JOURNAL OF PURE AND APPLIED ALGEBRA

ELSEVIER

Journal of Pure and Applied Algebra 155 (2001) 257–270

Absolutely convex modules and Saks spaces

D. Pumplün

Fachbereich Mathematik, Fernuniversität – GHS, D-58084 Hagen, Germany

Received 10 March 1999; received in revised form 7 May 1999
Communicated by P.T. Johnstone

Dedicated to Helmut Röhrl on the occasion of his 70th birthday

Abstract

Absolutely or totally convex modules are a canonical generalization of absolutely or countably absolutely convex sets in linear spaces. There are canonical connections between the category of absolutely convex modules and the category of Saks spaces, each of which is given by a pair of adjoint functors. Corresponding results hold for totally convex modules. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 52A01; 46A70; 46M15; 46A55

0. Introduction

Absolutely convex modules were introduced in [11,12], where they are still called *finitely totally convex spaces*. For a non-empty set M let $M^{(\mathbb{N})}$ denote the set of all finite sequences of elements of M. We define $\Omega_{\mathrm{ac}} := \{\hat{\alpha} \mid \hat{\alpha} \in \mathbb{K}^{(\mathbb{N})} \text{ and } \sum_{i=1}^{\infty} |\alpha_i| \leq 1\}$, where here and in the following, for $\hat{\alpha} = (\alpha_1, \dots, \alpha_n)$, the formal infinite sum $\sum_{i=1}^{\infty} \alpha_i$ is introduced as a convenient notation as $\sum_{i=1}^{\infty} \alpha_i = \sum_{i=1}^{n} \alpha_i$ and is often written simply as $\sum_i \alpha_i$. An absolutely convex module C is a non-empty set together with a family of operations $\hat{\alpha}_C : C^{(\mathbb{N})} \to C$, for any $\hat{\alpha} \in \Omega_{\mathrm{ac}}$. Writing $\sum_i \alpha_i c_i := \sum_{i=1}^{\infty} \alpha_i c_i := \sum_{i=1}^{n} \alpha_i c_i := \hat{\alpha}_C(c_i, \dots, c_n)$ for $\hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Omega_{\mathrm{ac}}$ and $(c_1, \dots, c_n) \in C^{(\mathbb{N})}$, these operations are, moreover, required to satisfy the following set of equations:

$$\sum_i \delta_{ik} c_i = c_k, \qquad \text{(AC1)}$$

E-mail address: dieter.pumpluen@fernuni-hagen.de (D. Pumplün).

PII: S0022-4049(99)00103-6

$$\sum_i \alpha_i \left(\sum_k \beta_{ik} c_k \right) = \sum_k \left(\sum_i \alpha_i \beta_{ik} \right) c_k \tag{AC2}$$

for any $\hat{\alpha}$, $\beta_i = (\beta_{ik} \mid 1 \le k \le m_i) \in \Omega_{\mathrm{ac}}$ and $c_i \in C$, $i \in \mathbb{N}$, for which the sums are defined. Together with absolutely affine mappings, i.e. mappings preserving these formal sums $f(\sum_{i=1}^n \alpha_i c_i) = \sum_{i=1}^n \alpha_i f(c_i)$, category **AC** of absolutely convex modules. Absolutely convex modules are a natural generalization of absolutely convex sets in linear spaces, they reflect the algebraic component of convexity. Here and in the following $\mathbb{K}$ denotes the field $\mathbb{R}$ of real or $\mathbb{C}$ of complex numbers. So actually one talks about two categories, namely the categories of real or complex absolutely convex modules, respectively.

A totally convex module C (cf. [11,12]) is defined in an analogous manner as a non-empty Ω-algebra, where $\Omega := \{\hat{\alpha} \mid \hat{\alpha} = (\alpha_i \mid i \in \mathbb{N}),\ \alpha_i \in \mathbb{K}, \sum_{i=1}^{\infty} |\alpha_i| \le 1\}$, satisfying the same equations (AC1), (AC2). This yields the category **TC** of totally convex modules and totally affine mappings, i.e. mappings preserving these formal sums $f(\sum_{i=1}^{\infty} \alpha_i c_i) = \sum_{i=1}^{\infty} \alpha_i f(c_i)$ (cf. [3,8]).

If E is a normed $\mathbb{K}$-vector space, its unit ball $\bigcirc(E) := \{x \mid x \in E, ||x|| \le 1\}$ is an absolutely convex set and hence an absolutely convex module which is denoted by $\widehat{\bigcirc}_{\mathrm{ac}}(E)$. This induces a full and faithful functor $\widehat{\bigcirc}_{\mathrm{ac}} : \mathbf{Vec}_1 \to \mathbf{AC}$, $\mathbf{Vec}_1$ the category of normed $\mathbb{K}$-vector spaces and linear contractions. In the same way, the unit ball $\bigcirc(E)$ of a $\mathbb{K}$-Banach space E induces a full and faithful functor $\widehat{\bigcirc} : \mathbf{Ban}_1 \to \mathbf{TC}$, $\mathbf{Ban}_1$ the category of $\mathbb{K}$-Banach spaces and linear contractions. The close connection between the theory of absolutely convex modules and the theory of normed vector spaces, or the theory of totally convex modules and the theory of Banach spaces, respectively, is shown by the fact that both functors have a left adjoint $S_0 : \mathbf{AC} \to \mathbf{Vec}_1$, $S_1 : \mathbf{TC} \to \mathbf{Ban}_1$, which means that any absolutely (totally) convex module generates a unique normed vector (Banach) space (cf. [11,12]).

The existence of S_0 and S_1 is guaranteed by a general theorem and a constructive proof of existence by "blowing up" is given in [11]. Because of their importance in establishing a close connection between functional analytic theories and purely algebraic ones it is of interest to get information about the structure of $S_0(C)$ ($S_1(C)$) for an absolutely (totally) convex module C. In the following a method used by Semadeni in [15,16], to prove the existence of a *universal compactification* of a bounded convex set in a locally convex vector space and, in the theory of (approximate) order unit Banach spaces (cf. [17, 9.11,9.13]), is used to give a new existence proof for S_0 and S_1. One forms the vector space of absolutely or totally affine mappings, respectively. Affine mappings were already used to prove the existence of the corresponding functor S for convex modules in [13] and superconvex modules in [10]. This approach yields more than an existence proof for S_0 and S_1, namely a canonical functor S_* to the category of Saks spaces [2].

For topological totally convex spaces, S_* turns out to be a left adjoint. In [6,7] Kleisli and Künzi have used a closely related method for proving the existence of $S_1(C)$ for their topological totally convex modules with the strong topology.

1. The associated Banach space

The main method used in the following is dualization. To illustrate its usefulness for absolutely convex modules and also for other problems, we give at first a short proof for the reflection of **AC** into the subcategory of singular **AC**-modules.

