Abstract

We prove existence and uniqueness of $\mathbb{L}^p$ solutions, $p \in [1, 2]$, of reflected backward stochastic differential equations with $p$-integrable data and generators satisfying the monotonicity condition. We also show that the solution may be approximated by the penalization method. Our results are new even in the classical case $p = 2$.

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1. Introduction

Nonlinear backward stochastic differential equations (BSDEs) were considered for the first time by Pardoux and Peng [13]. In the paper [6] by El Karoui et al., the so called reflected BSDEs (RBSDEs) were introduced. By a solution of the RBSDE with terminal value $\xi$, generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and obstacle $L = \{L_t, t \in [0, T]\}$ we understand a triple $(Y, Z, K)$ of $(\mathcal{F}_t)$ adapted processes such that

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s + K_T - K_t, \quad t \in [0, T], \\
Y_t &\geq L_t, \quad t \in [0, T], \\
K &\text{ is nondecreasing, continuous, } K_0 = 0, \quad \int_0^T (Y_t - L_t) \, dK_t = 0,
\end{align*}
\] (1.1)

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where $W$ is a standard $d$-dimensional Wiener process and $(\mathcal{F}_t)$ is the standard augmentation of the natural filtration generated by $W$. It is assumed here that $\xi$ is $\mathcal{F}_T$ measurable and $L$ is an $(\mathcal{F}_t)$ progressively measurable continuous process such that $L_T \leq \xi$ a.s. Condition in (1.1) says that the first component $Y$ of the solution is forced to stay above $L$. The role of $K$ is to push $Y$ upwards in order to keep it above $L$. We also require that $K$ is minimal in the sense of (1.1)$_3$, i.e. $K$ increases only when $Y = L$. Note that usual BSDEs may be considered as special case of RBSDEs with $L \equiv -\infty$ (and $K \equiv 0$).

In [13] it is proved that if $\xi \in L^2$, $\int_0^T (f(s, 0, 0))^2 \, ds \in L^1$ and $f$ is Lipschitz continuous in both variables $y, z$ then there exists a unique solution $(Y, Z)$ of BSDE with data $\xi, f$ such that $Y \in S^2, Z \in \mathcal{H}^2$, i.e. $Y$ is continuous and adapted, $Z$ is progressively measurable, and $Y^*_t \in L^2, \left(\int_0^T |Z_t|^2 \, dt\right)^{1/2} \in L^2$ (here and later on we use the notation $X^*_t = \sup_{s \leq t} X_s, t \in [0, T]$).

In [6] existence and uniqueness of a solution $(Y, Z, K)$ of (1.1) such that $Y, K \in S^2, Z \in \mathcal{H}^2$ is proved under the additional assumption that $L^+ = \max(L, 0) \in S^2$.

The assumptions on the data in [6,13] are sometimes too strong for applications (see, e.g., [5,7] for applications in economics and finance and [2,16] for applications to PDEs). Therefore many attempts have been made to weaken the integrability conditions imposed in [6,13] on $\xi$ and $f$ or weaken the assumption that $f$ is Lipschitz continuous. For instance, Briand and Carmona [2] and Pardoux [12] consider square-integrable solutions (i.e. $Y \in S^2, Z \in \mathcal{H}^2$) of BSDEs with generators which are Lipschitz continuous with respect to $z$ while with respect to $y$ are continuous and satisfy the monotonicity condition and the general growth condition of the form $|f(t, y, z)| \leq |f(t, 0, z)| + \varphi(|y|)$. In [2] $\varphi$ is a polynom, whereas in [12] an arbitrary positive continuous increasing function. In [7] conditions ensuring existence and uniqueness of $L^p$ solutions (i.e. $Y, Z \in S^p, Z \in \mathcal{H}^p$ for $p > 1$ of BSDEs with Lipschitz continuous generator with respect to both $y$ and $z$ are given. The strongest results in this direction are given by Briand et al. [3], where $L^p$ solutions of BSDEs for $p \in [1, 2]$ are considered. It is proved there that in case $p \in (1, 2]$ if

$$\xi \in L^p, \quad \int_0^T |f(s, 0, 0)| \, ds \in L^p,$$

$$\forall r > 0 \int_0^T \sup_{|y| \leq r} |f(s, y, 0) - f(s, 0, 0)| \, ds < +\infty$$

and $f$ is Lipschitz continuous in $z$ and continuous and monotone in $y$ then there exists a unique $L^p$ solution. Similar result is proved for $p = 1$ in case $f$ does not depend on $z$ and in the general case under some additional assumption (Assumption (H5) in Section 5). Finally, let us mention that many papers are devoted to BSDEs with quadratic growth generators in $z$ (see, e.g., [9] and the references given there).

In [11] existence of square-integrable solutions of RBSDEs with continuous generators satisfying the linear growth condition is proved. Square-integrable solutions of RBSDEs under monotonicity and the general growth condition with respect to $y$ were considered by Lepeltier et al. in [10]. In [8] existence and uniqueness of $L^p$ solutions of RBSDEs is proved in case $p \in (1, 2)$ for $\xi \in L^p, L^+ \in S^p$ and generators which are Lipschitz continuous in $y$ and $z$ and satisfy the condition $\int_0^T |f(s, 0, 0)| \, ds \in L^p$. Similar result for generators satisfying the monotonicity condition and the linear growth condition with respect to $y$ is proved by Aman [1]. $L^1$ solutions of some generalized Markov type RBSDEs with random terminal time are considered in [16].
In the present paper, we study $\mathbb{L}^p$ solutions of RBSDEs of the form (1.1) for $p \in [1, 2]$. Our main theorems on existence and uniqueness of solutions may be summarized by saying that if $\xi, f$ satisfy assumptions from [3] and the obstacle $L$ satisfies the assumptions

$$L_T^{+, *} \in \mathbb{L}^p, \quad \int_0^T |f(s, L_s^{+, *}, 0)| ds \in \mathbb{L}^p,$$

then there exists a unique $\mathbb{L}^p$ solution of (1.1). It is worth noting that as in [3] we do not assume that $f$ satisfies the general growth condition in $y$. Therefore our results strengthen known results proved in [1,10] even in the classical case $p = 2$ (see Remark 4.5) and results proved in [1,8] in case $p \in (1, 2)$. We also show that the solution $(Y, Z, K)$ to (1.1) may be approximated by the penalization method if $p \in (1, 2)$ and if $p = 1$ and $f$ is independent of $z$. More precisely, if $p \in (1, 2]$ then

$$\|Y^n - Y\|_{S^p} \to 0, \quad \|Z^n - Z\|_{\mathcal{H}^p} \to 0, \quad \|K^n - K\|_{S^p} \to 0,$$

where $(Y^n, Z^n)$ is a solution of the BSDE

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n, Z_s^n) \, ds - \int_t^T Z_s^n \, dW_s + K^n_T - K^n_t, \quad t \in [0, T]$$

with

$$K^n_t = n \int_0^t (Y^n_s - L_s)^- \, ds, \quad t \in [0, T].$$

This generalizes and at the same time strengthens corresponding result proved in [10] in case $p = 2$. In case $p = 1$ we show that (1.2) holds in the spaces $S^\beta, \mathcal{H}^\beta$ with $\beta \in (0, 1)$.

The paper is organized as follows. Section 2 contains basic notation and definitions. A useful a priori estimate for stopped solutions of RBSDEs is also given. In Section 3 we prove main estimates in case $p \in (1, 2)$. In Section 4 we apply the above mentioned estimates to prove convergence of penalization scheme in case $p \in (1, 2)$. Section 5 is devoted to the case where $p = 1$. For generator $f$ not depending on $z$ we give some a priori estimates similar to those proved in case $p > 1$ and we show convergence of the penalization scheme. In the general case, following [3, Section 6] we prove existence and uniqueness of solutions of (1.1) under some additional assumption on $f$. In this case the solution is a limit of solutions of appropriately chosen RBSDEs with generators not depending on $z$.

### 2. Notation and preliminary estimates

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. $\mathbb{L}^p, p > 0$, is the space of random variables $X$ such that $\|X\|_p = E(|X|^p)^{1/p} < +\infty$. $X^*_t = \sup_{s \leq t} |X_s|, t \in [0, T]$, $S^p$ denotes the set of adapted and continuous processes $X$ such that $\|X\|_{S^p} = \|X^*_p\|_p < +\infty$. Let $W$ be a standard $d$-dimensional Wiener process on $(\Omega, \mathcal{F}, P)$ and let $(\mathcal{F}_t)$ be the standard augmentation of the natural filtration generated by $W$. $\mathcal{H}^p$ denotes the set of progressively measurable $d$-dimensional processes $X$ such that $\|X\|_{\mathcal{H}^p} = \left\| \left( \int_0^T |X_s|^2 \, ds \right)^{1/2} \right\|_p < +\infty$. It is well known that $S^p$ and $\mathcal{H}^p$ are Banach spaces for $p \geq 1$. If $p < 1$ then $\mathbb{L}^p, S^p$ and $\mathcal{H}^p$ are complete metric spaces with metrics defined by $\| \cdot \|_p, \| \cdot \|_{S^p}$ and $\| \cdot \|_{\mathcal{H}^p}$, respectively.

We will assume that we are given an $\mathcal{F}_T$ measurable random variable $\xi$, a generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ measurable with respect to $\text{Prog} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$, where $\text{Prog}$
denotes the $\sigma$-field of progressive subsets of $[0, T] \times \Omega$ and a barrier $L$, which is an $(\mathcal{F}_t)$ adapted continuous process. We will always assume that $\xi \geq L_T$. We will need the following assumptions on $f$.

(H1) There is $\lambda \geq 0$ such that $|f(t, y, z) - f(t, y, z')| \leq \lambda |z - z'|$ for $t \in [0, T], y \in \mathbb{R}, z, z' \in \mathbb{R}^d$.

(H2) There is $\mu \in \mathbb{R}$ such that $(y - y')(f(t, y, z) - f(t, y', z)) \leq \mu (y - y')^2$ for $t \in [0, T], y, y' \in \mathbb{R}, z \in \mathbb{R}^d$.

In (H1), (H2) and in the sequel we understand that the inequalities hold true $P$-a.s.

**Proposition 2.1.** Assume that $f$ satisfies (H1), (H2) and let $(Y, Z, K)$ be a solution of (1.1). Then for every $p > 0$ there exists $C > 0$ depending only on $p$ and $\mu, \lambda, T$ such that for every stopping time $\tau$ ($\tau \leq T$),

$$
E \left( \left( \int_0^\tau |Z_s|^2 ds \right)^{p/2} + K_\tau^p \right) 
\leq C E \left( (Y_s^\tau)^p + (L_{\tau}^{+,*})^p + \left( \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \right)^p \right). 
$$

(2.1)

**Proof.** Let $a \in \mathbb{R}$ and let $\tilde{Y}_t = e^{at} Y_t, \tilde{Z}_t = e^{at} Z_t, \tilde{K}_t = \int_0^t e^{as} dK_s$ and $\tilde{\xi} = e^{aT} \xi, \tilde{f}(t, y, z) = e^{at} f(t, e^{-at} y, e^{-at} z) - ay$. Observe that $(\tilde{Y}, \tilde{Z}, \tilde{K})$ solves the RBSDE

$$
\tilde{Y}_t = \tilde{\xi} + \int_t^T \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^T \tilde{Z}_s dW_s + \tilde{K}_T - \tilde{K}_t, \quad t \in [0, T]
$$

with the reflecting barrier $\tilde{L}_t = e^{aT} L_t$, and that there exist constants $C_1, C_2 > 0$ depending only on $p, a, T$ such that

$$
\left( \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \right)^p + (L_{\tau}^{+,*})^p \leq C_1 \left( \left( \int_0^\tau |\tilde{f}(s, \tilde{L}_s^{+,*}, 0)| ds \right)^p + (\tilde{L}_{\tau}^{+,*})^p \right)
\leq C_2 \left( \left( \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \right)^p + (L_{\tau}^{+,*})^p \right).
$$

It follows that (2.1) is satisfied if and only if it is satisfied for the solution $(\tilde{Y}, \tilde{Z}, \tilde{K})$ and the data $\tilde{\xi}, \tilde{f}, \tilde{L}$ (with some constant $C$ depending also on $a$). Therefore choosing $a$ appropriately we may assume that (H2) is satisfied with arbitrary but fixed $\mu \in \mathbb{R}$. In the rest of the proof we will assume that $\mu = 0$. Moreover, without loss of generality we may and will assume that $Y^*_\tau, L^{+,*}_\tau, \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \in \mathbb{L}^p$.

