstrated stability for some such waves, and Aronson and Weinberger [11] have considered solutions of (1), again in the context of the advance of favorable genes. Stability of steady-state solutions of (1) on finite intervals was investigated by Bradford and Philip [12].

Kolmogorov *et al.* [2] showed that, from an initial unit step function $u(x, 0)$, a moving wave front is formed which approaches in shape and velocity a steady-state wave solution $V(x + m^\circ t)$ of (1). This wave has the minimum velocity possible for such waves. There is a necessary assumption that $f'(u) \leqslant f'(0)$, $0 \leqslant u \leqslant 1$. It is not, however, possible to show convergence to a specific wave solution.

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Kanel [3, 5] extended this result to functions $f(u)$ required to be nonpositive in some right neighborhood of 0, together with some other conditions. A further extension [4, 6] allowed the initial function $u(x, 0)$ to be a general monotone function on some finite interval, but still required it to take zero values to the left of that interval, and unit values to the right. However, together with the nonpositivity condition, a stronger result is obtained; the solution actually converges to a certain wave solution. Monotone initial functions are physically somewhat unreal, especially when applied to populations. A semi-infinite distribution cannot be achieved with any finite disturbance, and is only approximated after a long time. Aronson and Weinberger [11] investigated solutions generated by initial functions positive only on a bounded interval. Unfortunately, the methods used by Kolmogorov *et al.* and Kanel do not work in this case; using maximum principle methods they obtained somewhat looser bounds on the behavior of solutions for large time.

In this paper, a different method based on a sharpened comparison theorem is used. It is perhaps simpler than the approach of Kolmogorov *et al.* [2], but, more importantly, it is not sensitive to the form of the initial function used. A result analogous to that of Kolmogorov *et al.* can be proved for initial functions nonzero only on a finite interval. The method is also insensitive to the form of $f(u)$, and, in particular, covers nonconvex positive functions.

Preliminaries

The following consequence of the maximum principle for parabolic equations is needed:

THEOREM 1. *Let $u(x, t)$ and $v(x, t)$ be any solutions of* (1) *defined in some neighborhood of a point (x_1, t_1), with $0 \leqslant u(x_1, t_1) = v(x_1, t_1) \leqslant 1$. Then for any neighborhood $[a, b]$: $a < x_1 < b$, either $u(x, t_1) = v(x, t_1)$ on $[a, b]$, or there exists positive σ for which $u(x, t)$ and $v(x, t)$ have at least one intersection for some x in $[a, b]$ for each t: $t_1 - \sigma \leqslant t \leqslant t_1$.*

Remark. For this theorem and its applications, "intersection" means simple crossover. To say that two continuous functions $u(x)$ and $v(x)$ intersect at x_1 means that the difference $w(x) = u(x) - v(x)$ is zero at x_1, and in some interval $[a, b]$, with $a < x_1 < b$, $w(x)$ is nonnegative to one side of x_1, nonpositive to the other, and nonzero at a and b.

Proof. If $u(a, t_1) - v(a, t_1)$ and $u(b, t_1) - v(b, t_1)$ are of opposite sign, the theorem follows from the continuity of the solutions in x and t. If the differences are both, say, positive, then this remains true on some interval $[t_1 - \sigma, t_1]$, again invoking continuity. But if for some t_2: $t_1 - \sigma \leqslant t_2 < t_1$, $u(x, t_2) > v(x, t_2)$ on $[a, b]$, then by a theorem of Aronson and Weinberger [11], itself an extension of the maximum principle, $u(x, t_1) > v(x, t_1)$ on $[a, b]$. This gives a contradiction.

If the differences at t_1 are negative instead of positive, a similar argument holds. Finally, if $u(x, t_1) = v(x, t_1)$ at $x = a$ or b, then the proof can be applied on some appropriate subinterval, except in the degenerate case when $u = v$ on $[a, b]$.

The following lemma is needed:

LEMMA 1. *If* $0 \leqslant u(x, 0) \leqslant 1$, $u(x, 0)$ *is continuous and positive for some* x, *then* $0 < u(x, t) \leqslant 1$ *for all* $t > 0$, *where* $u(x, t)$ *is a solution of* (1).