For an absolutely convex module C let $\mathbf{AC}(C,\mathbb{K})$ denote the (hom-) set of all absolutely affine morphisms from C to $\mathbb{K}$. With the pointwise operations $\mathbf{AC}(C,\mathbb{K})$ is a $\mathbb{K}$-vector space and let $\mathbf{AC}^*(C,\mathbb{K})$ be its dual space. $\tilde{\rho}_C : C \to \mathbf{AC}^*(C,\mathbb{K})$ is defined by $\tilde{\rho}_C(x)(f) := f(x)$, for $x \in C$, $f \in \mathbf{AC}^*(C,\mathbb{K})$. An absolutely convex module C is called *singular* (cf. [12, 14.10]) if the "norm" of C is trivial. It was proved in [12] that the singular absolutely convex modules coincide with the *injective* objects of **AC** and with the $\mathbb{K}$-vector spaces. The full subcategory $\mathbf{AC}_{\text{sing}}$ of singular convex modules is ext-mono-coreflective ([12, 14.11]). Writing $R(C)$ for the subspace of $\mathbf{AC}^*(C,\mathbb{K})$ generated by $\tilde{\rho}_C(C)$ and ρ_C for the co-restriction of $\tilde{\rho}_C$ to $R(C)$ one gets the

Proposition 1.1. *For* $C \in \mathbf{AC}$, $\rho_C : C \to R(C)$ *is the reflection of* **AC** *into* $\mathbf{AC}_{\text{sing}}$. $\mathbf{AC}_{\text{sing}}$ *is not an epi-reflective subcategory of* **AC**.

Proof. Let $\varphi : C \to V$ be a morphism in **AC** and V be singular i.e. a $\mathbb{K}$-vector space. $\rho_C(x) = \rho_C(y)$, for $x, y \in C$, $\varphi(x) \neq \varphi(y)$ would imply the existence of a $\lambda \in V^*$, the dual of V, with $\lambda\varphi(x) \neq \lambda\varphi(y)$, which is a contradiction. Hence, there exists a unique **AC**-morphism $\bar{\varphi}_0 : \rho_C(C) \to V$ with $\varphi = \bar{\varphi}_0 \rho_C$. $\bar{\varphi}_0$ can be extended canonically to a vector space homomorphism, i.e. an **AC**-morphism, $\varphi_0 : R(C) \to V$ with $\varphi = \varphi_0 \rho_C \cdot \varphi_0$ is obviously uniquely determined by φ and this equation.

If $\mathbf{AC}_{\text{sing}}$ were an epi-reflective subcategory of **AC** it would be closed under taking subobjects, which would imply that an absolutely convex subset of a vector space is a vector space. □

Definition 1.2. For an absolutely convex module C define

$$\mathrm{Aff}_0(C) := \{f \mid f \in \mathbf{AC}(C,\mathbb{K}) \text{ and } f \text{ bounded}\}.$$

$\mathrm{Aff}_0(C)$ is a vector subspace of $\mathbf{AC}(C,\mathbb{K})$ and a Banach space with the norm

$$||f||_\infty := \sup\{|f(c)| \mid c \in C\}, \quad f \in \mathrm{Aff}_0(C).$$

For a topological $\mathbb{K}$-vector space E, E' will denote the topological dual of E, i.e. the space of all continuous linear forms on E. For a normed space E, E' is a Banach space with the norm $||\Box||_\infty$.

If $E \in \mathbf{Vec}_1$, the category of normed $\mathbb{K}$-vector spaces and linear contractions, the unit ball $\bigcirc(E)$ of E is an absolutely convex subset of E, hence an object of **AC**, which will be denoted by $\widehat{\bigcirc}_{ac}(E)$. This induces a covariant, full and faithful functor $\widehat{\bigcirc}_{ac} : \mathbf{Vec}_1 \to \mathbf{AC}$ (cf. [11]).

For $C \in \mathbf{AC}$, define $\tilde{\sigma}_C : C \to (\mathrm{Aff}_0(C))'$ by $\tilde{\sigma}_C(x)(f) := f(x)$, $x \in C$, $f \in \mathrm{Aff}_0(C)$. Then $\tilde{\sigma}_C$ is an **AC**-morphism. $\tilde{\sigma}_C(C)$ is an absorbent, absolutely convex subset of the subspace $S_0(C)$ generated by $\tilde{\sigma}_C(C)$ in $(\mathrm{Aff}_0(C))'$.

Proposition 1.3 (cf. Pumplün and Röhrl [11, 7.10]). *With the Minkowski functional* $\|\Box\|$ *of* $\tilde{\sigma}_C(C)$, $S_0(C)$ *is a normed* $\mathbb{K}$*-vector space and induces a functor* $S_0 : \mathbf{AC} \to \mathbf{Vec}_1$. S_0 *is left adjoint to* $\widehat{\bigcirc}_{ac}$ *with front adjunction* $\sigma_C^0 : C \to \widehat{\bigcirc}_{ac} \circ S_0(C)$, *where* σ_C^0 *denotes the co-restriction of* $\tilde{\sigma}_C$ *to* $S_0(C)$. *Moreover,* $\mathring{\bigcirc}(S_0(C)) \subset \sigma_C^0(C) \subset \bigcirc(S_0(C))$ *holds, where* $\mathring{\bigcirc}$ *denotes the open unit ball.*

Proof. One has $S_0(C) = \mathbb{K}\tilde{\sigma}_C(C)$ and $\|\Box\|$ is a norm, because $\tilde{\sigma}_C(C)$ is absolutely convex and bounded in $(\mathrm{Aff}_0(C))'$ with respect to $\|\Box\|_\infty$. If $E \in \mathbf{Vec}_1$ and $\varphi : C \to \widehat{\bigcirc}_{ac}(E)$ is an $\mathbf{AC}$-morphism one shows, using the Hahn–Banach Theorem, as in the proof of 1.1 that there is an $\mathbf{AC}$-morphism $\tilde{\varphi} : \tilde{\sigma}_C^0(C) \to \widehat{\bigcirc}_{ac}(E)$ which can be extended uniquely to a linear mapping $\varphi_0 : S_0(C) \to E$. As $\sigma_C^0(C) \subset \bigcirc(\mathrm{S}_0(C))$ and $\sigma_C^0(C)$ is dense in $\bigcirc(S_0(C))$, because of $\mathring{\bigcirc}(S_0(C)) \subset \sigma_C(C)$, $\varphi_0(\bigcirc(S_0(C)) \subset \bigcirc(E)$ follows, i.e. φ_0 is a linear contraction. φ_0 is uniquely determined by φ and $\varphi = \widehat{\bigcirc}_{ac}(\varphi_0)\sigma_C^0$ for the same reason. □

If X is a bounded subset of a Banach space B, the smallest totally convex subset of B containing X, the *totally convex closure* totconv(X) exists. It may be described as the intersection of all totally convex subsets containing X or as $\{b \in B \mid b = \sum_{i=1}^\infty \alpha_i x_i,\ \hat{\alpha} = (\alpha_i \mid i \in \mathbb{N}) \in \Omega,\ x_i \in X,\ i \in \mathbb{N}\}$. Totally convex subsets of arbitrary vector spaces may prove useful in functional analysis.