Set $\tau_n = \inf \left\{ t; \int_0^t |Z_s|^2 ds \geq n \right\} \land \tau, n \in \mathbb{N}$. Obviously $P(\tau_n = \tau) \to 1$. By Itô’s formula applied to the continuous semimartingale $Y - L^{+,*}$, for $n \in \mathbb{N}$ we have

$$
(Y_0 - L^{+,*}_0)^2 + \int_0^{\tau_n} |Z_s|^2 ds = (Y_{\tau_n} - L^{+,*}_{\tau_n})^2 + 2 \int_0^{\tau_n} (Y_s - L^{+,*}_s) f(s, Y_s, Z_s) ds
- 2 \int_0^{\tau_n} (Y_s - L^{+,*}_s) Z_s dW_s
+ 2 \int_0^{\tau_n} (Y_s - L^{+,*}_s)(dK_s + dL^{+,*}_s).
$$
Since $K$ is increasing only on the set $\{s : Y_s = L_s\}$,
\[
\int_0^\tau (Y_s - L_s^{+,*}) dK_s \leq \int_0^\tau (Y_s - L_s^{+,*}) 1_{\{Y_s > L_s^{+,*}\}} dK_s
\]
\[
= \int_0^\tau (Y_s - L_s^{+,*}) 1_{\{L_s > L_s^{+,*}\}} dK_s = 0.
\] (2.2)

By the above and (H1), (H2),
\[
\int_0^\tau |Z_s|^2 ds \leq (Y_\tau - L_\tau^{+,*})^2 + 2 \int_0^\tau |Y_s - L_s^{+,*}| |f(s, L_s^{+,*}, 0)| ds
\]
\[
+ 2\lambda \int_0^\tau |Y_s - L_s^{+,*}| |Z_s| ds + 2 \int_0^\tau (Y_s - L_s^{+,*}) Z_s dW_s
\]
\[
+ 2 \sup_{t \leq \tau} |Y_t - L_t^{+,*}| L_t^{+,*}
\]
\[
\leq \sup_{t \leq \tau} (Y_t - L_t^{+,*})^2 + 2 \sup_{t \leq \tau} |Y_t - L_t^{+,*}| \int_0^\tau |f(s, L_s^{+,*}, 0)| ds
\]
\[
+ \int_0^\tau \left( \frac{(2\lambda |Y_s - L_s^{+,*}|)^2}{2} + \frac{|Z_s|^2}{2} \right) ds
\]
\[
+ 2 \int_0^\tau (Y_s - L_s^{+,*}) Z_s dW_s + \sup_{t \leq \tau} (Y_t - L_t^{+,*})^2 + (L_{\tau_n}^{+,*})^2
\]
\[
\leq (3 + 2\lambda^2 T) \sup_{t \leq \tau} (Y_t - L_t^{+,*})^2 + \left( \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \right)^2
\]
\[
+ \frac{1}{2} \int_0^\tau |Z_s|^2 ds + (L_{\tau_n}^{+,*})^2 + 2 \int_0^\tau (Y_s - L_s^{+,*}) Z_s dW_s.
\]

Hence there is $C' > 0$ such that
\[
\int_0^\tau |Z_s|^2 ds \leq C' \left( (Y_\tau^{+,*})^2 + (L_\tau^{+,*})^2 + \left( \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \right)^2 \right.
\]
\[
\left. + \left| \int_0^\tau (Y_s - L_s^{+,*}) Z_s dW_s \right| \right),
\]
which implies that for some $C'_p > 0$,
\[
\left( \int_0^\tau |Z_s|^2 ds \right)^{p/2} \leq C'_p \left( (Y_\tau^{+,*})^p + (L_\tau^{+,*})^p \right.
\]
\[
+ \left( \int_0^\tau |f(s, L_s^{+,*}, 0)| ds \right)^p + \left| \int_0^\tau (Y_s - L_s^{+,*}) Z_s dW_s \right|^{p/2} \).
\]

By the Burkholder–Davis–Gundy inequality,
\[
E \left( \int_0^\tau (Y_s - L_s^{+,*}) Z_s dW_s \right)^{p/2} \leq c_p E \left( \int_0^\tau (Y_s - L_s^{+,*})^2 |Z_s|^2 ds \right)^{p/4}
\]
\[
\leq c'_p E \left( (Y_\tau^{+,*})^p + (L_\tau^{+,*})^p \right) + \frac{1}{2} E \left( \int_0^\tau |Z_s|^2 ds \right)^{p/2}.
\]
Putting together the last two estimates we see that there is $C > 0$ such that

$$E\left(\left(\int_0^\tau |Z_s|^2 \, ds\right)^{p/2}\right) \leq C E\left((Y_\tau^*)^p + (L_{\tau}^{+,*})^p + \left(\int_0^\tau |f(s, L_s^{+,*}, 0)| \, ds\right)^p\right)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using Fatou’s lemma we conclude that

$$E\left(\left(\int_0^\tau |Z_s|^2 \, ds\right)^{p/2}\right) \leq C E\left((Y_\tau^*)^p + (L_{\tau}^{+,*})^p + \left(\int_0^\tau |f(s, L_s^{+,*}, 0)| \, ds\right)^p\right). \quad (2.3)$$

In order to get estimates on $K$ we first observe that by (1.1),

$$K_t = Y_0 - Y_t - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s \, dW_s, \quad t \in [0, T].$$

Hence $dK_t = -dY_t - f(s, Y_s, Z_s) \, ds + Z_s \, dW_s$. From this, (H1) and the fact that $K$ is increasing only on the set $\{s : L_s = Y_s\}$ it follows that

$$K_\tau = \int_0^\tau 1_{\{Y_s \leq L_s^{+,*}\}} \, dK_s = -\int_0^\tau 1_{\{Y_s \leq L_s^{+,*}\}} \, dY_s - \int_0^\tau f(s, Y_s, Z_s) 1_{\{Y_s \leq L_s^{+,*}\}} \, ds$$

$$+ \int_0^\tau Z_s 1_{\{Y_s \leq L_s^{+,*}\}} \, dW_s$$

$$\leq -\int_0^\tau 1_{\{Y_s \leq L_s^{+,*}\}} \, dY_s - \int_0^\tau f(s, Y_s, 0) 1_{\{Y_s \leq L_s^{+,*}\}} \, ds$$

$$+ \lambda T^{1/2} \left(\int_0^\tau |Z_s|^2 \, ds\right)^{1/2} + \int_0^\tau Z_s 1_{\{Y_s \leq L_s^{+,*}\}} \, dW_s. \quad (2.4)$$

By the classical Itô–Tanaka formula applied to the function $g(x) = (x)^- = \max(-x, 0)$ and the continuous semimartingale $Y - L^{+,*}$,

$$-\int_0^\tau 1_{\{Y_s \leq L_s^{+,*}\}} \, dY_s = -(Y_0 - L_0^{+,*})^- + (Y_\tau - L_\tau^{+,*})^-$$

$$-\int_0^\tau 1_{\{Y_s \leq L_s^{+,*}\}} \, dL_s^{+,*} - \frac{1}{2} L_\tau^0 (Y - L^{+,*})$$

$$\leq Y_\tau^* + L_\tau^{+,*},$$

where $L_0^0 (Y - L^{+,*})$ denotes the usual local time of $Y - L^{+,*}$ at 0. On the other hand, by (H2),

$$-f(s, Y_s, 0) 1_{\{Y_s \leq L_s^{+,*}\}} \leq -f(s, L_s^{+,*}, 0) 1_{\{Y_s \leq L_s^{+,*}\}} \leq |f(s, L_s^{+,*}, 0)|.$$

From the above we deduce that there is $C_p > 0$ such that

$$E(K_\tau)^p \leq C_p E\left((Y_\tau^*)^p + (L_{\tau}^{+,*})^p + \left(\int_0^\tau |f(s, L_s^{+,*}, 0)| \, ds\right)^p\right)$$

$$+ \left(\int_0^\tau |Z_s|^2 \, ds\right)^{p/2} + \int_0^\tau Z_s^p 1_{\{Y_s \leq L_s^{+,*}\}} \, dW_s \right)^p.$$
3. Main estimates in the case \( p > 1 \)

Let \( g : \mathbb{R} \to \mathbb{R} \) be a difference of two convex functions and let \( X \) be a continuous semimartingale. We will use the following form of the Itô–Tanaka formula

\[
g(X_t) = g(X_0) + \int_0^t \frac{1}{2}(g'_- + g'_+)(X_s)dX_s + \frac{1}{2} \int_\mathbb{R} \tilde{L}_t^a(X)g''(da) \tag{3.1}
\]

(see [15, Exercise VI.1.25]). Here \( \tilde{L}_t^a(X) \) denotes the symmetric local time of \( X \) at \( a \in \mathbb{R} \) and \( g''(da) \) is a measure determined by the second derivative of \( g \) in the sense of distributions. Note that \( \tilde{L}_t^a(X) \) is a unique increasing process such that

\[
|X_t - a| = |X_0 - a| + \int_0^t \text{sgn}(X_s - a)dX_s + \tilde{L}_t^a(X), \tag{3.2}
\]

where \( \text{sgn}(x) \) is equal to 1 if \( x > 0 \), \(-1 \) if \( x < 0 \) and 0 if \( x = 0 \) (see [15, Exercise VI.1.25]). One can observe that \( \tilde{L}_t^a(X) = (L_t^a(X) + L_t^a(-X))/2 \), where \( L_t^a(X) \) denotes the usual local time of \( X \) at \( a \). If \( g'' \) is absolutely continuous, i.e. if \( g''(da) = g''(a)da \), then by the occupation time formula, \( \int_\mathbb{R} L_t^a(X)g''(da) = \int_0^T g''(X_s)d[X]_s \). In this section, we will apply (3.1) to functions of the form \( g(x) = |x|^p \) or \( g(x) = (x^+)^p \). If \( p > 1 \) then in both cases the second derivative of \( g \) is absolutely continuous. Therefore if \( p > 1 \) then the backward Itô–Tanaka formula has the form

\[
g(X_t) + \frac{1}{2} \int_t^T g''(X_s)d[X]_s = g(X_T) - \int_t^T \frac{1}{2}(g'_- + g'_+)(X_s)dX_s. \tag{3.3}
\]

We can now prove basic a priori estimate and comparison result for \( \mathbb{L}^p \) solutions of (1.1).

