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Absolute continuity of stable foliations for systems on Banach spaces

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ABSTRACT

We prove the absolute continuity of stable foliations for $C^{1,\alpha}$ maps of Banach spaces satisfying a globally defined infinitesimal invariant cones condition. Proofs of regularity for center and stable manifolds needed for the main theorem are included. Our results are applicable to dynamical systems generated by ordinary, partial, or functional differential equations, including non-autonomous differential equations that are periodic in time.

0. Introduction

When an infinite dimensional dynamical system has an attracting finite dimensional center manifold W^c, one can view the large-time dynamics starting from any initial condition in the phase space X as being captured by a finite dimensional system, namely that on W^c. The presence of a (strong) stable foliation will allow one to say more, namely that the trajectory starting from every initial condition $x \in X$ is asymptotically close to that of some $\psi(x) \in W^c$. Via this association, we propose to introduce a notion of "typical initial condition" in X, calling $x \in X$ typical if $\psi(x)$ is a typical point with respect of the Lebesgue measure on W^c. While such a notion can be defined in the abstract, it is hard to know *a priori* what it means: For example, what constitutes a typical set in a k-parameter family of initial conditions? Is it, under reasonable conditions, compatible with Lebesgue measure in parameter space? The property that would provide an answer to such questions is the *absolutely continuity of the stable foliation*, and that is the main result of the present paper.

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We will present our results in the framework of $C^{1,\alpha}$ maps of Banach spaces satisfying a globally defined *infinitesimal invariant cones* condition. In addition to its relevance to differential equations (see below), this setting is the simplest in which to present our main result. To discuss the absolute continuity of the stable foliation $\mathcal{F}^s$, we naturally need to first establish the existence of $\mathcal{F}^s$. Also, while our main result does not rely on the existence of center manifolds, similar ideas are used in the first steps of its proof. For these reasons, we will include in the present manuscript the proofs of existence and regularity of W^c and $\mathcal{F}^s$. We do not pretend that our results on W^c and $\mathcal{F}^s$ are novel: versions of them – especially in the case of C^1 center manifolds – have been proved a number of times before (see references below). We are not aware, however, of global results in the generality of (A1)–(A4) in Section 1 which provide the regularity needed for our purposes.

An important motivation for this work comes from finite or infinite dimensional dynamical systems generated by ordinary, partial, or functional differential equations. We will give examples to show that our assumptions are satisfied by time-T maps of certain classes of differential equations on suitable function spaces. Our results are also applicable to non-autonomous differential equations that are periodic in time. In order to include parabolic type PDEs which are not time reversible, we do not assume that our maps are locally invertible.

As for the invariant cones condition, we drew our motivation both from differential equations and from abstract hyperbolic theory.

From the differential equations side, in contexts where center manifolds have been proved, the dynamical systems is often described in terms of a dominant linear operator along with controllable nonlinear parts, and a spectral gap for the linear operator is assumed; see e.g. [4–6,10]. On the other hand, as was pointed out in [13], a more essential condition for the existence of invariant manifolds is the invariance of cones accompanied by certain expanding or contracting properties, and the spectral gap condition serves only as a more accessible condition to ensure the cone invariance. In the construction in [2,13,7,17,8], for example, the cone condition is verified for the differential equations in question by showing, roughly speaking, that the vector field on the boundaries of the cones point inward.

In abstract hyperbolic theory, one often starts with a compact invariant set the tangent bundle over which is equipped with a splitting into a direct sum of Df-invariant subbundles.[3] From this setup one deduces the existence of local invariant disks tangent to the invariant linear subspaces, and global invariant manifolds are obtained from local ones by iterating forwards or backwards. In the uniform hyperbolic setting, the standard reference for invariant manifolds theory is [11]; see also [3]. The results here differ from previous ones in that they are genuinely global in nature, and no assumptions are made on the existence of invariant splittings, which are in fact shown to exist only on a set determined by the dynamics.

We postpone our discussion on regularity of stable foliations to Section 1, after the relevant definitions have been introduced.

This paper is organized as follows: Section 1 contains the formal setting and statement of results. In Section 2 we illustrate how our conditions are verified for certain C^0 semigroups. Section 3 contains the proofs of Theorems 1 and 2 on center manifolds and stable foliations. This sets us up for the proof of Theorem 3, our main result, which asserts the absolute continuity of $\mathcal{F}^s$ and is proved in Section 4.

1. Statement of results

We state our results in a global setting, obtained usually from standard cutoff and extension arguments in concrete situations involving differential equations.

Setting. Let $(X, |\cdot|)$ be a Banach space, and let $f \in C^{1,\alpha_0}(X, X)$ for some $\alpha_0 > 0$, i.e., $f \in C^1$, and $\|Df\|_{C^{\alpha_0}} < \infty$. We assume $\|Df\|_{C^0} < e^a$, and there is a reference splitting of X into $X = E^c \oplus E^s$ where E^c and E^s are closed subspaces the projection operators associated with which satisfy

[3] We will not discuss the nonuniformly hyperbolic case, which from our point of view resembles the hyperbolic case except that it involves a host of other issues not relevant to the present context.