Definition 1.4. A set $M \neq \emptyset$ is called a *totally convex set*, if it is a subset of a vector space E and a totally convex module, such that for any $\hat{\alpha} \in \Omega$ with finite support, i.e. $\alpha_i = 0$ for $i > n$ for some n,

$$\sum_{i=1}^{\infty} \alpha_i x_i = \sum_{i=1}^{n} \alpha_i x_i$$

holds, if $x_i \in M$, $i \in \mathbb{N}$, where the sum on the right-hand side of this equation is just the ordinary (absolutely convex) sum in E.

The totally convex sets are up to isomorphism, exactly the totally convex modules, which can be embedded into a vector space by an injective $\mathbf{AC}$-morphism. A necessary condition for an absolutely convex subset of a vector space to be a totally convex subset is linear boundedness. Using a method of Rodé with which he proved the corresponding result for convex sets in [14], one can show that an absolutely convex set has at most one structure of a totally convex set. The following result should prove useful and is stronger than Lemma 1.2 in [2] for Banach balls.

Lemma 1.5. *If* $C \subset E$ *is a totally convex subset of a vector space* E, *then the subspace* $E_0 := \mathbb{K}C$ *is a Banach space with the Minkowski functional of* C *as norm.*

Proof. (cf. Pumplün and Röhrl [11, 7.4]): We may assume $E_0 = E$. For $x \in E$,

$$\|x\| := \inf\{\alpha > 0 \mid x \in \alpha C\}$$

is the Minkowski functional, i.e. a seminorm. In order to see that $||\square||$ is a norm consider $a \in E$ with $||a|| = 0$. Then, for any $n \in \mathbb{N}$, there is a $c_n \in C$ with $a = 2^{-n}c_n$. With $c_0 := a$, we introduce the "telescopic" formal totally convex sum

$$z := \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} c_{n-1}.$$

As $c_{n+1} = 2c_n$, $n \in \mathbb{N}$, the computational rules 2.4 of Pumplün and Röhrl [11] imply

$$z = \frac{1}{2}\left(\frac{1}{2}a + \frac{1}{2}\sum_{n=2}^{\infty} \frac{1}{2^{n-1}} c_{n-1}\right) = \frac{1}{4}a + \sum_{n=2}^{\infty} \frac{1}{2^{n+1}} c_{n-1}$$
$$= \frac{1}{4}a + \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} c_n = \frac{1}{4}a + \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} (2c_{n-1}) = \frac{1}{4}a + z.$$

Hence $a = 0$ follows, i.e. $||\square||$ is a norm.

Let $x_n \in E$, $n \in \mathbb{N}$, $\sigma := \sum_{n=1}^{\infty} ||x_n|| < \infty$ and assume $x_n \neq 0$, for $n \in \mathbb{N}$. Then $c_n := (2||x_n||^{-1})x_n \in C$, $n \in \mathbb{N}$, Putting $\alpha_n := \sigma^{-1}||x_n||$, $n \in \mathbb{N}$, results in $(\alpha_n \mid n \in \mathbb{N}) \in \Omega$, hence

$$c := \sum_{n=1}^{\infty} \alpha_n c_n$$

is well defined and $c \in C$. Using again 2.4 in [11] one obtains

$$2\sigma c - \sum_{i=1}^{n} x_i = 2\sigma \sum_{i=1}^{\infty} \alpha_i c_i - 2\sum_{i=1}^{n} ||x_i|| c_i$$
$$= 2\sigma\left(\sum_{i=1}^{\infty} \alpha_i c_i - \sum_{i=1}^{n} \alpha_i c_i\right).$$

With $t_n := \sum_{v=1}^{n} \alpha_v$, $n \in \mathbb{N}$,

$$\sum_{i=1}^{\infty} \alpha_i c_i - \sum_{i=1}^{n} \alpha_i c_i = t_n \sum_{i=1}^{n} (t_n^{-1}\alpha_i)c_i + (1-t_n)\sum_{i=n+1}^{\infty} (1-t_n)^{-1}\alpha_i c_i$$
$$- t_n \sum_{i=1}^{n} (t_n^{-1}\alpha_i)c_i = \sum_{i=n+1}^{\infty} \alpha_i c_i$$

follows. The norm inequality [11, 6.2], implies

$$\left\|2\sigma c - \sum_{i=1}^{n} x_i\right\| \leq 2\sigma \sum_{i=n+1}^{\infty} \alpha_i ||c_i|| \leq 2\sigma \sum_{i=n+1}^{\infty} \alpha_i,$$

hence $\sum_{i=1}^{\infty} x_i$ converges to $2\sigma c$ and E is a Banach space. $\square$

The proof shows that $\mathring{\bigcirc}(E) \subset C \subset \bigcirc(E)$ holds. The open unit ball of a Banach space supplies an example of a totally convex set, which is neither sequentially complete nor closed. Conversely, if $C \subset E$ is absolutely convex, absorbent and E is a Banach space with the Minkowski functional $\|\square\|$ of C, then $\|\square\|$ is also the Minkowski functional of $\bar{C} = \bigcirc(E)$, i.e. of a totally convex set in E. This shows that the result 1.5 is sharp. Denoting the restriction of $\widehat{\bigcirc}_{ac}$ to the subcategory $\mathbf{Ban}_1$ by $\widehat{\bigcirc}$ one gets the following

Proposition 1.6 (cf. Pumplün and Röhrl [11, Section 7], Kleisli and Künzi [6, 3.6]). *For $C \in \mathbf{AC}$, $S_1(C) := \mathbb{K}\,\mathrm{totconv}(\tilde{\sigma}_C(C))$ is a Banach space with norm the Minkowski functional of* $\mathrm{totconv}(\tilde{\sigma}_C(C))$. *$S_1(C)$, $C \in \mathbf{AC}$, induces a functor $S_1 : \mathbf{AC} \to \mathbf{Ban}_1$ left adjoint to $\widehat{\bigcirc} : \mathbf{Ban}_1 \to \mathbf{AC}$.*

Proof. $\tilde{\sigma}_C(C)$ is a bounded subset of $(\mathrm{Aff}_0(C))'$ hence its totally convex closure exists and the first assertion follows from Lemma 1.5. Now, for an $\mathbf{AC}$-morphism $\varphi : C \to \widehat{\bigcirc}(B)$, $B \in \mathbf{Ban}_1$, consider the $\mathbf{AC}$-morphism $\tilde{\varphi} : \tilde{\sigma}_C(C) \to \widehat{\bigcirc}(B)$ in the proof of 1.3. If $\tilde{\varphi}$ can be extended to an $\mathbf{AC}$-morphism $\tilde{\varphi}_1 : \mathrm{totconv}(\tilde{\sigma}_C(C)) \to \widehat{\bigcirc}(B)$ it must satisfy

$$\tilde{\varphi}_1\left(\sum_{i=1}^{\infty} \alpha_i c_i\right) = \sum_{i=1}^{\infty} \alpha_i \varphi(c_i), \tag{$*$}$$