**Proposition 3.1.** Assume that \( f \) satisfies (H1), (H2) and let \( (Y, Z, K) \) be a solution of (1.1) such that \( Y \in S^p \) for some \( p > 1 \). There exists \( C > 0 \) depending only on \( p, \mu, \lambda, T \) such that

\[
E(Y_T^p)^p \leq CE \left( |\xi|^p + (L_T^+, L_T^*)^p + \left( \int_0^T |f(s, L_s^+, 0)|ds \right)^p \right).
\]

**Proof.** We follow the proof of [3, Proposition 3.2]. The reasoning used at the beginning of the proof of Proposition 2.1 shows that we may assume that \( \mu = -\lambda^2/(p - 1) \) and \( \xi, L_T^+, L_T^* \), \( \int_0^T |f(s, L_s^+, 0)|ds \in \mathbb{L}^p \). By (3.3),

\[
|Y_t - L_t^+|^p + \frac{p(p - 1)}{2} \int_t^T |Y_s - L_s^+|^p \mathbf{1}_{\{Y_s \neq L_s^+\}} |Z_s|^2 ds
\]

\[= |\xi - L_T^+|^p + p \int_t^T |Y_s - L_s^+|^p \text{sgn}(Y_s - L_s^+) f(s, Y_s, Z_s) ds
\]

\[+ p \int_t^T |Y_s - L_s^+|^p \text{sgn}(Y_s - L_s^*) (dK_s + dL_s^*)
\]

\[- p \int_t^T |Y_s - L_s^+|^p \text{sgn}(Y_s - L_s^+) Z_s dW_s.
\]

By (H2) and the fact that \( K \) is increasing only on the set \( \{s : Y_s = L_s\} \),

\[
\text{sgn}(Y_s - L_s^+) f(s, Y_s, Z_s) \leq |f(s, L_s^+, 0)| + \mu |Y_s - L_s^+| + \lambda |Z_s|
\]

\[
\leq |f(s, L_s^+, 0)| + \mu |Y_s - L_s^+| + \lambda |Z_s|.
\]
\[ \text{sgn}(Y_s - L_s^{+,*}) \, dK_s \leq 0. \]

Hence
\[
\begin{align*}
|Y_t - L_t^{+,*}|^p &+ \frac{p(p-1)}{2} \int_t^T |Y_s - L_s^{+,*}|^{p-2} \mathbb{1}_{\{Y_s \neq L_s^{+,*}\}} |Z_s|^2 \, ds \\
&\leq |\xi - L_T^{+,*}|^p + p \int_t^T |Y_s - L_s^{+,*}|^{p-1} (|f(s, L_s^{+,*}, 0)| \, ds + dL_s^{+,*}) \\
&\quad + p\mu \int_t^T |Y_s - L_s^{+,*}|^p \, ds + p\lambda \int_t^T |Y_s - L_s^{+,*}|^{p-1} |Z_s| \, ds \\
&\quad - p \int_t^T |Y_s - L_s^{+,*}|^{p-1} \text{sgn}(Y_s - L_s^{+,*}) Z_s \, dW_s.
\end{align*}
\]

Since
\[
p\lambda |Y_s - L_s^{+,*}|^{p-1} |Z_s| \leq \frac{p\lambda^2}{p-1} |Y_s - L_s^{+,*}|^p \\
+ \frac{p(p-1)}{4} |Y_s - L_s^{+,*}|^{p-2} \mathbb{1}_{\{Y_s \neq L_s^{+,*}\}} |Z_s|^2
\]
for \( s \in [0, T] \), we have
\[
|Y_t - L_t^{+,*}|^p + \frac{p(p-1)}{2} \int_t^T |Y_s - L_s^{+,*}|^{p-2} \mathbb{1}_{\{Y_s \neq L_s^{+,*}\}} |Z_s|^2 \, ds \leq X - M_t, \quad (3.4)
\]
where
\[ X = |\xi - L_T^{+,*}|^p + p \int_0^T |Y_s - L_s^{+,*}|^{p-1} (|f(s, L_s^{+,*}, 0)| \, ds + dL_s^{+,*}) \]
and
\[ M_t = \int_0^t |Y_s - L_s^{+,*}|^{p-1} \text{sgn}(Y_s - L_s^{+,*}) Z_s \, dW_s, \quad t \in [0, T]. \]

Since \( Y \in \mathcal{S}^p \) and, by Proposition 2.1, \( Z \in \mathcal{H}^p \), applying Young’s inequality we obtain
\[
\begin{align*}
E X &\leq E|\xi - L_T^{+,*}|^p + E \left( \sup_{t \leq T} |Y_t - L_t^{+,*}|^{p-1} \left( \int_0^T |f(s, L_s^{+,*}, 0)| \, ds + L_T^{+,*} \right) \right) \\
&\leq E|\xi - L_T^{+,*}|^p + \frac{p-1}{p} E \sup_{t \leq T} |Y_t - L_t^{+,*}|^p \\
&\quad + \frac{2^{p-1}}{p} E \left( \int_0^T |f(s, L_s^{+,*}, 0)| \, ds \right)^p + (L_T^{+,*})^p < +\infty
\end{align*}
\]
and
\[
\begin{align*}
E(|M|^1_\mathcal{F}) &\leq E \left( \sup_{t \leq T} |Y_t - L_t^{+,*}|^{p-1} \left( \int_0^T |Z_s|^2 \, ds \right)^{1/2} \right)^{p/2} \\
&\leq \frac{(p-1)2^{p-1}}{p} E((Y_T^p + (L_T^{+,*})^p)^{p/2} + \frac{1}{p} E \left( \int_0^T |Z_s|^2 \, ds \right)^{p/2} < +\infty.
\end{align*}
\]
In particular, $M$ is a uniformly integrable martingale and hence, by (3.4),

$$\frac{p(p-1)}{2} E \int_0^T |Y_s - L^{+,+}_s|^p |Z_s|^2 ds \leq EX. \tag{3.5}$$

From (3.4), (3.5), the Burkholder–Davis–Gundy inequality and the definition of $M$ it follows that there is $c_p$ such that

$$E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p \leq EX + c_p E[M]^{1/2}_T$$

$$\leq EX + c_p E \left( \sup_{t \leq T} |Y_t - L^{+,+}_t|^p \int_0^T |Y_s - L^{+,+}_s|^p (1_{\{Y_s \neq L^{+,+}_s\}} |Z_s|^2 ds)^{1/2} \right)$$

$$\leq EX + \frac{1}{2} E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p + \frac{c_p^2}{2} E \int_0^T |Y_s - L^{+,+}_s|^p 1_{\{Y_s \neq L^{+,+}_s\}} |Z_s|^2 ds$$

$$\leq \left( 1 + \frac{c_p^2}{p(p-1)} \right) EX + \frac{1}{2} E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p.$$

By the above, the definition of $X$ and Young’s inequality,

$$E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p \leq c_p' EX$$

$$\leq c_p' \left( E|\xi - L^{+,+}_T|^p + pE \int_0^T |Y_s - L^{+,+}_s|^p (|f(s, L^{+,+}_s, 0)| ds + dL^{+,+}_s) \right)$$

$$\leq c_p' E|\xi - L^{+,+}_T|^p + pc_p' E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p \left( \int_0^T |f(s, L^{+,+}_s, 0)| ds + L^{+,+}_T \right)$$

$$\leq \frac{1}{2} E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p$$

$$+ c_p'' \left( E|\xi - L^{+,+}_T|^p + E \left( \int_0^T |f(s, L^{+,+}_s, 0)| ds + L^{+,+}_T \right)^p \right).$$

Hence

$$E \sup_{t \leq T} |Y_t - L^{+,+}_t|^p \leq 2c_p'' \left( E|\xi - L^{+,+}_T|^p + \left( \int_0^T |f(s, L^{+,+}_s, 0)| ds + L^{+,+}_T \right)^p \right),$$

from which the required estimate for $Y^n_T$ follows. \qed

**Proposition 3.2.** Let $(Y, Z, K)$ be a solution of (1.1) with $f$ satisfying (H1), (H2) and let $(Y', Z', K')$ be a solution of (1.1) with data $\xi', f'$, $L'$ such that $\xi \leq \xi', f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t)$ and $L_t \leq L'_t$, $t \in [0, T]$. If $Y, Y' \in \mathcal{S}^p$ for some $p > 1$ then $Y_t \leq Y'_t$, $t \in [0, T]$.

**Proof.** Assume that $\mu = -\lambda^2/(p-1)$. Then by (H1), (H2),

$$(Y_s - Y'_s)^{p-1} (f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s))$$

$$\leq -\frac{\lambda^2}{p-1} ((Y_s - Y'_s)^+)^p + \lambda((Y_s - Y'_s)^+)^{p-1} |Z_s - Z'_s|$$
for $s \in [0, T]$. Hence, by (3.3),

$$
((Y_t - Y'_t)^+) + \frac{p(p - 1)}{2} \int_t^T ((Y_s - Y'_s)^+) \, ds = ((\xi - \xi')^+) + \int_t^T (Y_s - Y'_s)^+ \, ds
$$

where $\xi$ is a uniformly integrable martingale. Therefore from (3.6) it follows that

$$
M \leq \int_t^T ((Y_s - Y'_s)^+) \, ds - p \int_t^T ((Y_s - Y'_s)^+) \, ds.
$$

Since

$$
p\lambda \leq \frac{p\lambda^2}{p - 1} \leq \frac{p\lambda^2}{(Y_s - Y'_s)^+} \, ds + \frac{p(p - 1)}{4} ((Y_s - Y'_s)^+) \, ds
$$

it follows that

$$
((Y_t - Y'_t)^+) + \frac{p(p - 1)}{4} \int_t^T ((Y_s - Y'_s)^+) \, ds
$$

Finally, as in the proof of Proposition 3.1 one can check that $M$ defined by

$$
M_t = \int_0^t (Y_s - Y'_s)^+ \, ds, \quad t \in [0, T]
$$

is a uniformly integrable martingale. Therefore from (3.6) it follows that $E((Y_t - Y'_t)^+)^p = 0, t \in [0, T]$. ∎

By repeating arguments from the proof of Proposition 3.2 one can obtain the following version of the comparison theorem for nonreflected BSDEs.

**Corollary 3.3.** Let $(Y, Z)$ be a solution of nonreflected (1.1) (i.e., where $L = -\infty$ and $K = 0$) with $f$ satisfying (H1), (H2) and let $(Y', Z')$ be a solution of nonreflected (1.1) with data $\xi', f'$, such that $\xi \leq \xi'$ and $f(t, Y'_t, Z'_t) \leq f'(t, Y'_t, Z'_t), t \in [0, T]$. If $Y, Y' \in S^p$ for some $p > 0$ then $Y_t \leq Y'_t, t \in [0, T]$. Note that Corollary 3.3 generalizes the comparison result proved in [12] for square-integrable solutions of nonreflected BSDEs.

4. **Existence and uniqueness of solutions in the case $p > 1$**

We begin with a general uniqueness result.
Proposition 4.1. If \( f \) satisfies (H1), (H2) then there is at most one solution \((Y, Z, K)\) of (1.1) such that \( Y \in S^p \) for some \( p > 1 \).

Proof. Follows from Proposition 3.2. \( \square \)

The problem of existence of solutions is more delicate. In the present section, we will assume additionally that

(H3) (a) \( E|\xi|^p < +\infty \),
(b) for every \( t \in [0, T] \) and \( z \in \mathbb{R}^d \), \( y \mapsto f(t, y, z) \) is continuous,
(c) \( E \left( \int_0^T |f(s, 0, 0)| ds \right)^p < +\infty \),
(d) for every \( r > 0 \), \( \int_0^T \sup_{|y| \leq r} |f(s, y, 0) - f(s, 0, 0)| ds < +\infty \),

(H4) (a) \( E(L_{t}^{+,*})^p < +\infty \),
(b) \( E \left( \int_0^T |f(s, L_s^{+,*}, 0)| ds \right)^p < +\infty \).