Proof. Since $f(u)$ has a right derivative at $u = 0$, and is continuous on $[0, 1]$, there is an $\alpha: f(u) > \alpha u$ on $[0, 1]$. Then by the comparison theorem $u(x, t) \geqslant v(x, t)$, where $u(x, 0) = v(x, 0)$ and v is a solution of $v_t = v_{xx} + \alpha v$. But

$$v(x, t) = \int_{-\infty}^{\infty} \frac{\exp[\alpha t - (x - x')^2/4t]}{(2\pi t)^{1/2}} u(x', 0)\, dx'$$

$$> 0, \qquad \text{for all } x \text{ and all } t > 0.$$

The proof that $u(x, t) \leqslant 1$ is similar.

STEADY-SHAPE SOLUTIONS

Solutions of (1) of the form $u(x + mt + \alpha)$ are sought for various m and arbitrary α. For such functions $u(\sigma)$, (1) implies that

$$(d^2u/d\sigma^2) - m(du/d\sigma) + f(u) = 0$$

or, using the transformation $p = du/d\sigma$,

$$p(dp/du) = mp - f(u). \tag{2}$$

The relevant theory of solutions of (2) has been treated extensively elsewhere [1, 2, 10, 11, 13]. It is also fairly elementary, so it will be merely outlined here.

To each trajectory $p(u)$ of (2) in $0 \leqslant u \leqslant 1$ there correspond solutions $u(x + mt + \alpha)$ of (1); these can be derived by inverting the transformation $x - x_0 = \int_{u_0}^{u} (du/p)$. The u-axis contains singular points of (2); the singularities at $(0, 0)$ and $(1, 0)$ are of special interest. Since $f'(1) < 0$ there is, for any m, a pair of solutions of (2) through $(1, 0)$ with slopes of opposite sign. Only that with negative slope is of interest here. If $f'(0) > 0$, and $m^2 \geqslant 4f'(0)$ then there is a solution through $(0, 0)$ with slope $[m/2]\,[1 + (1 - 4f'(0)/m^2)^{1/2}]$ and an infinite family of solutions with slope $[m/2]\,[1 - (1 - 4f'(0)/m^2)^{1/2}]$.

If $m^2 < 4f'(0)$, then solutions approaching $(0, 0)$ do so in a spiral fashion; for present purposes that means that no solutions pass through $(0, 0)$. If

$f'(0) < 0$, then there are, for each m, two solutions through (0, 0) with slopes of opposite sign.

If $f'(0) > 0$, then there is a certain minimum value $m^\circ > 0$ such that for $m \geqslant m^\circ$ there is one solution of (2) passing through (0, 0) and (1, 0), which is positive for $0 < u < 1$. The solution with $m = m^\circ$ is the maximum of all such solutions. If $f'(0) < 0$ there is only one m° for which (2) has a solution passing through (0, 0) and)1. 0). In either case the m°-solution is important, and is denoted $P(u)$.

If $f'(u) \leqslant f'(0)$ on [0, 1] then $m^\circ = 2[f'(0)]^{1/2}$ (see, e.g., [10]). Otherwise, $m^{\circ 2} \geqslant 4f'(0)$.

In the following theorems wave solutions $u(x + mt + \alpha)$ are considered which lie between 0 and 1 only for some subinterval of values on the x axis. Their behavior outside $0 \leqslant u \leqslant 1$ is unimportant, and for present purposes it does not matter if the solution reenters the $0 \leqslant u \leqslant 1$ band for some other values of x. However, for simplicity it will be assumed that this does not happen.

For any solution $u(x, t)$ of (1) tending to zero as $x \to -\infty$, a function $\gamma(u, t)$ is defined as the value of $u_x(x, t)$ at the least x for which $u(x, t) = u$.

Convergence to Wave-Front Shape

With $\gamma(u, t)$ as just defined, it is shown in this section that given certain initial functions $u(x, 0)$, $\gamma(u, t) \to P(u)$. This means that the left-moving front acquires with time the shape and speed of the stationary wave of minimum velocity; it is analogous to the result of Kolmogorov *et al.* [2]. It is not generally true that the wave front converges to any particular wave.

The proofs are rather technical, and are assisted by the following definitions, applicable with respect to a particular solution $u(x, t)$ of (1):

An overfunction is a solution $V(x + mt)$ of (1) for which $V(x) > u(x, 0)$ on the interval on which $V(x)$ lies between 0 and 1. An underfunction is defined analogously.