$c_i \in C$, $i \in \mathbb{N}$, $\hat{\alpha} \in \Omega$. To see that $(*)$ may be used as a definition for $\tilde{\varphi}_1$, consider an equation

$$\sum_{i=1}^{\infty} \alpha_i \cdot \tilde{\sigma}_C(c_i) = \sum_{i=1}^{\infty} \beta_i \cdot \tilde{\sigma}_C(d_i),$$

c_i, $d_i \in C$, $i \in \mathbb{N}$, $\hat{\alpha}$, $\hat{\beta} \in \Omega$. This implies

$$\sum_{i=1}^{\infty} \alpha_i f(c_i) = \sum_{i=1}^{\infty} \beta_i f(d_i)$$

for every $f \in \mathrm{Aff}_0(C)$. Now, for any $\lambda \in B'$, $\lambda\varphi \in \mathrm{Aff}_0(C)$ holds, hence

$$\sum_{i=1}^{\infty} \alpha_i (\lambda\varphi)(c_i) = \sum_{i=1}^{\infty} \beta_i (\lambda\varphi)(d_i)$$

follows, which implies

$$\sum_{i=1}^{\infty} \alpha_i \varphi(c_i) = \sum_{i=1}^{\infty} \beta_i \varphi(d_i),$$

i.e. $\tilde{\varphi}_1$ is well-defined by $(*)$. $\tilde{\varphi}_1$ extends uniquely to a linear contraction $\varphi_1 : S_1(C) \to B$ satisfying $\varphi = \widehat{\bigcirc}(\varphi_1)\hat{\sigma}_C$, where $\hat{\sigma}_C$ is the co-restriction of $\tilde{\sigma}_C$.

It is well known that $\mathbf{Ban}_1$ is a dense-reflective, full subcategory of $\mathbf{Vec}_1$ with reflection $T : \mathbf{Vec}_1 \to \mathbf{Ban}_1$ the completion. Hence $T \circ S_0$ is also a left adjoint of $\widehat{\bigcirc}$, which means that $S_1(C)$ is the completion of $S_0(C)$ in $(\mathrm{Aff}_0(C))'$ and the inclusion $S_0(C) \subset S_1(C)$ is an isometry. $\square$

The full subcategory $\mathbf{TC}$ of $\mathbf{AC}$ of totally convex modules is of particular interest, so we investigate the above construction now for totally convex modules.

Definition 1.7. Let $C \in \mathbf{TC}$, $E \in \mathbf{Vec}_1$, then a mapping $f : C \to E$ is called *totally affine*, if for any $(\alpha_i \,|\, i \in \mathbb{N}) \in \Omega$ and any $x_i \in C$, $i \in \mathbb{N}$,

$$f\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{\infty} \alpha_i f(x_i)$$

holds. Of course, this equation is to mean that the right side is convergent in the norm of E.

Lemma 1.8. *For $C \in \mathbf{TC}$, $E \in \mathbf{Vec}_1$, a mapping $f : C \to E$ is totally affine if and only if f is absolutely affine and bounded.*

Proof. If f is totally affine, it is obviously absolutely affine. Assume f not to be bounded, then for any $n \in \mathbb{N}$, there is $x_n \in C$ with $||f(x_n)|| \geq 2^n$ and $\sum_{n=1}^{\infty} 2^{-n} f(x_n)$ is convergent in the norm, hence $\lim_{n\to\infty}(2^{-n}||f(x_n)||) = 0$, which is a contradiction.

Conversely, assume f to be absolutely affine and bounded. Let $(\alpha_i \,|\, i \in \mathbb{N}) \in \Omega$, $x_i \in C$, $i \in \mathbb{N}$ and put

$$A_n := \sum_{i=1}^{n} |\alpha_i|.$$

One may assume $0 < A_n < 1$ for all $n \in \mathbb{N}$. Because of (AC2) in the infinite case

$$\sum_{i=1}^{\infty} \alpha_i x_i = A_n \sum_{i=1}^{n} \frac{\alpha_i}{A_n} x_i + (1 - A_n) \sum_{i=n+1}^{\infty} \frac{\alpha_i}{1 - A_n} x_i$$

for $n \in \mathbb{N}$, which implies

$$f\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) - \sum_{i=1}^{n} \alpha_i f(x_i) = (1 - A_n) f\left(\sum_{i=n+1}^{\infty} \frac{\alpha_i}{1 - A_n} x_i\right)$$

or, as f is bounded by some $M > 0$, i.e. $||f(x)|| \leq M$, for $x \in C$ (cf. [11, 6.2]):

$$\left\| f\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) - \sum_{i=1}^{n} \alpha_i f(x_i) \right\| \leq M \sum_{i=n+1}^{\infty} |\alpha_i|.$$

Hence, $\sum_{i=1}^{\infty} \alpha_i f(x_i)$ is convergent to $f(\sum_{i=1}^{\infty} \alpha_i x_i)$ in the norm of E. □

Lemma 1.9. *Let $f : C \to E$ be totally affine, $C \in \mathbf{TC}$, $E \in \mathbf{Vec}_1$, then $V := \mathbb{K} f(C)$ is a $\mathbb{K}$-Banach space with norm the Minkowski functional of $f(C)$:*

$$||z||_0 := \inf\{\alpha > 0 \,|\, z \in \alpha f(C)\}.$$

This follows immediately from Lemma 1.5 because $f(C)$ is obviously a totally convex subset of E (cf. Lemma 1.8).

Definition 1.7, Lemmas 1.8 and 1.9 remain valid if E is a Hausdorff locally convex $\mathbb{K}$-vector space (cf. [2, 1.2, p. 4]), but they are needed here only for normed spaces.

The unit ball $\bigcirc(B)$ of a $\mathbb{K}$-Banach space has a canonical totally convex structure and induces a full and faithful functor $\widehat{\bigcirc} : \mathbf{Ban}_1 \to \mathbf{TC}$. Because of Lemma 1.8, Definition 1.2 remains unchanged for $\mathbf{TC}$ and so does the definition of $S_0(C)$ in 1.3.

Corollary 1.10 (cf. Pumplün and Röhrl [11, 7.7], Kleisli and Künzi [6, 3.6]). *For $C \in \mathbf{TC}$, $S_0(C) = S_1(C)$ holds and $\sigma_C^1 = \sigma_C^0$ is a* **TC**-*morphism. S_1 is left adjoint to $\widehat{\bigcirc}$ with front adjunction $\sigma_C^1 : C \to \widehat{\bigcirc} \circ S_1(C)$.*

Proof. Because of Lemma 1.8 $\tilde{\sigma}_C$ is totally affine, hence $\tilde{\sigma}_C(C)$ is totally convex (cf. Lemma 1.9), i.e. $\tilde{\sigma}_C(C) = \mathrm{totconv}(\tilde{\sigma}_C(C))$ and the assertion follows from Proposition 1.6. □

2. The associated Saks space

$(\mathrm{Aff}_0(C))'$, for $C \in \mathbf{AC}$, also contains another set, which, in the classical case of a bounded, convex subset C of a locally convex vector space, is called the *affine compactification* of C by Semadeni in [15]. The underlying structure of this construction turns out to be a functor from **AC** to the category $\mathbf{Saks}_1$ of Saks spaces and continuous linear contractions.

Definition 2.1 (cf. Cooper [2, 3.2, p. 28]). A *Saks space* is a triple $(||\square||, E, \mathfrak{T})$ where $(||\square||, E)$ is a normed $\mathbb{K}$-vector space, $(E, \mathfrak{T})$ is a (Hausdorff) locally convex $\mathbb{K}$-vector space with topology $\mathfrak{T}$ and the unit ball $\bigcirc(E)$ is bounded and closed in $\mathfrak{T}$.