From Theorem 4.2 and Remark 4.3 in [3] one can deduce that under (H1)–(H3), (H4a) for every \( n \in \mathbb{N} \) there exists a unique solution \( Y^n \in S^p \), \( Z^n \in H^p \) of the BSDE (1.3).

Proposition 4.2. Let \( f \) satisfy (H1), (H2) and let \((Y^n, Z^n, K^n)\), \( n \in \mathbb{N} \), be a solution of (1.3). Then for every \( p > 0 \) there exists \( C > 0 \) depending only on \( p \), \( \mu \), \( \lambda \), \( T \) such that for every stopping time \( \tau \leq T \) and \( n \in \mathbb{N} \),

\[
E \left( \left( \int_0^\tau |Z^n_s|^2 \, ds \right)^{p/2} + (K^n_\tau)\right) \leq C E \left( (Y^{n,*})^p + (L_{t}^{+,*})^p + \left( \int_0^\tau |f(s, L_s^{+,*}, 0)| \, ds \right)^p \right) .
\]

Proof. The proof is similar to that of Proposition 2.1. Set \( \bar{Y}_t^n = e^{at} Y_t^n \), \( \bar{Z}_t^n = e^{at} Z_t^n \) and \( \bar{L}_t^n = e^{at} L_t^n, \bar{\xi} = e^{aT} \xi, \bar{f}(t, y, z) = e^{at} f(t, e^{-at} y, e^{-at} z) - ay \). Then \((\bar{Y}_n, \bar{Z}_n)\) solves the BSDE

\[
\bar{Y}_t^n = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s^n, \bar{Z}_s^n) d s - \int_t^T \bar{Z}_s^n d W_s + \bar{K}_t^n - \bar{\bar{K}}_t^n, \quad t \in [0, T]
\]

with the penalization term

\[
\bar{K}_t^n = \int_0^t e^{as} d K_s^n = n \int_0^t (\bar{Y}_s^n - \bar{L}_s)^- \, ds \quad t \in [0, T].
\]

Therefore without loss of generality we may assume that \( \mu = 0 \). Since \( K^n \) is increasing only on the set \( \{ s : Y_s^n < L_s \} \),

\[
\int_0^t (Y_s^n - L_s^{+,*}) d K_s^n \leq \int_0^t (Y_s^n - L_s^{+,*}) 1_{\{Y_s^n > L_s^{+,*}\}} d K_s^n = 0
\]

and

\[
K_t^n = \int_0^T 1_{\{Y_s^n \leq L_s^{+,*}\}} d K_s^n \leq Y_t^{n,*} + L_t^{+,*} + \int_0^T |f(s, L_s^{+,*}, 0)| \, ds + \lambda T^{1/2} \left( \int_0^T |Z_s^n|^2 \, ds \right)^{1/2} + \int_0^T Z_s^n 1_{\{Y_s^n \leq L_s^{+,*}\}} d W_s .
\]
To get the desired estimate it suffices now to repeat step by step arguments from the proof of Proposition 2.1, the only difference being in using the above estimates involving $K^n$ instead of $(2.2), (2.4)$. □

**Proposition 4.3.** Let Assumptions (H1)–(H4) hold and let $(Y^n, Z^n, K^n)$ be a solution of (1.3). Then for every $p > 1$ there exists $C > 0$ depending only on $p$ and $\mu, \lambda, T$ such that for every $n \in \mathbb{N}$,

$$
E \left( (Y^n_T)^p + \left( \int_0^T |Z^n_s|^2 \, ds \right)^{p/2} + (K^n_T)^p \right) 
\leq C E \left( |\xi|^p + (L^n_T)^p + \left( \int_0^T |f(s, L^n_s^+, 0)| \, ds \right)^p \right).
$$

**Proof.** Since $n \in \mathbb{N}$, then for every $p > 1$.

$$
p \int_t^T |Y^n_s - L^n_s|^p |s|^{p-1} \text{sgn}(Y^n_s - L^n_s) \, dK^n_s
\leq p \int_t^T |Y^n_s - L^n_s|^p \, ds 1_{\{Y^n_s > L^n_s\}} 1_{\{Y^n_s < L^n_s\}} \, dK^n_s = 0,
$$

applying the Itô–Tanaka formula to the function $g(x) = |x|^p$ and the semimartingale $Y^n - L^n$ we can estimate $E(Y^n_T)^p$ in the same way as in Proposition 3.1 (by the results from [3] we know that $Y^n \in \mathcal{S}^p, n \in \mathbb{N}$). Therefore the desired result follows from Proposition 4.2 with $\tau = T$. □

**Theorem 4.4.** Assume that (H1)–(H4) are satisfied. If $(Y^n, Z^n, K^n), n \in \mathbb{N}$, is a solution of BSDE (1.3), then

$$
\|Y^n - Y\|_{\mathcal{S}^p} \to 0, \quad \|Z^n - Z\|_{\mathcal{H}^p} \to 0, \quad \|K^n - K\|_{\mathcal{S}^p} \to 0,
$$

where $(Y, Z, K)$ is a unique solution of the reflected BSDE (1.1) such that $Y \in \mathcal{S}^p, Z \in \mathcal{H}^p$ and $K \in \mathcal{S}^p$.

**Proof.** Without loss of generality we may assume that $\mu = 0$. Let $(Y^n, Z^n, K^n)$ be a solution of (1.3). By Corollary 3.3, $Y^n_t \leq Y^{n+1}_t, n \in \mathbb{N}$. Therefore for every $t \in [0, T]$ there exists $Y_t$ such that $Y^n_t \not\rightarrow Y_t$. The rest of the proof is divided into 3 steps.

**Step 1.** We show that $Y$ is a càdlàg process. To see this let us first note that for every $t \in [0, T]$ there exists $V_t$ such that

$$
0 \leq V^n_t = \sup_{s \leq t} (Y^n_s - Y^1_s) \not\rightarrow V_t.
$$

By Fatou's lemma, $Y_t, V_t$ are finite. Indeed, $|Y_t| \leq V_t + Y^{1, *}_t, t \in [0, T]$, and by Proposition 4.3,

$$
E(V_T) \leq \liminf_{n \to \infty} E(V^n_T) \leq 2 \sup_n E(Y^{n, *}_T) \leq 2 \sup_n \|Y^{n, *}_T\|_p < \infty.
$$

Therefore $V$ is a progressively measurable nondecreasing process. Since the filtration $(\mathcal{F}_t)_{t \geq 0}$ is right-continuous, setting

$$
V'_t = \inf_{t' > t} V_{t'}, \quad t \in [0, T] \quad \text{and} \quad V'_T = V_T
$$

...
we get a progressively measurable càdlàg process $V'$. Obviously $V_t \leq V'_t$, so $Y^{n,*}_t \leq V'_t + Y^{1,*}_t$, $t \in [0, T]$, $n \in \mathbb{N}$. For $k \in \mathbb{N}$ set now

$$
\tau_k = \inf \left\{ t; \min \left( V'_t + Y^{1,*}_t, L^{*,*}_t, \int_0^t |f(s, L^{*,*}_s, 0)| \, ds \right) > k \right\} \wedge T. \tag{4.1}
$$

Clearly $\tau_k \leq \tau_{k+1}$, $k \in \mathbb{N}$, and $P(\tau_k = T) \not\rightarrow 1$. Since $Y^n$ is a continuous process,

$$
Y^{n,*}_{\tau_k} = Y^{n,*}_{\tau_k^-} \leq V'_{\tau_k^-} + Y^{1,*}_{\tau_k} \leq \max(k, c), \quad k \in \mathbb{N},
$$

where $c = \sup_n (Y^n_0)^+$ with the convention that $Y^{n,*}_0 = Y^{n,*}_0$, $V'_0 = V'_0$ ($c$ is a nonnegative constant because $Y^n_0$, $n \in \mathbb{N}$, are deterministic and by Proposition 4.3, $|Y^n_0| \leq C E \left( |\xi|^p + (L^+/T^n)^p + \left( \int_0^T |f(s, L^{*,*}_s, 0)| \, ds \right)^p \right)$ for $n \in \mathbb{N}$). Moreover, $L^{*,*}_T \leq \max(k, c)$ and $\int_0^{\tau_k} |f(s, L^{*,*}_s, 0)| \, ds \leq k$. Putting $p = p' > 2$ and $\tau = \tau_k$ in Proposition 4.2 we get

$$
\sup_n E \left( \left( \int_0^{\tau_k} |Z^n_s|^2 \, ds \right)^{p'/2} + (K^{n}_{\tau_k})^{p'} \right) \leq 3 C \max(k, c)^{p'} < +\infty, \tag{4.2}
$$

and consequently,

$$
\sup_n E \left| \int_0^{\tau_k} f(s, Y^n_s, Z^n_s) \, ds \right|^{p'} < +\infty. \tag{4.3}
$$

Since $f$ is Lipschitz continuous with respect to $z,$

$$
\int_0^t f(s, Y^n_s, Z^n_s) \, ds = \int_0^t h^n_s \, ds + \int_0^t f(s, Y^n_s, 0) \, ds, \quad t \in [0, T],
$$

where $h^n_s = (f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, 0)) I_{|Z^n_s| > 0} = C^n_s |Z^n_s|$ and $C^n_s$ is a one-dimensional progressively measurable process bounded by $\lambda$. By (4.2), $\sup_n E \left( \int_0^{\tau_k} (h^n_s)^2 \, ds \right)^{p'/2} \leq +\infty$. Since the sequences $\{1_{s \leq \tau_k} h^n_s\}_{n \in \mathbb{N}}, \{1_{s \leq \tau_k} Z^n_s\}_{n \in \mathbb{N}}$ are bounded in $\mathcal{H}^2$, there exist a subsequence $(n') \subset (n),$ a one-dimensional progressively measurable process $h$ and a $d$-dimensional progressively measurable process $Z$ such that $1_{s \leq \tau_k} h^n_s \rightarrow h$ and $1_{s \leq \tau_k} Z^n_s \rightarrow Z$ weakly in $\mathcal{H}^2$, i.e. for any one-dimensional $h' \in \mathcal{H}^2$,

$$
E \int_0^T 1_{(s \leq \tau_k)} h^n_s h'_s \, ds \rightarrow E \int_0^T h_s h'_s \, ds \tag{4.4}
$$

and for any $d$-dimensional process $Z' \in \mathcal{H}^2$,

$$
E \int_0^T 1_{(s \leq \tau_k)} Z^n_s Z'_s \, ds \rightarrow E \int_0^T Z_s Z'_s \, ds. \tag{4.5}
$$