A test function is a solution $V(x + mt)$ of (1) for which $V(x)$ has only one intersection with $u(x, 0)$ in some interval of interest.

By Theorem 1, over- and underfunctions retain their properties as t increases, and stratagems can be devised so that test functions do also.

A test region, defined for some time t, is the region in $R \times [0, 1]$ occupied by graps of test functions. It is usually a connected set.

Theorems 2, 3, and 3′ are proved by ensuring that after some time some important part of the graph of $u(x, t)$ lies in a test region.

THEOREM 2. *Let $u(x, t)$ be a solution of* (1) *with $0 \leqslant u(x, 0) \leqslant 1$ for all x and suppose $u_x(x, 0) \to 0$ as $x \to \pm\infty$. Then for any $\sigma > 0$ there exists t_1: $\gamma(u, t) \leqslant P(u) + \sigma$ for $t \geqslant t_1$ and for all u for which $\gamma(u, t)$ exists.*

Proof. For some range of values of m close to m°, there exist positive solutions $p(u)$ of (2) on [0, 1] for which $P(u) < p(u) < P(u) + \sigma$. Let $p_1(m_1, u)$, $p_2(m_2, u)$ be two such solutions, with $m_1 > m_2$. Let $V_1(X)$, $V_2(X)$ be functions: for $i = 1, 2$, $V_i(X)$ is defined on some interval $[0, a_i]$, $V_i(0) = 0$, $V_i(a_i) = 1$, and $V_i(x + m_i t + \alpha_i)$ for arbitrary α_i gives the family of solutions of (1) corresponding to $p_i(m_i, u)$. Then for each i, $V_i(X)$ is increasing and has a positive minimum slope on $[0, a_i]$.

Since $u_x(x, 0) \to 0$ as $x \to \infty$, there is a β_1: for all $\alpha_1 \leqslant \beta_1$, $V_1(x + \alpha_1)$ and $u(x, 0)$ have only one intersection, since for large negative α_1 the interval of definition of $V_1(x + \alpha_1)$ is far to the right, V_1' has positive lower bound, and $u_x(x, 0) \to 0$. Similarly, since $u_x(x, 0) \to 0$ as $x \to -\infty$ there exists β_2: for all $\alpha_2 \geqslant \beta_2$, $V_2(x + \alpha_2)$ and $u(x, 0)$ have only one intersection.

By Theorem 1, $V_1(x + \alpha_1 + m_2 t)$ has only one intersection with $u(x, t)$ for all $t > 0$, if $\alpha_1 \leqslant \beta_1$, and is therefore a test function. So is $V_2(x + m_2 t + \alpha_2)$, if $\alpha_2 \geqslant \beta_2$. Next it is shown that after some time the test regions overlap, and fill $R \times [0, 1]$.

Let $t_1 = \mathrm{Max}[(\alpha_1 + \beta_2 - \beta_1)/(m_1 - m_2), 0]$. Then for any (x_2, t_2): $t_2 \geqslant t_1$, there exist α_1, α_2 for which $u(x_2, t_2) = V_1(x_2 + \alpha_1 + m_1 t_2) = V_2(x_2 + \alpha_2 + m_2 t_2)$.

Since $0 < u(x_2, t_2) < 1$, $0 < x_2 + \alpha_i + m_i t_2 < a_i$, $i = 1, 2$. Therefore, $\alpha_2 - \alpha_1 + t_2(m_2 - m_1) > -a_1$. Then $\alpha_2 - \alpha_1 > (m_1 - m_2)\, t_1 - a_1 > \beta_2 - \beta_1$. So either $\alpha_2 > \beta_2$, or $\alpha_1 < \beta_1$.

If $\beta_2 < \alpha_2$, then $V_2(x + \alpha_2 + m_2 t)$ has only one intersection with $u(x, t)$ and so $u_x(x_2, t_2) \leqslant V_2'(x_2 + \alpha_2 + m_2 t_2)$. If $\beta_1 > \alpha_1$, then likewise $u_x(x_2, t_2) \leqslant V_1'(x_2 + \alpha_1 + m_1 t_2)$. In either case, $\gamma(u, t) = u_x(x, t) \leqslant P(u) + \sigma$ for $t \geqslant t_1$.
Q.E.D.