For the sake of simplicity we will denote a Saks space by a letter e.g. E and, if several spaces are involved, its norm by $||\square||_E$ and its topology by $\mathfrak{T}_E$. A *morphism $f : E_1 \to E_2$ of Saks spaces* is a linear contraction with respect to the norms, such that its restriction to the unit balls $\bigcirc(f) : \bigcirc(E_1) \to \bigcirc(E_2)$ is $\mathfrak{T}_{E_1} - \mathfrak{T}_{E_2}$ continuous. The Saks spaces together with their morphisms form the category $\mathbf{Saks}_1$ of Saks spaces. A Saks space E is called *complete* if $\bigcirc(E)$ is $\mathfrak{T}_E$-complete, from which it follows that $(||\square||, E)$ is a Banach space (cf. Lemma 1.5; [2, 1.2]). A Saks space E is called *compact*, if $\bigcirc(E)$ is $\mathfrak{T}_E$-compact; a compact Saks space is obviously complete. The full subcategory of compact Saks spaces will be denoted by $\mathbf{CompSaks}_1$.

In the following $\sigma((\mathrm{Aff}_0(C))', \mathrm{Aff}_0(C))$ will denote the weak $*$-topology of $(\mathrm{Aff}_0(C))'$; $(\mathrm{Aff}_0(C))'$ with the supremum norm and this topology is a Saks space. For a subset $M \subset (\mathrm{Aff}_0(C))'$, $\mathrm{cl}_*(M)$ will denote its *weak $*$-closure*. If $\mathrm{cl}_\infty(\square)$ is the *closure operator* of the $||\square||_\infty$*-topology* in $(\mathrm{Aff}_0(C))'$, one has $\mathrm{cl}_\infty(\tilde{\sigma}_C(C)) \subset \mathrm{cl}_*(\tilde{\sigma}(C))$, because any weakly $*$-closed set is $||\square||_\infty$-closed. One defines $S_*(C) := \mathbb{K}\,\mathrm{cl}_*(\tilde{\sigma}_C(C))$, for $C \in \mathbf{AC}$. $S_*(C)$ is obviously a $\mathbb{K}$-vector space and the Minkowski functional

$$||z||_* := \inf\{\alpha > 0 \mid z \in \alpha\,\mathrm{cl}_*(\tilde{\sigma}_C(C)\}, \quad z \in S_*(C)).$$

is a norm on $S_*(C)$ with

$$||z||_\infty \le ||z||_*, \quad z \in S_*(C).$$

$\mathfrak{T}_C^*$ is defined as the topology induced in $S_*(C)$ by $\sigma((\mathrm{Aff}_0(C))', \mathrm{Aff}_0(C))$.

If E is a Saks space its unit ball $\bigcirc(E)$ is an absolutely convex subset hence, with this structure, an absolutely convex module. This induces a full and faithful functor $\widehat{\bigcirc}_S : \mathbf{Saks}_1 \to \mathbf{AC}$.

Proposition 2.2. *For* $C \in \mathbf{AC}$ *the following statements hold*:
(i) $(\|\Box\|_*, S_*(C), \mathfrak{T}_C^*)$ *is a compact Saks space with* $\bigcirc(S_*(C)) = \mathrm{cl}_*(\tilde{\sigma}_C(C))$.
(ii) The $S_*(C)$, $C \in \mathbf{AC}$, *induce a functor* $S_* : \mathbf{AC} \to \mathbf{Saks}_1$ *and the* $\tilde{\sigma}_C : C \to (\mathrm{Aff}_0(C))'$ (cf. Proposition 1.3) *induce a natural transformation* $\sigma^* : \mathbf{AC} \to \widehat{\bigcirc}_S \circ S_*$.

Proof. (i) $\mathrm{cl}_*(\tilde{\sigma}_C(C)) \subset \bigcirc(S_*(C))$ follows from the definition of $\|\Box\|_*$. On the other hand, $\mathrm{cl}_*(\tilde{\sigma}_C(C))$ is a weakly $*$-closed subset of $S_*(C)$ hence also $\mathfrak{T}_C^*$-closed. This yields $\bigcirc(S_*(C)) \subset \mathrm{cl}_*(\tilde{\sigma}_C(C))$ (cf. [4, 6.4]). As $\mathrm{cl}_*(\tilde{\sigma}_C(C))$ is weakly $*$-compact hence $\mathfrak{T}_C^*$-compact, (i) is proved.

To show (ii) let $\varphi : C_1 \to C_2$ be an **AC**-morphism, then

$$(\mathrm{Aff}_0(\varphi))' : (\mathrm{Aff}_0(C_1))' \to (\mathrm{Aff}_0(C_2))'$$

is a $\mathbf{Saks}_1$-morphism. Now, for $c_1 \in C_1$ and $f_2 \in \mathrm{Aff}_0(C_2)$ we have

$$\begin{aligned}((\mathrm{Aff}_0(\varphi))'\tilde{\sigma}_{C_1})(c_1)(f_2) &= ((\mathrm{Aff}_0(\varphi))'(\tilde{\sigma}_{C_1}(c_1)))(f_2) = (\tilde{\sigma}_{C_1}(c_1)\mathrm{Aff}_0(\varphi))(f_2)\\ &= \tilde{\sigma}_{C_1}(c_1)(f_2\varphi) = f_2(\varphi(c_1)) = \tilde{\sigma}_{C_2}(\varphi(c_1))(f_2).\end{aligned}$$

This implies $(\mathrm{Aff}_0(\varphi))'\tilde{\sigma}_{C_1} = \tilde{\sigma}_{C_2}\varphi$, i.e. $\tilde{\sigma}_C : C \to (\mathrm{Aff}_0(C))'$ induces a natural transformation. As $(\mathrm{Aff}_0(\varphi))'$ is weakly $*$-continuous

$$(\mathrm{Aff}_0(\varphi))'(\mathrm{cl}_*(\tilde{\sigma}_{C_1}(C_1))) \subset \mathrm{cl}_*(\tilde{\sigma}_{C_2}(C_2))$$

hence $(\mathrm{Aff}_0(\varphi))'$ can be restricted to $S_*(C_1)$ and $S_*(C_2)$ and one defines $S_*(\varphi)$ as this restriction. This proves (ii). □

As $\mathrm{cl}_*(\tilde{\sigma}_C(C))$ is weakly $*$-compact, it is a totally convex subset of $(\mathrm{Aff}(C))'$, hence totconv $(\tilde{\sigma}_C(C)) \subset \mathrm{cl}_*(\tilde{\sigma}_C(C))$, $S_1(C) \subset S_*(C)$ and $\|z\|_* \le \|z\|_1$, for $z \in S_1(C)$, follows. For totally convex spaces these results and proofs remain essentially the same. The only change one has to make is that one has to consider the unit ball functor $\widehat{\bigcirc}_{CS} : \mathbf{CSak}_1 \to \mathbf{TC}$, the restriction of $\widehat{\bigcirc}_S$ to the full subcategory of complete Saks spaces $\mathbf{CSaks}_1$.