From (4.5) and (4.4) it follows that $h$, $Z$ are equal to 0 on the set $\{s > \tau_k\}$. Moreover, for every stopping time $\sigma \leq \tau_k$,

$$
\int_0^\sigma h^n_s \, ds \rightarrow \int_0^\sigma h_s \, ds, \quad \int_0^\sigma Z^n_s \, dW_s \rightarrow \int_0^\sigma Z_s \, dW_s \tag{4.6}
$$
Let $Y \in \mathbb{L}^2$. Then $h' = E(Y|\mathcal{F}_t) \in \mathcal{H}^2$ since $E \int_0^T |E(Y|\mathcal{F}_s)|^2 ds \leq T EY^2 < +\infty$. Hence, by (4.4) and Fubini’s theorem,

$$E \left( Y \int_0^T 1_{(s \leq \sigma)} h''_s ds \right) = E \left( \int_0^T Y 1_{(s \leq \sigma)} h''_s ds \right) = \int_0^T E(Y 1_{(s \leq \sigma)} h''_s) ds$$

$$= \int_0^T E(Y|\mathcal{F}_s) 1_{(s \leq \sigma)} h''_s ds$$

$$= \int_0^T E(Y|\mathcal{F}_s) 1_{(s \leq \sigma)} h_s ds = E \left( Y \int_0^T 1_{(s \leq \sigma)} h_s ds \right),$$

which means that $\int_0^\sigma h''_s ds \to \int_0^\sigma h_s ds$ weakly in $\mathbb{L}^2$. By the representation theorem, $Y = \int_0^T Z'_s dW_s$ for some $d$-dimensional process $Z' \in \mathcal{H}^2$. Hence, by (4.5),

$$E \left( Y \int_0^T Z'_s dW_s \right) = \int_0^T 1_{(s \leq \sigma)} Z'_s Z'_s ds \to \int_0^T 1_{(s \leq \sigma)} Z_s Z'_s ds$$

$$= E \left( Y \int_0^\sigma Z_s dW_s \right),$$

which proves the second convergence in (4.6). By (H3b)–(H3d) and the Lebesgue dominated convergence theorem, for every stopping time $\sigma \leq \tau_k$, $\int_0^\sigma f(s, Y_s, 0) ds \to \int_0^\sigma f(s, Y_s, 0) ds$ $P$-a.s., and by (4.2) and (4.3), the last convergence holds in $\mathbb{L}^2$, too. Since $Y^n \not\rightarrow Y_\sigma$ in $\mathbb{L}^2$ as well, for every stopping time $\sigma \leq \tau_k$,

$$Y_\sigma = Y_0 - \int_0^\sigma f(s, Y_s, 0) ds - \int_0^\sigma h_s ds + \int_0^\sigma Z_s dW_s - K_\sigma,$$  \hspace{1cm} (4.7)

where $K_\sigma$ is a weak limit in $\mathbb{L}^2$ of $\{K^n_\sigma\}$. From the proof of the monotone limit theorem for BSDE (see [14, Lemma 2.2]) it follows that $Y$ is càdlàg and $K$ is nondecreasing càdlàg on the stochastic interval $[0, \tau_k]$. Since $P(\tau_k = T) \not\rightarrow 1$, it follows that $P$-almost all trajectories of $Y$ are càdlàg on the whole interval $[0, T]$.

**Step 2.** We show that $Y_t \geq L_t$, $t \in [0, T]$ and $(Y^n - L)^{-} \to 0$ $P$-a.s. By (H3a), (H4) and Proposition 4.3 there is $C > 0$ such that $E \left( \int_0^T (Y^n_s - L_s)^{-} ds \right)^p \leq C/n^p$. Hence, by Fatou’s lemma,

$$E \int_0^T (Y_s - L_s)^{-} ds \leq \liminf_{n \to \infty} E \int_0^T (Y^n_s - L_s)^{-} ds = 0,$$

which implies that $\int_0^T (Y_s - L_s)^{-} ds = 0$. Since $Y - L$ is a càdlàg process, $(Y_t - L_t)^{-} = 0$ for $t \in [0, T)$ and hence $Y_t \geq L_t$ for $t \in [0, T)$. Moreover, $Y_T = Y^n_T = \xi \geq L_T$. Hence $(Y^n_t - L_t)^{-} \searrow 0$ for $t \in [0, T]$ and by Dini’s theorem, $(Y^n - L)^{-} \to 0$ $P$-a.s.

**Step 3.** We show that $\{(Y^n, Z^n, K^n)\}_{n\in\mathbb{N}}$ converges in $\mathcal{S}^P \times \mathcal{H}^P \times \mathcal{S}^P$ to $(Y, Z, K)$, where $(Y, Z, K)$ is a unique solution of (1.1). Let $\{\tau_k\}$ be a sequence of stopping times defined in
Step 1. By Itô’s formula, (H1) and (H2) with $\mu = 0$,

$$
(Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m)^2 + \int_{t \wedge \tau_k}^{\tau_k} |Z_s^n - Z_s^m|^2 \, ds \\
= (Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m)^2 + 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m)) \, ds \\
+ 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(dK_s^n - dK_s^m) - 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(Z_s^n - Z_s^m) \, dW_s \\
\leq (Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m)^2 + 2\lambda \int_{t \wedge \tau_k}^{\tau_k} |Y_s^n - Y_s^m| |Z_s^n - Z_s^m| \, ds \\
+ 2(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^m + 2(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^n - 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(Z_s^n - Z_s^m) \, dW_s \\
\leq (Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m)^2 + 2\lambda \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)^2 \, ds + 2(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^m + 2(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^n \\
+ \frac{1}{2} \int_{t \wedge \tau_k}^{\tau_k} |Z_s^n - Z_s^m|^2 \, ds - 2 \int_{t \wedge \tau_k}^{\tau_k} (Y_s^n - Y_s^m)(Z_s^n - Z_s^m) \, dW_s.
$$

Hence

$$
E \int_0^{\tau_k} |Z_s^n - Z_s^m|^2 \, ds \leq 2E(Y_{t \wedge \tau_k}^n - Y_{t \wedge \tau_k}^m)^2 + 4\lambda^2 E \int_0^{\tau_k} (Y_s^n - Y_s^m)^2 \, ds \\
+ 4E(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^m + 4E(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^n.
$$

By Fubini’s theorem,

$$
E \int_0^{\tau_k} (Y_s^n - Y_s^m)^2 \, ds = \int_0^T E(Y_s^n 1_{s \leq \tau_k} - Y_s^m 1_{s \leq \tau_k})^2 \, ds,
$$

which converges to 0 as $m, n \to \infty$. By Step 2 and (4.2),

$$
E(Y^n - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^m \leq \|(Y^n - L)_{t \wedge \tau_k}^{-,*}\|_2 \|K_{t \wedge \tau_k}^m\|_2 \leq \|(Y^n - L)_{t \wedge \tau_k}^{-,*}\|_2 \sup_m \|K_{t \wedge \tau_k}^m\|_2,
$$

which converges to 0 as $n \to \infty$. Similarly, $E(Y^m - L)_{t \wedge \tau_k}^{-,*} K_{t \wedge \tau_k}^m \to 0$ as $m \to \infty$. Since $E(Y^n_{t \wedge \tau_k} - Y^m_{t \wedge \tau_k})^2 \to 0$ as $m, n \to \infty$, it is clear that $\{1_{s \leq \tau_k} Z^n_s\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}^2$. Let $Z^{(k)}$ denote its limit. By using standard arguments based on the Burkholder–Davis–Gundy inequality one can show that in fact $E \sup_{t \leq \tau_k} |Y_t^n - Y_t^m|^2 \to 0$ as $n, m \to \infty$, which implies that $\sup_{t \leq \tau_k} |Y_t^n - Y_t^m| \to 0$ (here $\to_p$ stands for the convergence in probability $P$). Since $P(\tau_k = T) \not\to 1$,

$$
\sup_{t \leq T} |Y_t^n - Y_t|^2 \to 0, \quad \text{(4.8)}
$$

and consequently, $Y$ has continuous trajectories. Similarly, if we set $\tau_0 = 0$, $Z_t = Z_t^{(k)}$, $t \in (\tau_{k-1}, \tau_k)$, $k \in \mathbb{N}$ and $Z_0 = 0$, then

$$
\int_0^T |Z_s^n - Z_s^m|^2 \, ds \to 0. \quad \text{(4.9)}
$$
To see this let us fix $\varepsilon > 0$. By Chebyshev’s inequality, for each $k \in \mathbb{N}$,

$$P \left( \int_0^T |Z^n_s - Z_s|^2 \, ds > \varepsilon \right) \leq P \left( \int_0^{\tau_k} |Z^n_s - Z_s|^2 \, ds > \varepsilon, \, T = \tau_k \right) + P(T > \tau_k) \leq \varepsilon^{-2} E \int_0^{\tau_k} |Z^n_s - Z_s^{(k)}|^2 \, ds + P(T > \tau_k).$$

Since we know that $1_{[s \leq \tau_k]}(Z^n_s - Z_s^{(k)}) \to 0$ in $\mathcal{H}^2$, it follows that

$$\lim_{n \to \infty} \sup_k P \left( \int_0^T |Z^n_s - Z_s|^2 \, ds > \varepsilon \right) \leq P(T > \tau_k),$$

which proves (4.9) since $P(T > \tau_k) \downarrow 0$. By (H3c) and (H3d), $|f(s, Y^n_s, Z_s^n)| \leq g_k(s) + \lambda |Z_s^n|$, $s \leq \tau_k$, where $g_k$ is an integrable function. Hence, by (H3b) and (4.9),

$$\int_0^{t \wedge \tau_k} f(s, Y^n_s, Z^n_s) \, ds \to_p \int_0^{t \wedge \tau_k} f(s, Y_s, Z_s) \, ds$$

for every $k \in \mathbb{N}$. Letting $k \to \infty$ shows that we can omit $\tau_k$ in the upper limit of integration. From the above we deduce that

$$K^n_t \to K_t = Y_0 - Y_t - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s \, dW_s, \quad t \in [0, T],$$

where $K$ is a continuous nondecreasing process such that $K_0 = 0$. It is clear that in fact,

$$\sup_{t \leq T} |K^n_t - K_t| \to 0. \quad (4.10)$$

By the above and (4.8), $0 \geq \int_0^T (Y^n_t - L_t) \, dK^n_t \to_p \int_0^T (Y_t - L_t) \, dK_t$, which when combined with Step 2 implies that $\int_0^T (Y_t - L_t) \, dK_t = 0$. Putting together the facts mentioned above we deduce that $(Y, Z, K)$ is a solution of the reflected BSDE (1.1).

In order to complete the proof we have to show that $(Y^n, Z^n, K^n)$ converges to $(Y, Z, K)$ in $\mathcal{S}^p \times \mathcal{H}^p \times \mathcal{S}^p$. To see this let us first observe that by (4.8)–(4.10), Proposition 4.3 and Fatou’s lemma, $(Y^n, Z^n, K^n) \in \mathcal{S}^p \times \mathcal{H}^p \times \mathcal{S}^p$, which together with Proposition 4.1 implies that $(Y, Z, K)$ is a unique solution of (1.1) in $\mathcal{S}^p \times \mathcal{H}^p \times \mathcal{S}^p$. Since

$$\sup_n \sup_{t \leq T} |Y^n_t| \leq \sup_{t \leq T} |Y_t| + \sup_{t \leq T} |Y^n_1|,$$

applying the Lebesgue dominated convergence theorem shows that $\|Y^n - Y\|_{\mathcal{S}^p} \to 0$. Now, we are going to estimate $Z^n - Z$ in the norm of $\mathcal{H}^p$. By Itô’s formula,

$$\int_0^T |Z^n_s - Z_s|^2 \, ds = (Y^n_0 - Y_0)^2 + 2 \int_0^T (Y^n_s - Y_s)(f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)) \, ds$$

$$+ 2 \int_0^T (Y^n_s - Y_s)(dK^n_s - dK_s)$$

$$- 2 \int_0^T (Y^n_s - Y_s)(Z^n_s - Z_s) \, dW_s.$$
Hence, by (H1) and (H2) with \( \mu = 0 \),
\[
\int_0^T |Z^n_s - Z_s|^2 ds \leq 2(Y^n_0 - Y_0)^2 + 4\lambda^2 E \int_0^T (Y^n_s - Y_s)^2 ds
\]
\[
+ 4(Y^n - L)_{T^-}^* K_T + 4 \int_0^T (Y^n_s - Y_s)(Z^n_s - Z_s) dW_s.
\]