THEOREM 3. *Let $u(x, t)$ be a solution of* (1) *for which*

(a) $u(x, t) \to 1$ *for all x as $t \to \infty$,*

(b) $0 \leqslant u(x, 0) \leqslant 1$ *for all x,*

(c) *for some* x_1, $u_x(x, 0) \geqslant u(x, 0)\{m^\circ + [m^{\circ 2} - 4f'(0)]^{1/2}\}/2$ *for all* $x \leqslant x_1$.

If in addition $f'(0) > 0$, then for any c: $0 < c < 1$ and any $\sigma > 0$ there is a $t_1 \geqslant 0$ for which $\gamma(u, t) \geqslant P(u) - \sigma$ for any $t \geqslant t_1$ and $0 < u \leqslant c$.

Proof. Let c be any number: $0 < c < 1$ and σ any positive constant. A value $m_1 > m^\circ$ and a corresponding solution $p_1(u)$ of (2) can be found for which:

$$p_1(0) = 0,$$

$$p_1(u) \geqslant P(u) - \sigma, \qquad 0 < u \leqslant c,$$

and for some b:

$$c < b < 1, \qquad p_1(b) = 0.$$

To $p_1(u)$ corresponds a function $V_1(X)$ such that for any α, $V_1(x + m_1 t - \alpha)$ is a solution of (1). $V_1(X)$ has a single maximum value b at some $X = X_1$ in the band $0 \leqslant V_1 \leqslant 1$, and

$$\frac{V_1'(X)}{V_1(X)} \to p_1'(0) = \frac{m_1 - [m_1^2 - 4f'(0)]^{1/2}}{2} \qquad \text{as} \qquad X \to -\infty$$

Now $u(x_1, t) \to 1$ as $t \to \infty$, so for some t_2 and all $t \geqslant t_2$, $u(x_1, t) > b$. For $0 \leqslant t \leqslant t_2$, $u(x_2, t) > 0$ by Lemma 1, and so has a positive lower bound b_1. Then there is a number α_1 such that for all $\alpha \geqslant \alpha_1$ the following are true:

(i) $0 < V_1(x_1 - \alpha + m_1 t_2) < b_1$;

(ii) $V_1'(x_1 - \alpha + m_1 t_2) > 0$;

(iii) $0 < \dfrac{V_1'(x - \alpha)}{V_1(x - \alpha)} < \dfrac{m^\circ + [m^{\circ 2} - 4f'(0)]^{1/2}}{2}$ for all $x \leqslant x_1$.

(iii) holds because $V_1'(X)/V_1(X) \to p_1'(0) < \frac{1}{2}(m^\circ + [m^{\circ 2} - 4f'(0)]^{1/2})$ as $X \to -\infty$.

From (i), noting that $u(x_1, t) > b > V_1(x_1 - \alpha + m_1 t)$ for $t > t_1$, it follows that $u(x_1, t) > V_1(x_1 - \alpha + m_1 t)$ for all $t \geqslant 0$. And from (iii) it follows that $V_1(x - \alpha)$ and $u(x, 0)$ have one intersection to the left of x_1. So by Theorem 1, $V_1(x - \alpha + m_1 t)$ and $u(x, t)$ have at most one intersection to the left of x_1, if $\alpha \geqslant \alpha_1$. Then V_1 is a test function, and the test region is the set $\{(x, u): x \leqslant x_1, 0 < u \leqslant V(x - \alpha_1 + m_1 t) \text{ if } x - \alpha_1 + m_1 t \leqslant X_1, \text{ and } 0 < u \leqslant b \text{ if } X_1 + \alpha_1 - m_1 t \leqslant x \leqslant x_1\}$.