Corollary 2.3. $S_*(C)$, *for* $C \in \mathbf{TC}$, *is a compact Saks space and induces a functor* S_*: $\mathbf{TC} \to \mathbf{CSaks}_1$. $\tilde{\sigma}_C$, $C \in \mathbf{TC}$, *induce a natural transformation* $\sigma^* : \mathbf{TC} \to \widehat{\bigcirc}_{CS} \circ S_*$.

The construction of S_* gives an interesting relation between **AC** or **TC** and $\mathbf{Saks}_1$. An obvious question, especially with respect to the results of Section 1, is, under which conditions S_* is universal, i.e. a left adjoint of $\widehat{\bigcirc}_S$ or $\widehat{\bigcirc}_{CS}$. The key to the solution of this problem is the observation that these two unit ball functors “forget” too much, namely the locally convex topology $\mathfrak{T}_E$ on $\bigcirc(E)$, $E \in \mathbf{Saks}_1$. Hence, we now introduce topological absolutely or totally convex modules, respectively. By far more interesting, however, is an analogous construction for topological absolutely resp. totally convex modules, which will be presented now.

Definition 2.4 (cf. Kirschner [5], where a different notion is introduced). C is called a *topological absolutely convex module*, if C is an absolutely convex module and $\mathfrak{T}_C$ is a topology on C, such that the mapping

$$\mu_C : \Omega_{\mathrm{ac}} \times C^{(\mathbb{N})} \to C$$

defined by $\mu_C(\hat{\alpha}, \mathbf{c}) := \sum_i \alpha_i c_i$, $(\hat{\alpha}, \mathbf{c}) \in \Omega_{\mathrm{ac}} \times C^{(\mathbb{N})}$, $C^{(\mathbb{N})} := \{\mathbf{c} \mid \mathbf{c} \in C^{\mathbb{N}}$, support of $\mathbf{c}$ finite$\}$, is continuous. Here, $C^{(\mathbb{N})}$ carries the subspace topology induced by the product topology on $C^{\mathbb{N}}$ and $\Omega_{\mathrm{ac}} = \bigcirc(\mathbb{K}^{(\mathbb{N})})$ the subspace topology of $\mathbb{K}^{(\mathbb{N})}$. A mapping $f : C_1 \to C_2$ between topological absolutely convex modules is called a morphism if it is an **AC**-morphism and $\mathfrak{T}_{C_1}$-$\mathfrak{T}_{C_2}$ continuous. The topological absolutely convex modules and their morphisms constitute the category **TopAC**.

The category **TopTC** of topological totally convex modules is defined analogously. $C \in$ **TC** is called *topological*, if it has a topology $\mathfrak{T}_C$ such that

$$\mu_C : \Omega \times C^{\mathbb{N}} \to C,$$

$\mu_C(\hat{\alpha}, \mathbf{c}) := \sum_i \alpha_i c_i$, is continuous, where $\Omega = \bigcirc(l_1(\mathbb{N}))$ carries the subspace topology of $l_1(\mathbb{N})$ and $C^{\mathbb{N}}$ the product topology.

The unit balls $\bigcirc(E)$, for $E \in \mathbf{Saks}_1$ resp. $E \in \mathbf{CSaks}_1$, induce two canonical, faithful functors

$$\widehat{\bigcirc}^* : \mathbf{Saks}_1 \to \mathbf{TopAC},$$

$$\widehat{\bigcirc}_{\mathrm{C}}^* : \mathbf{CSaks}_1 \to \mathbf{TopTC}.$$

That $\widehat{\bigcirc}^*(E)$ is in **TopAC** for $E \in$ **AC** is obvious, the assertion for $\widehat{\bigcirc}_{\mathrm{C}}^*(E)$, $E \in \mathbf{CSaks}_1$, follows from

Proposition 2.5. *For a complete Saks space E, $\widehat{\bigcirc}(E)$ is a topological totally convex module.*

Proof. Consider a net $\{(\hat{\alpha}^i, \mathbf{x}^i) \mid i \in I\}$ in $\Omega \times \widehat{\bigcirc}(E)^{\mathbb{N}}$ converging to $(\hat{\alpha}, \mathbf{x})$. This means that, for $\varepsilon > 0$, there is an $i_0(\varepsilon) \in I$ such that, for all $i \geq i_0(\varepsilon)$, $||\hat{\alpha} - \hat{\alpha}^i|| < \varepsilon$ in the norm of $l_1(\mathbb{N})$. Let q be an element of a defining family of seminorms for the locally convex topology $\mathfrak{T}_E$. Then, for any $\varepsilon > 0$, and any $k \in \mathbb{N}$, there is an $i_1(\varepsilon, q, k) \in I$, such that for every $i \geq i_1(\varepsilon, q, k)$

$$q\left(\tfrac{1}{2}x_j^i - \tfrac{1}{2}x_j\right) < \varepsilon$$

holds for $1 \leq j \leq k$. Besides, one has

$$q\left(\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n^i x_n^i - \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n x_n\right) \leq q\left(\sum_{n=1}^{\infty}\alpha_n^i\left(\frac{1}{2}x_n^i - \frac{1}{2}x_n\right)\right) + q\left(\sum_{n=1}^{\infty}\left(\frac{1}{2}\alpha_n^i - \frac{1}{2}\alpha_n\right)x_n\right).$$

If $k(\varepsilon) \in \mathbb{N}$ is chosen such that $\sum_{n>k(\varepsilon)}|\alpha_n| < \varepsilon$, $\sum_{n>k(\varepsilon)}|\alpha_n^i| < 2\varepsilon$ follows for all $i \geq i_0(\varepsilon)$. This implies, for $i \geq \max\{i_0(\varepsilon), i_1(\varepsilon, q, k(\varepsilon))\}$,

$$\begin{aligned} q\left(\sum_{n=1}^{\infty}\alpha_n^i\left(\frac{1}{2}x_n^i - \frac{1}{2}x_n\right)\right) &\leq \sum_{n=1}^{\infty}|\alpha_n^i| q\left(\frac{1}{2}x_n^i - \frac{1}{2}x_n\right) \\ &= \sum_{n\leq k(\varepsilon)}|\alpha_n^i| q\left(\frac{1}{2}x_n^i - \frac{1}{2}x_n\right) + \sum_{n>k(\varepsilon)}|\alpha_n^i| q\left(\frac{1}{2}x_n^i - \frac{1}{2}x_n\right) \\ &< \varepsilon + M_q\varepsilon = \varepsilon(1 + M_q), \end{aligned}$$

where $M_q := \sup\{q(x) \mid x \in \bigcirc(E)\} < \infty$. This implies

$$q\left(\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n^i x_n^i - \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n x_n\right) \leq \varepsilon\left(1 + \frac{3}{2}M_q\right)$$

and hence the assertion. □

Definition 2.6. For $C \in \mathbf{TopAC}$ define

$$\mathrm{Aff}_0^{\mathrm{c}}(C) := \{f \mid f \in \mathrm{Aff}_0(C) \text{ and } f \text{ continuous}\}.$$

Lemma 2.7. *Let* $C \in \mathbf{TopAC}$, *then the following statements hold*:

(i) *With the supremum norm* $\|\Box\|_\infty$ $\mathrm{Aff}_0^{\mathrm{c}}(C)$ *is a Banach subspace of* $\mathrm{Aff}_0(C)$.