By Step 2, \((Y^n - L)_{T^-}^* \leq (Y^n - Y)^*_{T^-} + (Y - L)_{T^-}^* = (Y^n - Y)^*_{T^-}\). Hence
\[
\left( \int_0^T |Z^n_s - Z_s|^2 ds \right)^{p/2} \leq C((Y^n - Y)^*_{T^-})^p + 2((Y^n - Y)^*_{T^-} K_T)^{p/2}
\]
\[
+ 2 \left| \int_0^T (Y^n_s - Y_s)(Z^n_s - Z_s) dW_s \right|^{p/2}.
\]

Using the Burkholder–Davis–Gundy inequality we deduce from the above that
\[
E \left( \int_0^T |Z^n_s - Z_s|^2 ds \right)^{p/2} \leq C'(E((Y^n - Y)^*_{T^-})^p + \|Y^n - Y\|_{T^-}^p \|K_T\|_p) \to 0.
\]

On the other hand, there is \( C > 0 \) depending only on \( \lambda \) and \( T \) such that
\[
\left\| \int_0^T (f(s, Y^n, Z^n_s) - f(s, Y_s, Z_s)) ds \right\|_{S^p}
\]
\[
\leq \left\| \int_0^T (f(s, Y^n, Z_s) - f(s, Y_s, Z_s)) ds \right\|_{S^p} + C\|Z^n - Z\|_{H^p}.
\]

By monotonicity of the mapping \( y \mapsto f(s, y, z) \),
\[
\sup_n \sup_{t \leq T} \left| \int_0^t f(s, Y^n, Z_s) ds \right| \leq U,
\]
where \( U = \sup_{t \leq T} \left| \int_0^t f(s, Y_s, Z_s) ds \right| + \sup_{t \leq T} \left| \int_0^t f(s, Y^1_s, Z_s) ds \right| \in L^p \). Hence, by the Lebesgue dominated convergence theorem,
\[
\left\| \int_0^T (f(s, Y^n, Z_s) - f(s, Y_s, Z_s)) ds \right\|_{S^p} \to 0.
\]

Finally, putting together all the above convergences it is clear that \( \|K^n - K\|_{S^p} \to 0 \) and the proof of Theorem 4.4 is complete. \( \square \)

**Remark 4.5.** Let us remark that if \( f \) satisfies (H3c) and the general increasing growth condition considered in [10,12], i.e.
\[
|f(t, y, 0)| \leq |f(t, 0, 0)| + \varphi(|y|), \quad t \in [0, T], \; y \in \mathbb{R}, \tag{4.11}
\]
where \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a deterministic continuous increasing function, and if \( \varphi(L^{+,*}_{T^-}) \in L^p \) then condition (H4b) is satisfied. Moreover, if we assume (H3c) and that (4.11) holds true for some measurable \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \int_0^T \varphi(L^{+,*}_s) ds \in L^p \), then (H4b) is satisfied and the
conclusion of Theorem 4.4 is still in force. Therefore Theorem 4.4 generalizes and strengthens the corresponding results of [10] proved under condition (4.11) in case \( p = 2 \) only.

**Corollary 4.6.** Under the assumptions of Proposition 3.2, if moreover \( L = L' \) and \( \xi, f, L \) and \( \xi', f', L \) satisfy (H3) and (H4), \( dK_s \geq dK_s' \).

**Proof.** By Proposition 3.2,

\[
K^n_{t_2} - K^n_{t_1} = n \int_{t_1}^{t_2} (Y^n_s - L_s)^- \, ds \geq n \int_{t_1}^{t_2} (Y'^n_s - L_s)^- \, ds = K'^n_{t_2} - K'^n_{t_1}
\]

for every \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq t_2 \leq T \). Since \( \|K^n - K\|_{\mathcal{S}^p} \to 0 \) and \( \|K'^n - K'\|_{\mathcal{S}^p} \to 0 \) by Theorem 4.4, it follows that \( K^n_{t_2} - K^n_{t_1} \geq K'^n_{t_2} - K'^n_{t_1} \) for every \( 0 \leq t_1 \leq t_2 \leq T \), which proves the desired result.

**Remark 4.7.** Of course (H2) is satisfied if \( f \) is Lipschitz continuous with respect to \( y \), i.e. when

\[
|f(t, y, z) - f(t, y', z)| \leq C|y - y'|, \quad t \in [0, T], \ y, y' \in \mathbb{R}, \ z \in \mathbb{R}^d
\]

for some \( C \geq 0 \). Moreover, (4.12) together with (H3c), (H4a) implies (H4b). Therefore conclusions of Theorem 4.4 and Corollary 4.6 hold true if (H1), (H3a), (H3c), (H4a) and (4.12) are satisfied. Thus, Theorem 4.4 and Corollary 4.6 strengthen the corresponding stability and comparison results for (1.1) proved in [8], where (4.12) is assumed.

5. \( L^1 \) solutions of reflected BSDEs

Throughout this section we will assume that \( p = 1 \) in conditions (H3), (H4).

Let us recall that a process \( X \) belongs to the class \( \mathcal{D} \) if the family of random variables \( \{X_\sigma; \sigma \text{ stopping time, } \sigma \leq T\} \) is uniformly integrable. In [4, p. 90] it is observed that the space of continuous (càdlàg), adapted processes from \( \mathcal{D} \) is complete under the norm \( \|X\|_{\mathcal{D}} = \sup\{E|X_\sigma|; \sigma \text{ stopping time, } \sigma \leq T\} \).

First we consider the case where \( f \) does not depend on \( z \).

**Proposition 5.1.** Assume that \( f \) satisfies (H2) and does not depend on \( z \), and let \( (Y, Z, K) \) be a solution of (1.1) such that \( Y \in \mathcal{D} \).

(i) There exists \( C > 0 \) depending only on \( \mu, T \) such that

\[
\|Y\|_{\mathcal{D}} \leq C E\left( |\xi| + L_T^{+,*} + \int_0^T |f(s, L_s^{+,*})| \, ds \right).
\]

(ii) For every \( \beta \in (0, 1) \) there exists \( C > 0 \) depending only on \( \beta, \mu, T \) such that

\[
E\left( (Y_T^*)_\beta + \left( \int_0^T |Z_s|^2 \, ds \right)^{\beta/2} + K_T^\beta \right) \leq C \left( E\left( |\xi| + L_T^{+,*} + \int_0^T |f(s, L_s^{+,*})| \, ds \right) \right)^\beta.
\]
Proof. We may and will assume that $\mu = 0$. Let $\tau_n = \inf \{ t; \int_0^t |Z_s|^2 ds \geq n \} \wedge T, n \in \mathbb{N}$. By (3.2),
\[
|Y_{\sigma \wedge \tau_n} - L_s^{+, \ast} | + L_s^{+, \ast} (Y - L_s^{+, \ast}) \\
= |Y_{\tau_n} - L_s^{+, \ast} | + \int_{\sigma \wedge \tau_n}^{\tau_n} \sigma(Y_s - L_s^{+, \ast}) f(s, Y_s) ds \\
+ \int_{\sigma \wedge \tau_n}^{\tau_n} \sigma(Y_s - L_s^{+, \ast}) (dK_s + dL_s^{+, \ast}) - \int_{\sigma \wedge \tau_n}^{\tau_n} \sigma(Y_s - L_s^{+, \ast}) Z_s dW_s.
\]
(5.1)
Since
\[
\int_{\sigma \wedge \tau_n}^{\tau_n} \sigma(Y_s - L_s^{+, \ast}) (dK_s + dL_s^{+, \ast}) \leq L_{\tau_n}^{+, \ast} - L_{\sigma \wedge \tau_n}^{+, \ast}
\]
and, by (H2),
\[
\int_{\sigma \wedge \tau_n}^{\tau_n} \sigma(Y_s - L_s^{+, \ast}) f(s, Y_s) ds \leq \int_{\sigma \wedge \tau_n}^{\tau_n} |f(s, L_s^{+, \ast})| ds,
\]
it follows from (5.1) that
\[
|Y_{\sigma \wedge \tau_n} | \leq |Y_{\tau_n} | + 2L_{\tau_n}^{+, \ast} + \int_{\sigma \wedge \tau_n}^{\tau_n} |f(s, L_s^{+, \ast})| ds - \int_{\sigma \wedge \tau_n}^{\tau_n} \sigma(Y_s - L_s^{+, \ast}) Z_s dW_s.
\]
Conditioning with respect to $\mathcal{F}_{\sigma \wedge \tau}$ and then letting $n \to \infty$ we deduce from the above that
\[
|Y_{\sigma} | \leq E \left( |\xi| + 2L^{+, \ast}_T + \int_0^T |f(s, L_s^{+, \ast})| ds |\mathcal{F}_{\sigma} \right), \tag{5.2}
\]
which implies (i).

By (5.2) and [3, Lemma 6.1],
\[
E(Y_T^{\ast})_\beta \leq (1 - \beta)^{-1} \left( E \left( |\xi| + 2L^{+, \ast}_T + \int_0^T |f(s, L_s^{+, \ast})| ds \right) \right)\beta
\]
for every $\beta \in (0, 1)$. Therefore (ii) follows from (2.1) with $\tau = T$. \hfill \Box

Let us note that by Proposition 5.1, if $(X, Z, K)$ satisfies (1.1) and $Y \in \mathcal{D}$ then $Z \in \bigcup_{\beta < 1} \mathcal{H}_\beta$ and $K \in \bigcup_{\beta < 1} \mathcal{S}_\beta$.

**Proposition 5.2.** Under the assumptions of Proposition 5.1 there exists at most one solution $(Y, Z, K)$ of (1.1) such that $Y \in \mathcal{D}$.

**Proof.** Without loss of generality we may assume that $\mu = 0$. Let $(Y, Z, K), (Y', Z', K')$ be two solutions of (1.1). Then from the Itô–Tanaka formula, (H2) and the inequality
\[
\text{sgn}(Y_s - Y'_s)(dK_s - dK'_s) \leq 0 \tag{5.3}
\]
it follows that for every \( t \in [0, T] \),
\[
|Y_t - Y'_t| + \bar{L}_t^0(Y - Y') = \int_t^T \text{sgn}(Y_s - Y'_s)(f(s, Y_s) - f(s, Y'_s)) \, ds \\
+ \int_t^T \text{sgn}(Y_s - Y'_s)(dK_s - dK'_s) \\
- \int_t^T \text{sgn}(Y_s - Y'_s)(Z_s - Z'_s) \, dW_s \\
\leq - \int_t^T \text{sgn}(Y_s - Y'_s)(Z_s - Z'_s) \, dW_s.
\]

By using the fact that \( Y, Y' \in D \), stopping at \( \tau_n = \inf \left\{ t; \int_0^t |Z_s - Z'_s|^2 \, ds \geq n \right\} \wedge T \) and then letting \( n \to \infty \) we deduce from the above that \( E|Y_t - Y'_t| = 0, t \in [0, T] \), i.e. \( Y = Y' \).

Consequently, \( \int_t^0 (Z_s - Z'_s) \, dW_s = (K_t - K'_t), t \in [0, T] \), which implies that \( Z = Z' \) and \( K = K' \). \( \square \)

**Proposition 5.3.** Let \((Y, Z, K)\) be a solution of (1.1) with \( f \) not depending on \( z \) and satisfying (H2), and let \((Y', Z', K')\) be a solution of (1.1) with data \( \xi', f', L' \) such that \( \xi \leq \xi' \), \( f' \) does not depend on \( z \), \( f(t, Y'_t) \leq f'(t, Y'_t) \) and \( L_t \leq L'_t, t \in [0, T] \). If \( Y, Y' \in D \) then \( Y_t \leq Y'_t, t \in [0, T] \).