Let $m_2 = (m_1 + m^\circ)/2$. There is a solution $p_2(u)$ of (2) with:

$$m = m_2$$

$$\frac{dp_2}{du}(0) = \frac{m_2 - [m_2^2 - 4f'(0)]^{1/2}}{2},$$

$$p_2(0) = 0 \qquad \text{and} \qquad p_2(u) > 0 \quad \text{in} \quad (0, 1).$$

The corresponding steady-shape wave solution of (1), $V_2(x + m_2 t - \alpha)$ is increasing, and

$$V_2'(X)/V_2(X) \to \{m_2 - [m_2^2 - 4f'(0)]^{1/2}\}/2 < \{m^\circ + [m^{\circ 2} - 4f'(0)]^{1/2}\}/2 \qquad \text{as} \qquad X \to \infty.$$

Such a solution can be found, for $\alpha = \alpha_2$, such that $V_2(x - \alpha_2) > u(x, 0)$ for all x where V_2 is defined. Consequently, $V_2(x - \alpha_2 + m_2 t) > u(x, t)$ for all $t \geqslant 0$, for V_2 is an overfunction. The various functions used are diagrammed in Figs. 1 and 2.

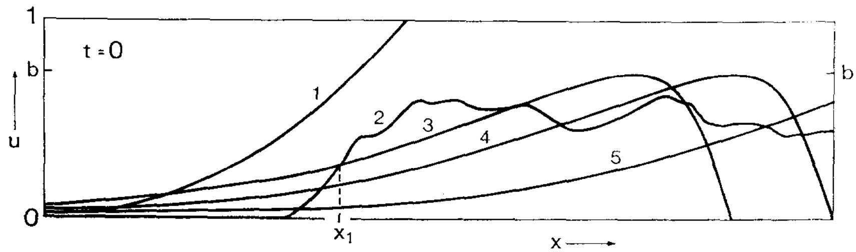

FIG. 1. Diagrams of some functions involved in the proof of Theorem 3. (1) $V_2(x - \alpha_2)$; (2) $u(x, 0)$; (3) $V_1(x - \alpha_1)$; (4) and (5) $V_1(x - \alpha)$ for various $\alpha > \alpha_1$.

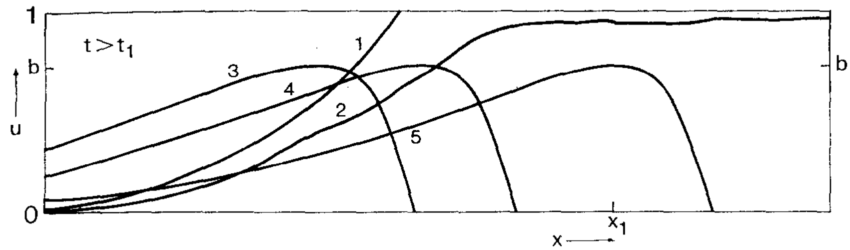

FIG. 2. Curves of Fig. 1 at some $t \geqslant t_1$. (1) $V_2(x - \alpha_2 + m_2t)$; (2) $u(x, t)$; (3) $V_1(x - \alpha_1 + m_1t)$; (4) and (5) $V_1(x - \alpha + m_1t)$.

Let X_2 be the value of X for which $V_2(X) = b$. Let

$$t_1 = \text{Max}[(X_1 - X_2 + \alpha_1 - \alpha_2)/(m_1 - m_2), 0].$$

For any $t_3 > t_1$, let $x_2 = X_1 + \alpha_1 - m_1t_3$. Then $(m_1 - m_2)\, t_3 > X_1 - X_2 + \alpha_1 - \alpha_2$, or $X_2 > x_2 + m_2t_3 - \alpha_2$, so $V_1(x_2 + m_1t_3 - \alpha_1) = V_1(X_1) = b = V_2(X_2) > V_2(x_2 + m_2t_3 - \alpha_2) > u(x_2, t_3)$, since V_2 is increasing and is an overfunction.

But $u(x_1, t_3) > b$, so at x_1, $u(x, t_3)$ exceeds the value of every test function. So the graph of $u(x, t_3)$ enters the test region somewhere between x_2 and x_3, at a point where $u(x, t_3) = b$. Through every point in the test region passes the positive slope part of the graph of a test function, so every point on the graph of $u(x, t_3)$ in the test region has a steeper gradient than the test function it intersects there. So $u(x, t_3)$ cannot leave the test region and is monotonic there, and for any number d, when $u(x, t_3) = d = V_1(X)$, then $u_x(x, t_3) > V_1'(X)$; i.e., $\gamma(d, t_3) > P(d) - \sigma$. This inequality holds for any d: $0 < d \leqslant c < b$, and any σ sufficiently small.