(ii) *The mapping* $\sigma_C : C \to (\mathrm{Aff}_0^{\mathrm{c}}(C))'$ *defined as* $\sigma_C(x)(f) := f(x)$, *for* $x \in C$, $f \in \mathrm{Aff}_0^{\mathrm{c}}(C)$, *is* $\mathfrak{T}_C$-*weakly* $*$-*continuous. The* σ_C, $C \in \mathbf{TopAC}$, *induce a natural transformation.*

Proof. (i) It is obvious that $\|\Box\|_\infty$ is a norm on $\mathrm{Aff}_0^{\mathrm{c}}(C)$. $\mathrm{Aff}_0^{\mathrm{c}}(C)$ is $\|\Box\|_\infty$-complete because the uniform limit of continuous functions is continuous.

(ii) For $f \in \mathrm{Aff}_0^{\mathrm{c}}(C)$, let $\Lambda(f) : (\mathrm{Aff}_0^{\mathrm{c}}(C))' \to \mathbb{K}$ denote the linear form $\Lambda(f)(\mu) := \mu(f)$, $\mu \in (\mathrm{Aff}_0^{\mathrm{c}}(C))'$. Then $\Lambda(f)\sigma_C = f$ is continuous for all $f \in \mathrm{Aff}_0^{\mathrm{c}}(C)$, hence σ_C is $\mathfrak{T}_C$-weakly $*$-continuous. For the last assertion we regard $(\mathrm{Aff}_0^{\mathrm{c}}(C))'$ as a locally convex $\mathbb{K}$-vector space with the weak $*$-topology. Let $\varphi : C_1 \to C_2$ be a **TopAC**-morphism. For $f_2 \in \mathrm{Aff}_0^{\mathrm{c}}(C_2)\,\mathrm{Aff}_0^{\mathrm{c}}(\varphi)(f_2) = f_2\varphi$ (cf. Proposition 2.2) hence $\mathrm{Aff}_0^{\mathrm{c}}(\varphi)(f_2) \in \mathrm{Aff}_0^{\mathrm{c}}(C_1)$ and $\mathrm{Aff}_0^{\mathrm{c}}(\Box)$ is a functor from **TopAC** to the category of locally convex vector spaces, hence so is $(\mathrm{Aff}_0^{\mathrm{c}}(\Box))'$. That the σ_C, $C \in \mathbf{TopAC}$, induce a natural transformation follows from the proof of Proposition 2.2 because σ_C and $\tilde{\sigma}_C$ have the same underlying function.

For $C \in \mathbf{TopAC}$ and a set $M \subset (\mathrm{Aff}_0^{\mathrm{c}}(C))'$, let $\mathrm{cl}_*^{\mathrm{c}}(M)$ denote the weak $*$-closure of M. Define

$$\mathrm{S}_*^{\mathrm{c}}(C) := \mathbb{K}\,\mathrm{cl}_*^{\mathrm{c}}(\sigma_C(C))$$

and introduce the Minkowski functional

$$||z||_*^{\mathrm{c}} := \inf\{\alpha > 0 \,|\, z \in \alpha\,\mathrm{cl}_*^{\mathrm{c}}(\sigma_C(C))\},$$

$z \in S_*^{\mathrm{c}}(C)$. Then $S_*^{\mathrm{c}}(C)$ is a subspace of $(\mathrm{Aff}_0^{\mathrm{c}}(C))'$ and $||\Box||_*^{\mathrm{c}}$ is a norm on $S_*^{\mathrm{c}}(C)$ with

$$||z||_\infty \le ||z||_*^{\mathrm{c}} \quad \text{for } z \in S_*^{\mathrm{c}}(C).$$

If $\hat{\mathfrak{T}}_C^*$ denotes the topology on $S_*^{\mathrm{c}}(C)$ induced by the weak $*$-topology of $(\mathrm{Aff}_0^{\mathrm{c}}(C))'$, then one gets, by a proof completely analogous to that of Proposition 2.2,

Proposition 2.8. (i) *For* $C \in \mathbf{TopAC}$ $(||\Box||_*^{\mathrm{c}}, S_*^{\mathrm{c}}(C), \hat{\mathfrak{T}}_C^*)$ *is a compact Saks space with*

$$\bigcirc(S_*^{\mathrm{c}}(C)) = \mathrm{cl}_*^{\mathrm{c}}(\sigma_C(C)).$$

(ii) *Let* σ_C^* *denote the co-restriction of* σ_C *to* $\widehat{\bigcirc}_{\mathrm{C}}(S_*^{\mathrm{c}}(C))$, *then the* $S_*^{\mathrm{c}}(C)$, $C \in \mathbf{TopAC}$, *induce a functor* $S_*^{\mathrm{c}} : \mathbf{TopAC} \to \mathbf{CSaks}_1$ *and the* σ_C^*, $C \in \mathbf{TopAC}$, *a natural transformation* $\sigma_C^* : \mathbf{TopAC} \to \widehat{\bigcirc}_{\mathrm{C}}^* \circ S_*^{\mathrm{c}}$.

Theorem 2.9. *If* $\widehat{\bigcirc}_{\mathrm{cp}}^* : \mathbf{CompSaks}_1 \to \mathbf{TopAC}$ *denotes the restriction of* $\widehat{\bigcirc}_{\mathrm{C}}^*$ *to* $\mathbf{CompSaks}_1$, S_*^{c} *is a left adjoint of* $\widehat{\bigcirc}_{\mathrm{cp}}^*$ *with front adjunction* $\sigma^* : \mathbf{TopAC} \to \widehat{\bigcirc}_{\mathrm{cp}}^* \circ S_*^{\mathrm{c}}$.