**Proof.** Assume that \( \mu = 0 \) and observe that by (3.1), (H2) and (5.3),
\[
(Y_t - Y'_t)^+ + \frac{1}{2} \bar{L}_t^0(Y - Y') = \int_t^T 1_{\{Y_s > Y'_s\}}(f(s, Y_s) - f(s, Y'_s)) \, ds \\
+ \int_t^T 1_{\{Y_s > Y'_s\}}(dK_s - dK'_s) \\
- \int_t^T 1_{\{Y_s > Y'_s\}}(Z_s - Z'_s) \, dW_s \\
\leq - \int_t^T 1_{\{Y_s > Y'_s\}}(Z_s - Z'_s) \, dW_s.
\]

From this as in the proof of **Proposition 5.2** we deduce that \( E(Y_t - Y'_t)^+ = 0, t \in [0, T] \), i.e. \( Y_t \leq Y'_t, t \in [0, T] \). \( \square \)

**Theorem 5.4.** Assume that \( f \) does not depend on \( z \) and (H2)–(H4) are satisfied. If \((Y^n, Z^n, K^n), n \in \mathbb{N}, \) is a solution of BSDEs (1.3) then for every \( \beta \in (0, 1) \),
\[
\|Y^n - Y\|_{\mathcal{S}^\beta} \to 0, \quad \|Z^n - Z\|_{\mathcal{H}^\beta} \to 0, \quad \|K^n - K\|_{\mathcal{S}^\beta} \to 0,
\]
where \((Y, Z, K)\) is a unique solution of the reflected BSDEs (1.1) such that \( Y \in D, Z \in \bigcup_{\beta < 1} \mathcal{H}^\beta \) and \( K \in \bigcup_{\beta < 1} \mathcal{S}^\beta \).

**Proof.** We may and will assume that \( \mu = 0 \). By Briand et al. [3, Proposition 6.4], for every \( n \in \mathbb{N} \) there exists a unique solution \((Y^n, Z^n, K^n)\) of BSDE (1.3) such that \( Y^n \in D, Z^n \in \bigcup_{\beta < 1} \mathcal{H}^\beta \)
and $K^n \in \bigcup_{\beta<1} S^\beta$. As in the proof of Proposition 4.3 one can observe that

$$|Y^n_\sigma| \leq E \left( |\xi| + 2L^n_T + \int_0^T |f(s, L^n_s)| \, ds | F_\sigma \right),$$

which implies that for $N > 0$,

$$E(|Y^n_\sigma| 1_{(|Y^n_\sigma| > N)}) \leq E \left( \left( |\xi| + 2L^n_T + \int_0^T |f(s, L^n_s)| \, ds \right) 1_{(|Y^n_\sigma| > N)} \right).$$

Since by Chebyshev's inequality, $\lim_{N \to \infty} \sup_{\sigma, n} P(|Y^n_\sigma| > N) = 0$, it is clear that

$$\{Y^n_\sigma; \sigma \text{ stopping time, } \sigma \leq T, \ n \in \mathbb{N} \} \text{ is uniformly integrable.} \quad (5.4)$$

Now, as in the proof of Proposition 4.3 one can check that for every $\beta \in (0, 1)$ there exists $C > 0$ depending only on $\mu, \lambda, T$ such that for every $n \in \mathbb{N},$

$$E \left( (Y^n_T)^\beta + \left( \int_0^T |Z^n_s|^2 \, ds \right)^{\beta/2} + (K^n_T)^\beta \right) \leq C \left( E \left( |\xi| + L^n_T + \int_0^T |f(s, L^n_s)| \, ds \right) \right)^\beta.$$

Moreover, arguing as in the proof of Proposition 5.3 shows that $Y^n_t \leq Y^{n+1}_t, n \in \mathbb{N}, t \in [0, 1]$. Therefore for every $t \in [0, T]$ there exists $Y_t$ such that $Y^n_t \not\to Y_t$. By the same method as in the proof of Theorem 4.4 we can show that $Y$ is càdlàg (the process $V$ need not be integrable and we only know that $E(V^n_T)^\beta \leq \liminf_{n \to \infty} E(V^n_T)^\beta \leq 2 \sup_n E(Y^n_T)^\beta$). By Fatou’s lemma,

$$E \left( \int_0^T (Y^n_s - L^n_s)^+ \, ds \right) \leq \liminf_{n \to \infty} E \left( \int_0^T (Y^n_s - L^n_s)^+ \, ds \right) = 0,$$

which implies that $Y_t \geq L_t, t \in [0, T]$, and $(Y^n - L)^+ \to 0$-a.s. As in the proof of Theorem 4.4 we also show that $\|Y^n - Y\|_{S^\beta} \to 0$, $\|Z^n - Z\|_{\mathcal{H}^\beta} \to 0$ and $\|K^n - K\|_{S^\beta} \to 0$, where $(Y, Z, K)$ is a solution of (1.1) such that $Y, K \in \bigcup_{\beta<1} S^\beta$ and $Z \in \bigcup_{\beta<1} \mathcal{H}^\beta$. In order to complete the proof we have to check that $Y \in \mathcal{D}$, but this is an easy consequence of (5.4).

The following corollary may be proved in the same way as Corollary 4.6.

**Corollary 5.5.** Under the assumptions of Proposition 5.3, if moreover $L = L'$ and $\xi, f, L$ and $\xi', f', L$ satisfy (H3) and (H4), $dK_s \geq dK'_s$.

We now consider reflected BSDEs of the form

$$Y_t = \xi + \int_t^T f(s, Y_s, V_s) \, ds - \int_t^T Z_s \, dW_s + K^r_t - K_t \quad t \in [0, T] \quad (5.5)$$

and

$$Y'_t = \xi + \int_t^T f(s, Y'_s, V'_s) \, ds - \int_t^T Z'_s \, dW_s + K'_t - K'_t, \quad t \in [0, T], \quad (5.6)$$

where $V, V'$ are arbitrary progressively measurable processes on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. 
Proposition 5.6. Let $f$ satisfy (H2) and let $(Y, Z, K), (Y', Z', K')$ be solutions of (5.5), (5.6), respectively, such that $Y, Y' \in D$. Then for every $p > 1$ there is $C > 0$ depending only on $\mu, T$ such that

$$
\|Y - Y'\|_{\mathcal{S}^p} + \|Z - Z'\|_{\mathcal{H}^p} \leq C \left\| \int_0^T |f(s, Y_s, V_s) - f(s, Y_s, V'_s)| \, ds \right\|_p.
$$

Proof. Without loss of generality we may and will assume that $\mu = 0$ and $U = \int_0^T |f(s, Y_s, V_s) - f(s, Y_s, V'_s)| \, ds \in L^p$. Arguing as in the proof of Proposition 5.2, i.e. using the fact that $Y, Y' \in D$, stopping at $\tau_n = \inf\{t; \int_0^t |Z_s - Z'_s|^2 \, ds \geq n\} \wedge T$ and then letting $n \to \infty$ we show that $|Y_t - Y'_t| \leq E(U|\mathcal{F}_t), t \in [0, T]$. Hence, by Doob’s maximal inequality,

$$
E((Y - Y')^+)^p \leq C_p E(U)^p. \tag{5.7}
$$

On the other hand, by the same method as in the proof of Proposition 2.1 one can show that for $n \in \mathbb{N}$ we have

$$
\int_0^{\tau_n} |Z_s - Z'_s|^2 \, ds = (Y_{\tau_n} - Y'_{\tau_n})^2 + 2 \int_0^{\tau_n} (Y_s - Y'_s)(f(s, Y_s, V_s) - f(s, Y'_s, V'_s)) \, ds
- 2 \int_0^{\tau_n} (Y_s - Y'_s)(Z_s - Z'_s) \, dW_s + 2 \int_0^{\tau_n} (Y_s - Y'_s)(dK_s - dK'_s).
$$

Since $K$ is increasing only on the set $\{s : Y_s = L_s\}$ and $K'$ is increasing only on the set $\{s : Y'_s = L'_s\}$,

$$
\int_0^{\tau_n} (Y_s - Y'_s)(dK_s - dK'_s) \leq 0.
$$

On the other hand, by (H2),

$$
\int_0^{\tau_n} (Y_s - Y'_s)(f(s, Y_s, V'_s) - f(s, Y'_s, V'_s)) \, ds \leq 0.
$$

By the above,

$$
\int_0^{\tau_n} |Z_s - Z'_s|^2 \, ds = (Y_{\tau_n} - Y'_{\tau_n})^2 + 2 \int_0^{\tau_n} (Y_s - Y'_s)(f(s, Y_s, V_s) - f(s, Y'_s, V'_s)) \, ds
- 2 \int_0^{\tau_n} (Y_s - Y'_s)(Z_s - Z'_s) \, dW_s
$$

and hence

$$
\left( \int_0^{\tau_n} |Z_s - Z'_s|^2 \, ds \right)^{p/2}
\leq c_p \left( |Y_{\tau_n} - Y'_{\tau_n}|^p + ((Y - Y')^+)^p(U)^{p/2} + \left| \int_0^{\tau_n} (Y_s - Y'_s)(Z_s - Z'_s) \, dW_s \right|^{p/2} \right)
\leq c'_p \left( ((Y - Y')^+)^p + (U)^p + \left| \int_0^{\tau_n} (Y_s - Y'_s)(Z_s - Z'_s) \, dW_s \right|^{p/2} \right).
$$
Using the Burkholder–Davis–Gundy inequality and letting \( n \to \infty \) we conclude from the above that
\[
E \left( \int_0^T |Z_s - Z'_s|^2 \, ds \right)^{p/2} \leq C_p E \left( ((Y - Y')_T^p) + (U)^p \right),
\]
which together with (5.7) implies the desired result. \( \square \)

By the arguments from the proof of the above proposition one can obtain similar estimates for processes on arbitrary intervals \([t, q]\) \( \subset [0, T] \).

**Proposition 5.7.** Under the assumptions of Proposition 5.6 for every \( p > 1 \) there is \( C > 0 \) depending only on \( \mu, T \) such that for every \( 0 \leq t < q \leq T \),
\[
\begin{align*}
\| (Y - Y')_1_{[t, q]} \|_{\mathcal{S}^p} + \| (Z - Z')_1_{[t, q]} \|_{\mathcal{H}^p} \\
\leq C \left( \| Y_q - Y'_q \|_p + \left\| \int_{t}^{q} |f(s, Y_s, V_s) - f(s, Y'_s, V'_s)| \, ds \right\|_p \right).
\end{align*}
\]

To deal with generators depending on \( z \) we will need the following condition introduced in [3]:

(H5) There exist constants \( \gamma \geq 0, \alpha \in (0, 1) \) and a nonnegative progressively measurable process \( g \) such that \( E \left( \int_0^T g_s \, ds \right) < +\infty \) and
\[
|f(t, y, z) - f(t, y, 0)| \leq \gamma (g_t + |y| + |z|)^\alpha, \quad t \in [0, T], \ y \in \mathbb{R}, \ z \in \mathbb{R}^d.
\]

**Theorem 5.8.** Let Assumptions (H1)–(H5) hold. Then there exists a unique solution \((Y, Z, K)\) of the reflected BSDEs (1.1) such that \( Y \in \mathcal{D}, Z \in \bigcup_{\beta < 1} \mathcal{H}^\beta \) and \( K \in \bigcup_{\beta < 1} \mathcal{S}^\beta \).