COROLLARY. *Let $P_1(u)$ be the positive derivative of $V_1(X)$ when $V_1(X) = u$. Then $\gamma(d, t) \geqslant P_1(d)$ for $0 < d \leqslant c$ and $t \geqslant t_1$. This means, inter alia, that for $t \geqslant t_1$, $u(x, t)$ increases from the left until it reaches the value c.*

Remarks. For many applications, $u(x, 0) = 0$ for large negative x, in which case (c) holds. It seems unlikely that there are any interesting cases where condition (a) does not hold (see [12]). That is not true if, as in the following theorem, $f'(0) \leqslant 0$. There are some theorems of Kanel [4, 6] and Aronson and Weinberger [11] which are helpful in that case. The following theorem is slightly weaker than Theorem 3:

THEOREM 3'. *If* $f'(0) \leqslant 0$, *then let* $u(x, t)$ *be a solution of* (1) *subject to conditions* (a)–(c) *of Theorem* 3. *Then for any numbers* c, d: $0 < d < c < 1$, *and any* $\sigma > 0$ *there is* $t_1 \geqslant 0$ *for which* $\gamma(u, t) \geqslant P(u) - \sigma$ *for any* $t \geqslant t_1$ *and* $d \leqslant u \leqslant c$.

Proof. Since $f'(0) \leqslant 0$, then $m^{\circ 2} > 4f'(0)$.

A number $m_1 < m^\circ$ can be found, with a corresponding solution $p_1(u)$ of (2), for which: $p_1(0) = 0$, $p_1(u) \geqslant P(u) - \sigma$, $0 \leqslant u \leqslant c$, and for some b: $c < b < 1$, $p_1(b) = 0$.

To $p_1(u)$ corresponds a function $V_1(x)$ such that for any α, $V_1(x + m_1 t + \alpha)$ is a solution of (1). $V_1(X)$ has a single maximum, with value b, in $0 \leqslant V_1 \leqslant 1$, and $V_1'(X)/V_1(X) \to (m_1 + [m_1^2 - 4f'(0)]^{1/2})/2$ as $X \to \infty$.

So there is a number α_1 such that for all $\alpha \geqslant \alpha_1$, $V_1(x + \alpha)$ and $u(x, 0)$ have only one intersection, since for any α_0, $V_1(x + \alpha) > u(x, 0)$ for large negative x. V_1 is the test function.

Let $m_2 = (m_1 + m^\circ)/2$. Then with $m = m_2$ there is a solution $p_2(u)$ of (2) for which:

$$p_2(0) > 0,$$

$$p_2(B) = 0, \qquad b < B < 1,$$

and

$$p_2(u) > 0, \qquad 0 \leqslant u < B.$$

The corresponding steady-shape wave solution of (2), $V_2(x + m_2 t + \alpha)$ is defined, and $0 \leqslant V_2 \leqslant B < 1$ on some closed interval I. So for some time t_2 and some $x = x_2$, $u(x, t_2) > V_2(x + m_2 t_2 + \alpha)$ on I, and hence everywhere. So V_2 is an underfunction. Let X_2 be the value of X for which $V_2(X) = B$, and X_1 the value for which $V_1(X) = d$, and $V_1'(X) \geqslant 9$.

Let $t_1 = \text{Max}[X_2 - X_1 + \alpha_1 - \alpha_2)/(m_2 - m_1), t_2]$. Then for $t_3 \geqslant t_1$, let $x_2 = X_2 - m_2 t_3 - \alpha_2$.

Then $V_1(x_2 + m_1 t_3 + \alpha_1) \leqslant d$, and $u(x_2, t_3) < V_2(x_2 + m_2 t_3 + \alpha_2) = B < b$. So for any z: $d \leqslant z \leqslant c$, the leftmost point x_3 for which $u(x_3, t_3) = z$ is to the left of x_2. So if α_3 is the number for which $V_1(x_3 + m_1 t_3 + \alpha_3) = z$, $V_1'(x_3 + m_1 t_3 + \alpha_3) > 0$, then $x_3 + m_1 t_3 + \alpha_3 > x_2 + m_2 t_3 + \alpha_2$, i.e., $\alpha_3 > \alpha_2$.

So $V_1(x + m_1 t_3 + \alpha_3)$ has only one intersection with $u(x, t_3)$, therefore $u_x(x_3, t_3) > V_1'(x_3 + m_1 t_3 + \alpha_3)$, i.e., $\gamma(z, t_3) > P(z) - \sigma$. Q.E.D.