Proof. To simplify notation we write $\widehat{\bigcirc}^*$ for $\widehat{\bigcirc}_{\mathrm{cp}}^*$. Let $\varphi : C \to \widehat{\bigcirc}^*(E)$, $E \in \mathbf{CompSaks}_1$, be a $\mathbf{TopAC}$-morphism and denote by $\tau : \widehat{\bigcirc}^*(E) \to \widehat{\bigcirc}^*(S_*^{\mathrm{c}} \circ \widehat{\bigcirc}^*(E))$ the mapping σ_C^* for $C = \widehat{\bigcirc}^*(E)$, to simplify notation. As τ is continuous (2.7) and $\widehat{\bigcirc}^*(E)$ compact, $\tau(\widehat{\bigcirc}^*(E))$ is weakly $*$-compact, hence $\tau(\widehat{\bigcirc}^*(E)) = \widehat{\bigcirc}^*(S_*^{\mathrm{c}} \circ \widehat{\bigcirc}^*(E))$, i.e. τ is surjective. As any continuous linear form on E induces a continuous totally affine mapping on $\widehat{\bigcirc}^*(E)$, τ is also injective. Hence, τ is a homeomorphism i.e. an isomorphism in $\mathbf{TopAC}$ (cf. [1,16]). The usual argument shows that τ can be uniquely extended to a linear contraction $\tau_0 : E \to S_*^{\mathrm{c}}(\widehat{\bigcirc}^*(E))$: For $z \in E$, $z \neq 0$, and any $\alpha \neq 0$ with $\alpha z \in \bigcirc(E)$ one defines $\tau_0(z) := \alpha^{-1}\tau(\alpha z)$. This argument also implies that $\widehat{\bigcirc}^*$ is full and faithful. Thus τ_0 is an isomorphism in $\mathbf{CompSaks}_1$. One puts $\varphi_0 := \tau_0^{-1} S_*^{\mathrm{c}}(\varphi)$ for any morphism $\varphi : C \to \widehat{\bigcirc}^*(E)$ in $\mathbf{TopAC}$, $\varphi_0 : S_*^{\mathrm{c}}(C) \to E$ and gets $\widehat{\bigcirc}^*(\varphi_0)\sigma_C^* = \tau^{-1}\widehat{\bigcirc}^*(S_*^{\mathrm{c}}(\varphi))\sigma_C^* = \varphi$. φ_0 is uniquely determined by φ and this equation because $\sigma_C^*(C)$ is dense in $\bigcirc(S_*^{\mathrm{c}}(C))$. This proves that S_*^{c} is left adjoint to $\widehat{\bigcirc}^*$ with front adjunction σ_*. $\Box$

Proposition 2.8 obviously holds also for $\mathbf{TopTC}$ and the $S_*^{\mathrm{c}}(C)$, $C \in \mathbf{TopTC}$, induce a functor again denoted by $S_*^{\mathrm{c}} : \mathbf{TopTC} \to \mathbf{CompSaks}_1$. The proof of Theorem 2.9 yields the

Corollary 2.10. S_*^c : **TopTC** $\to$ **CompSaks**$_1$ *is a left adjoint of* $\widehat{\bigcirc}_{cp}^*$: **CompSaks**$_1$ $\to$ **TopTC** *with front adjunction* σ^*.

For $E \in$ **CompSaks**$_1$, $\widehat{\bigcirc}_{cp}^*(E)$ is also a topological totally convex space in the sense of Kleisli and Künzi, but their notion of a topological totally convex module (called "space" in [6,7]) is different from the one used here. They define $C \in$ **TC** to be *topological*, if it carries a locally convex topology and if, for any $\hat{\alpha} \in \Omega$, the mapping $\hat{\alpha}_C : C^{\mathbb{N}} \to C$, $\hat{\alpha}(\mathbf{C}) := \sum_i \alpha_i c_i$, $\mathbf{c} \in C^{\mathbb{N}}$, is continuous, where $C^{\mathbb{N}}$ carries the product topology. These objects are called *pretopological* by Kirschner [5]. The proof of Kleisli and Künzi for a left adjoint of the unit ball functor from the subcategory of their topological totally convex modules given by the modules with the *strong topology* also uses duality and it is similar to the one presented here.

Semadeni proved a theorem on affine compactification for bounded convex sets (cf. [15,16]). The corresponding statement for absolutely resp. totally convex sets is:

For any bounded absolutely (*totally*) *convex subset* K *of a locally convex vector space* E *there is a compact convex absolutely subset* $R(K)$ $(R_\infty(K))$ *of a locally convex vector space and an injective morphism* $\tau_K : K \to R(K)$ $(\tau_{K\infty} : K \to R_\infty(K))$ *in* **TopAC** (**TopTC**), *such that for any morphism* $\varphi : K \to X$ *in* **TopAC** (**TopTC**), *with* X *any compact absolutely convex subset of a locally convex vector space*, *there is a unique morphism* $\varphi_0 : R(K) \to X$ $(\varphi_0 : R_\infty(K) \to X)$ *in* **TopAC**(**TopTC**) *with* $\varphi = \varphi_0 \tau_K$.

Proof. These assertions follow easily from Theorem 2.9 and Corollary 2.10 with $\tau_K = \sigma_K^*$, $R(K) = \widehat{\bigcirc}_{cp}^* \circ S_*^c(K)$ $(R_\infty(K) = \widehat{\bigcirc}_{cp}^* \circ S_*(C))$. □

For $C \in$ **AC**, $\sigma_C^1 : C \to \widehat{\bigcirc}_{ac}(S_1(C))$ may be regarded as its *universal completion* (1.5), *while*, *for* $C \in$ **TopAC**, $\sigma_C^* : C \to \widehat{\bigcirc}_{cp}^*(S_*^c(C))$ is simultaneously its *universal compactification* and a *topological completion*. An analogous statement applies to (topological) totally convex modules.

References

[1] H. Bauer, Silovscher Rand und Dirichletsches problem, Ann. Inst. Fourier (Grenoble) 11 (1961) 89–136.

[2] J.B. Cooper, Saks Spaces and Applications to Functional Analysis, North-Holland, Amsterdam, 1978.

[3] G.J.O. Jameson, Convex series, Proc. Cambridge Philos. Soc. 72 (1972) 37–47.

[4] H. Jarchow, Locally Convex Spaces, Teubner, Stuttgart, 1981.

[5] K. Kirschner, Über die Kategorie der topologischen absolutkonvexen Räume, Ph.D. Thesis, Fernuniversität – GHS, Fachbereich Mathematik, 1996.

[6] H. Kleisli, H.-P. Künzi, Topological totally convex spaces I, Appl. Catal. Struct. 2 (1994) 45–55.

[7] H. Kleisli, H.-P. Künzi, Topological totally convex spaces II, Cahiers Topologie Geom. Differentielie Categories XXXVI-1 (1995) 11–52.

[8] H. König, G. Wittstock, Superconvex spaces and superconvex sets and the embedding of convex and superconvex spaces, Note Math. X (Suppl. 2) (1990) 343–362.

[9] D. Pumplün, Regularly ordered Banach spaces and positively convex spaces, Res. Math. 7 (1984) 85–112.
[10] D. Pumplün, Banach spaces and superconvex modules, in: Behara, Fritsch, Lintz (Eds.), Symp. Gaussiana, de Gruyter, Berlin, 1995, pp. 323–338.
[11] D. Pumplün, H. Röhrl, Banach spaces and totally convex spaces I, Comm. Algebra 12 (1985) 953–1019.
[12] D. Pumplün, H. Röhrl, Banach spaces and totally convex spaces II, Comm. Algebra 13 (1985) 1047–1113.
[13] D. Pumplün, H. Röhrl, The Eilenberg–Moore algebras of base normed vector spaces, to appear.
[14] G. Rodé, Superkonvexität und schwache Kompaktheit, Arch. Math. 36 (1) (1981) 62–72.
[15] Z. Semadeni, Categorical methods in convexity, Proc. Coll. Convex, Copenhagen, 1965, 1967, pp. 281–307.
[16] Z. Semadeni, Free-compact convex sets, Bull. Acad. Polon. Sci. 13 (1965) 141–146.
[17] Y.-C. Wong, K.-F. Ng, Partially Ordered Topological Vector Spaces, Clarendon, Oxford, 1973.