**Proof.** Our method of proof will be adaptation of the proofs of Theorems 6.2 and 6.3 in [3]. Assume that \( \mu = 0 \). First we show that there exists at most one solution. Let \((Y, Z, K), (Y', Z', K')\) be two solutions of (1.1) such that \( Y, Y' \in \mathcal{D} \) and \( Z, Z' \in \bigcup_{\beta < 1} \mathcal{H}^\beta \). Let \( p > 1 \) be such that \( \alpha p < 1 \). Then
\[
E \left( \int_0^T |f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)| \, ds \right)^p \leq (2\gamma)^p T^{p-1} E \left( \int_0^T (g_s + |Y_s| + |Z_s| + |Z'_s|)^{\alpha p} \, ds \right) < +\infty
\]
and hence, by Proposition 5.6, \( E \left( \int_0^T |Z_s - Z'_s|^2 \, ds \right)^{p/2} < +\infty \). Moreover, by Proposition 5.6 and (H1),
\[
\begin{align*}
\| Y - Y' \|_{\mathcal{S}^p} + \| Z - Z' \|_{\mathcal{H}^p} & \leq C \left\| \int_0^T |f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s)| \, ds \right\|_p \\
& \leq C\lambda T^{1/2} \| Z - Z' \|_{\mathcal{H}^p}.
\end{align*}
\]
Therefore, if \( 2C\lambda T^{1/2} \leq 1 \) then \( \| Y - Y' \|_{\mathcal{S}^p} + \frac{1}{2} \| Z - Z' \|_{\mathcal{H}^p} \leq 0 \), which implies that \( Y = Y' \) and \( Z = Z' \), and consequently, \( K = K' \). If \( 2C\lambda T^{1/2} > 1 \), we divide the interval \([0, T]\) into
a finite number of intervals with mesh \( \delta > 0 \) such that \( 2C_\lambda \delta^{1/2} \leq 1 \) and prove uniqueness by induction using Proposition 5.7 instead of Proposition 5.6.

Now we are going to prove existence of solutions. Let \( Z^0 = 0 \). By (H5) and Theorem 5.4, for each \( n \in \mathbb{N} \) there exists a unique solution \((Y^n, Z^n, K^n)\) of the reflected BSDEs (with obstacle \( L \)) of the form

\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s^{n-1}) \, ds - \int_t^T Z^n_s \, dW_s + K^n_T - K^n_t, \quad t \in [0, T]
\]

(5.8)
such that \( Y^n \in \mathcal{D}, Z^n \in \bigcup_{\beta < 1} \mathcal{H}^\beta \) and \( Y^n, K^n \in \bigcup_{\beta < 1} \mathcal{S}^\beta \). Let \( p > 1 \) be such that \( \alpha > 1 \). Then by (H5) and Proposition 5.6,

\[
\|Y^{n+1} - Y^n\|_\mathcal{S}^p + \|Z^{n+1} - Z^n\|_{\mathcal{H}^p} \leq C \left( \int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z^{n-1}_s)| \, ds \right)^{1/p} \leq 2 \gamma T^{(p-1)/p} \left( E \left( \int_0^T (g_s + |Y^n_s| + |Z^n_s| + |Z^{n-1}_s|)^{p/\alpha} \, ds \right) \right)^{1/p} < +\infty.
\]

Thus, \((Y^{n+1} - Y^n) \in \mathcal{S}^p, (Z^{n+1} - Z^n) \in \mathcal{H}^p\), and hence, by elementary calculations, \((K^{n+1} - K^n) \in \mathcal{S}^p\). It follows that \((Y^n - Y^1) \in \mathcal{S}^p, (Z^n - Z^1) \in \mathcal{H}^p\) and \((K^n - K^1) \in \mathcal{S}^p, n \in \mathbb{N}\). As in the proof of uniqueness we first assume that \( 2C_\lambda T^{1/2} \leq 1 \). By (H1) and Proposition 5.6,

\[
\|Y^{n+1} - Y^n\|_\mathcal{S}^p + \|Z^{n+1} - Z^n\|_{\mathcal{H}^p} \leq \frac{1}{2} \|Z^n - Z^{n-1}\|_{\mathcal{H}^p}
\]

for \( n \in \mathbb{N} \). Hence

\[
\|Y^m - Y^n\|_\mathcal{S}^p + \|Z^m - Z^n\|_{\mathcal{H}^p} \leq 2 \left( \frac{1}{2} \right)^{n-1} \|Z^2 - Z^1\|_{\mathcal{H}^p}
\]

for all \( m \geq n \). Consequently, \( \{(Y^n - Y^1, Z^n - Z^1)\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{S}^p \times \mathcal{H}^p \) converging to some process \((\tilde{Y}, \tilde{Z})\). Since \((Y^1, Z^1) \in \mathcal{D} \times \bigcup_{\beta < 1} \mathcal{H}^\beta\), it follows that

\[
Y^n \rightarrow \tilde{Y} = Y^1 \quad \text{in} \mathcal{D}, \quad Z^n \rightarrow \tilde{Z} = Z^1 \quad \text{in} \mathcal{H}^\beta, \; \beta \in (0, 1).
\]

(5.9)

Using standard arguments one can also check that for every \( \beta \in (0, 1) \) the sequence \( \{K^n\} \) converges in \( \mathcal{S}^\beta \) to some nondecreasing continuous process \( K \) and that \((Y, Z, K)\) is a solution of the reflected BSDE (1.1).

If \( 2C_\lambda T^{1/2} > 1 \) we consider a partition \( 0 = t_0 < t_1 < \cdots < t_k = T \) of the interval \([0, T]\) such that \( t_i - t_{i-1} \leq \delta, i = 1, \ldots, k \), and \( 2C_\lambda \delta^{1/2} \leq 1 \). By arguments from the first part of the proof and Proposition 5.7,

\[
\|(Y^{n+1} - Y^n)1_{[t_{k-1}, T]}\|_{\mathcal{S}^p} + \|(Z^{n+1} - Z^n)1_{[t_{k-1}, T]}\|_{\mathcal{H}^p} \leq C \left( \int_{t_{k-1}}^T |f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z^{n-1}_s)| \, ds \right)^{1/p} \leq \frac{1}{2} \|(Z^n - Z^{n-1})1_{[t_{k-1}, T]}\|_{\mathcal{H}^p} \leq \left( \frac{1}{2} \right)^{n-1} \|(Z^2 - Z^1)1_{[t_{k-1}, T]}\|_{\mathcal{H}^p}.
\]
which implies that
\[ \|(Y^n - Y^m)\|_{[t_{k-1}, T]} \|_{\mathcal{S}^p} + \|(Z^n - Z^m)\|_{[t_{k-1}, T]} \|_{\mathcal{H}^p} \leq 2 \left( \frac{1}{2} \right)^{n-1} \|(Z^1 - Z^1)\|_{[t_{k-1}, T]} \|_{\mathcal{H}^p} \]
for all \( m \geq n \). Accordingly, \( \{(Y^n - Y^1)\}_{t_{k-1}, T}, (Z^n - Z^1)_{t_{k-1}, T} \) in \( \mathbb{N} \) is a Cauchy sequence in \( \mathcal{S}^p \times \mathcal{H}^p \). Therefore as in the proof of (5.9) one can show that there exist processes \( Y^{(k)}, Z^{(k)}, K^{(k)} \) such that \( Y^n_{t_{k-1}, T} \rightarrow Y^{(k)} \) in \( \mathcal{D} \), \( Z^n_{t_{k-1}, T} \rightarrow Z^{(k)} \) in \( \mathcal{H}^{\beta}, \beta \in (0, 1) \), and \( (K^n - K^n)_{t_{k-1}, T} \rightarrow K^{(k)} \) in \( \mathcal{S}^\beta, \beta \in (0, 1) \). Observe that \( Y^{(k)}_T = \xi \) and
\[
Y_t^{(k)} = Y_{t_{k-1}}^{(k)} - \int_{t_{k-1}}^t f(s, Y_s^{(k)}, Z_s^{(k)}) \, ds + \int_{t_{k-1}}^t Z_s^{(k)} \, dW_s - K_t^{(k)}, \quad t \in [t_{k-1}, T]. \tag{5.10}
\]
By (H5) and Theorem 5.4, for each \( n \in \mathbb{N} \) there exists a unique solution \( (Y^n, Z^n, K^n) \) of the reflected BSDEs (5.8) with \( [0, T] \) replaced by \( [0, t_{k-1}] \) and \( \xi \) replaced by \( Y^{(k)}_{t_{k-1}} \). Therefore in the same manner as before we can see that there exist processes \( Y^{(k-1)}, Z^{(k-1)}, K^{(k-1)} \) such that \( Y^n_{t_{k-2}, t_{k-1}} \rightarrow Y^{(k-1)} \) in \( \mathcal{D} \), \( Z^n_{t_{k-2}, t_{k-1}} \rightarrow Z^{(k-1)} \) in \( \mathcal{H}^{\beta}, \beta \in (0, 1) \), and \( (K^n - K^n)_{t_{k-2}, t_{k-1}} \rightarrow K^{(k-1)} \) in \( \mathcal{S}^\beta, \beta \in (0, 1) \). We continue in this fashion to obtain for \( i = k - 1, \ldots, 1 \) the triple of processes \( (Y^{(i)}, Z^{(i)}, K^{(i)}) \) such that \( Y_t^{(i)} = Y_t^{(i+1)} \) and
\[
Y_t^{(i)} = Y_{t_{i-1}}^{(i)} - \int_{t_{i-1}}^t f(s, Y_s^{(i)}, Z_s^{(i)}) \, ds + \int_{t_{i-1}}^t Z_s^{(i)} \, dW_s - K_t^{(i)}, \quad t \in [t_{i-1}, t_i]. \tag{5.11}
\]
It is clear that for \( i = 1, \ldots, k, \)
\[
L_t \leq Y_t^n \rightarrow Y_t^{(i)} \geq L_t, \quad t \in [t_{i-1}, t_i] \tag{5.12}
\]
and
\[
0 = \int_{t_{i-1}}^{t_i} (Y_t^n - L_t) \, dK_t^n \rightarrow \int_{t_{i-1}}^{t_i} (Y_t^{(i)} - L_t) \, dK_t^{(i)} = 0. \tag{5.13}
\]
Set \( Y_T = \xi, Z_T = 0 \) and
\[
Y_t = Y_t^{(i)}, \quad Z_t = Z_t^{(i)}, \quad t \in [t_{i-1}, t_i], \quad i = 1, \ldots, k, \quad K = \sum_{i=1}^k K^{(i)},
\]
and observe that \( Y, Z, K \) are progressively measurable, \( Y \in \mathcal{D}, Z \in \mathcal{H}^{\beta}, \beta \in (0, 1) \), and \( K \in \mathcal{S}^\beta, \beta \in (0, 1), K_0 = 0 \). Moreover, the process \( Y \) is continuous, \( K \) is continuous and nondecreasing, and by (5.10), (5.11) the triple \( (Y, Z, K) \) satisfies the forward equation
\[
Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s) \, ds + \int_0^t Z_s \, dW_s - K_t, \quad t \in [0, T].
\]
Since \( Y_T = \xi \), it satisfies the backward equation (1.1) as well. Finally, by (5.12), \( Y_t \geq L_t, t \in [0, T] \), whereas by (5.13),
\[
\int_0^T (Y_t - L_t) \, dK_t = \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (Y_t^{(i)} - L_t) \, dK_t^{(i)} = 0,
\]
i.e. (1.1)2 and (1.1)3 are satisfied. Thus, the triple \( (Y, Z, K) \) is a solution of the reflected BSDE (1.1). \( \square \)
Remark 5.9. Similarly to the case $p > 1$ (see Remark 4.7), if we assume that stronger than (H2) condition (4.12) is satisfied, then in Theorem 5.4, Corollary 5.5 and Theorem 5.8 Assumption (H4b) may be omitted.

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