The methods of Theorems 2 and 3 or 3′ could be extended to show that for initial conditions permitting a less rapid but still exponential diminution of $u(x, 0)$ as $x \to -\infty$, convergence to the shape of a different wave, with higher velocity, is obtained. This has not been done here, partly because such initial conditions do not seem to be important in applications, and partly because Sattinger (10) has obtained some convergence proofs by different (spectral) methods, although with a convexity-type restriction on $f(u)$.

While the theorems already proved allow deductions to be made about the shape of solutions in the wave-front area, an extension can be made to show uniform convergence of the whole solution to a certain shape. The following theorem is restricted in its application to initial functions of compact support; this is perhaps the most interesting case, and the method, once demonstrated, can be easily applied to other cases.

The next theorem is somewhat weaker than the preceding results, in that nothing is said about the behavior of the gradients of solutions.

THEOREM 4. *Let $u(x, t)$ be a solution of* (1) *with* $1 \geqslant u(x, 0) \geqslant 0$, *and* $u(x, 0) = 0$ *outside some compact interval. Suppose that, for each x, $u(x, t) \to 1$ as $t \to \infty$. Then for any $\sigma > 0$, there exists t_1 such that for all $t \geqslant t_1$, there are numbers $\alpha_1(t), \alpha_2(t)$ for which $| u(x, t) - V(x + \alpha'(t)) - V(-x + \alpha_2(t)) + 1 | < \sigma$ for all x, where $V(X)$ is the function determining the traveling wave solution of* (1) *with minimum speed.*

Proof. Let σ_1 be any number: $0 < \sigma_1 < 1 - u_+$ and let $\sigma = \sigma_1/2$. If the direction of the x-axis is reversed, a function $\gamma_-(u, t)$ may be defined analogously to $\gamma(u, t)$, and Theorems 2 and 3 or 3′ apply to show that $\gamma_-(u, t)$ converges to $-P(u)$.

Then there is some $t_1 > 0$ such that if $t \geqslant t_1$, $\gamma(u, t)$ and $\gamma_-(u, t)$ are defined for $\sigma < u < 1 - \sigma$, and

$$| 1/\gamma(u, t) - 1/P(u)| < \sigma/A,$$

$$| 1/\gamma_-(u, t) + 1/P(u)| < \sigma/A$$

where

$$A = \operatorname*{Max}_{0 \leqslant u \leqslant 1} P(u).$$

Define $c(t)$ and $c_-(t)$ to be respectively the least and greatest x for which $u(x, t) = 1 - \sigma$. Since $\gamma(1 - \sigma, t)$ and $\gamma_-(1 - \sigma, t)$ are nonzero for $t \geqslant t_1$, then c and c_- are well-defined and continuous functions of t, and $c(t) < c_-(t)$.

For some $m < m^\circ$ it is possible to find a solution $V_1(X)$ of (2) for which, for some positive a_1, a_2, $V_1(0) = V_1(a_2) = 0$, $V_1(a_1) = 1 - 2\sigma$, $V_1'(a_1) = 0$, and $V_1(X) \leqslant 1 - 2\sigma$ in $[0, a_2]$.

Then $V_1(-x + mt + \alpha)$ is a solution of (1), and if $\alpha \leqslant \alpha_1 = c(t_1) - mt_1$, its right zero is to the left of $c(t_1)$, and any intersections it has with $u(x, t_2)$ are to the left of $c(t_1)$. By Theorem 1, it is impossible for $V_1(-x + mt + \alpha)$ then to intersect $u(x, t)$ to the right of $c(t)$ for $t \geqslant t_1$, if $\alpha \leqslant \alpha_1$, since $u(c(t), t) = 1 - \sigma > V_1(-c(t) + m(t) + \alpha)$.

(i) Therefore, for $t \geqslant t_1$, if $c(t) \leqslant x \leqslant c(t_1) + m(t - t_1) - a_1$ then $u(x, t) > V_1(-x + mt + a_1 + x - mt) = 1 - 2\sigma$.

(ii) Similarly, if $c_-(t) \geqslant x \geqslant c_-(t_1) - m(t - t_1) + a_1$ then $u(x, t) > 1 - 2\sigma$.

If $t \geqslant t_2 = t_1 + (c_-(t_1) - c(t_1) + 2a_1)/m$, then either (i) or (ii) is true for any x lying in $[c(t), c_-(t)]$.

For $t \geqslant t_2$ let $\alpha_1(t)$, $\alpha_2(t)$ be numbers for which $V(c(t) + \alpha_1(t)) = 1 - \sigma$, $V(-c_-(t) + \alpha_2(t)) = 1 - \sigma$, respectively. Then when $c(t) \leqslant x \leqslant c_-(t)$,

$$2\sigma > 1 - V(x + \alpha_1(t)) + 1 - V(-x + \alpha_2(t)) - (1 - u(x, t)) > -2\sigma,$$

i.e., $| u(x, t) + 1 - V(x + \alpha_1(t)) + V(-x + \alpha_2(t))| < 2\sigma$ in $[c(t), c_-(t)]$.

Let x_1 be the least value of x for which $u(x, t) = \sigma$. Then for any x in $[x_1, c(t)]$, $\sigma \leqslant u(x, t) = u \leqslant 1 - \sigma$. Let $u_1 = V(x + \alpha_1(t))$. Then

$$x - c(t) = \int_u^{1-\sigma} \frac{dv}{\gamma(v, t)} = \int_{u_1}^{1-\sigma} \frac{dv}{P(v)}.$$

So

$$\int_u^{u_1} \frac{dv}{P(v)} = \int_u^{1-\sigma} \left[\frac{1}{\gamma(v, t)} - \frac{1}{P(v)}\right] dv,$$

and

$$| u_1 - u | \leqslant \left[\operatorname*{Max}_{0 \leqslant V \leqslant 1} P(V)\right] \cdot \left[\operatorname*{Max}_{0 \leqslant V \leqslant 1-\sigma} \left|\frac{1}{\gamma(V, t)} - \frac{1}{P(V)}\right|\right] < \sigma \quad \text{since} \quad t \geqslant t_1.$$

Since $x \leqslant c(t) < c_-(t)$, $V(-x + \alpha_2(t)) \geqslant V(-c_-(t) + \alpha_2(t)) = 1 - \sigma$. Therefore $| u(x, t) - V(x + \alpha_1(t)) + 1 - V(-x + \alpha_2(t))| < 2\sigma$, when $x_1 \leqslant x \leqslant c(t)$.

And when $x \leqslant x_1$, $0 < u(x, t) \leqslant \sigma$, and $0 < V(x + \alpha_1(t)) \leqslant V(x_1 + \alpha_1(t) \leqslant u(x_1, t) + \sigma \leqslant 2\sigma$ and again $V(-x + \alpha_2(t)) > 1 - \sigma$.

So for $x \leqslant x_1$, $-2\sigma < u(x, t) - V(x + \alpha_1(t)) + 1 - V(-x + \alpha_2(t)) < 2\sigma$ and $| u(x, t) + 1 - V(x + \alpha_1(t)) - V(-x + \alpha_2(t))| < 2\sigma = \sigma_1$ for all $t \geqslant t_2$, $x \leqslant c_-(t)$.

By arguments analogous to those above, the inequality can be verified for the interval $x \geqslant c_-(t)$.

Discussion

As Theorem 4 indicates, it is possible to deduce from Theorems 2 and 3 almost everything of interest about the eventual shape and speed of the wavelike solutions of (1). It is possible to refine the argument from conservation of intersections still further to show that, locally, the velocity of a level point of a solution converges. This is stronger than the result deduced from the limits imposed by the existence of over and underfunctions, which is that the velocity of the wave front as a whole tends to m°. That does not exclude the possibility that, for example, locally the velocity might incorporate an oscillation. The proof of that stronger result is lengthy, and is not given here.

The method used here does not easily give any indication of the rate at which a traveling-wave shape is assumed, or the rate at which the ultimate velocity is approached. This rate of convergence of velocity determines whether the solution actually converges to one of the traveling waves or not. Kanel's result [3], showing actual convergence under some circumstances, actually reflects a deeper distinction between waves for which $m^{\circ 2} = 4f'(0)$, and waves where $m^{\circ 2} > 4f'(0)$, convergence being obtained for certain boundary conditions in the latter case. The implications of this distinction are investigated in [14].

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