$\|\pi^c\|, \|\pi^s\| \leqslant M$. These subspaces are *not* assumed to be Df-invariant, and the relations between them and the dynamics are as follows:

(A1) There exist $\mu_c \in (0,1)$ and $\lambda_c \in \mathbb{R}$ such that for any $x_0, x \in X$ with $|\pi^s x| \leqslant \mu_c |\pi^c x|$, we have

$$\left|\pi^s (Df)_{x_0} x\right| \leqslant \mu_c \left|\pi^c (Df)_{x_0} x\right| \quad \text{and} \quad \left|\pi^c (Df)_{x_0} x\right| \geqslant e^{\lambda_c} \left|\pi^c x\right|.$$

(A2) There exist $\mu_s \in (0,1)$[4] and $\lambda_s < \lambda_c^- \triangleq \min\{0, \lambda_c\}$ such that for any $x_0, x \in X$, if $|\pi^c (Df)_{x_0} x| \leqslant \mu_s |\pi^s (Df)_{x_0} x|$, then

$$\left|\pi^c x\right| \leqslant \mu_s \left|\pi^s x\right| \quad \text{and} \quad \left|\pi^s (Df)_{x_0} x\right| \leqslant e^{\lambda_s} \left|\pi^s x\right|.$$

(A3) $\dim(E^c) < \infty$.

(A4) Let $B^s(0,R) = \{x \in E^s\colon\ |x| < R\}$. Then for any $R > 0$, there exists $R' > 0$ such that $f(E^c + B^s(0,R)) \subset (E^c + B^s(0,R'))$.

The constants $\alpha_0, a, M, \mu_c, \mu_s, \lambda_c$ and λ_s will be referred to in the rest of this paper as *system constants*.

We remark on these assumptions: (A1) and (A2) are *invariant cones* conditions. Notice that λ_c can be positive or negative, so that (A1) and (A2) define *center* and *stable cones* respectively. The bound in (A1) is usually stated as $|\pi^c (D^n f)_{x_0} x| \geqslant C e^{n\lambda_c} |\pi^c x|$ for all $n \geqslant 1$, and analogously for (A2). We have omitted the constant C since for our purposes we can work with f^k for some fixed $k \geqslant 1$ instead of f. Our main result, namely the absolute continuity of stable foliations (Theorem 3), requires $\dim(E^c) < \infty$, while our results on center manifolds and stable foliations (Theorems 1 and 2) are valid also when $\dim(E^c) = \infty$ and (A3) is replaced by the weaker condition

(A3′) For every $x \in X$, $\pi^c (Df)_x|_{E^c}$ maps E^c onto E^c.

In the case $\dim(E^c) < \infty$, this condition is implied by (A1).

Notation: Throughout this paper, the tangent space at a point in X is identified with X itself. In (A1), for example, x_0 is the base point and x is a tangent vector at x_0. Likewise, we identify the linear spaces E^c and E^s with the corresponding subsets of X, permitting statements such as "For $g: E^c \to E^s$, define $\operatorname{graph}(g) \triangleq \{y + g(y),\ y \in E^c\}$".

Results. Three results are presented. Theorems 1 and 2, which have been proved a number of times under technical assumptions different from ours (see the Introduction), provide the backdrop for Theorem 3, which is the main result of the present paper.

Theorem 1 *(Center manifolds theorem). Assume* (A1)–(A4), *with* (A3) *replaced by* (A3′). *Then there exist* α, $K_0 > 0$ *and a unique* $h^c \in C^{1,\alpha}(E^c, E^s)$ *such that*

(1) $f(W^c) = W^c$ *where* $W^c \triangleq \operatorname{graph}(h^c)$;
(2) $\|Dh^c\|_{C^0} \leqslant \mu_c$ *and* $\|Dh^c\|_{C^\alpha} \leqslant K_0 \|Df\|_{C^\alpha}$.

Here α depends only on system constants, and K_0 depends on system constants and on α.

Theorem 2 *(Stable foliations theorem). Assume* (A1)–(A4), *with* (A3) *replaced by* (A3′). *Then there exist* $\alpha, K_0 > 0$ (*with the same dependence as above*) *such that the following hold*:

(1) (Individual stable manifolds.) *Associated with every $x \in X$, there is a mapping $h^s_x : E^s \to E^c$ such that*

[4] Alternately, we may permit μ_c to be $\geqslant 1$ and assume $0 < \mu_s < \min\{1, \frac{1}{\mu_c}\}$.

(a) $h^s_x(\pi^s x) = \pi^c x$, *i.e.*, $x \in W^s_x \triangleq \mathrm{graph}(h^s_x)$;
(b) $f(W^s_x) \subset W^s_{fx}$;
(c) $\|Dh^s_x\|_{C^0} \leqslant \mu_s$ *and* $\|Dh^s_x\|_{C^\alpha} \leqslant K_0\|Df\|_{C^\alpha}$.

(2) (Stable foliation.) *The family* $\{W^s_x\}$ *defines a* C^α *foliation* $\mathcal{F}^s$ *on* X, *i.e., for* $x, y \in X$, *either* $W^s_x = W^s_y$ *or* $W^s_x \cap W^s_y = \emptyset$; *moreover the mapping defined by*

$$H^s(x) = h^s_{\pi^c x}\big(\pi^s x\big) + \pi^s x$$

is a homeomorphism, and $H^s \in C^{0,\alpha}(X, X)$.

We remark that smoothness of a foliation involves not only the smoothness of individual leaves but also how these leaves are "packed together". Theorem 2 asserts that individual stable manifolds W^s_x are $C^{1,\alpha}$, but the foliation $\mathcal{F}^s$ is only $C^{0,\alpha}$. Indeed, under the conditions (A1)–(A4), the stable foliation is, in general, not C^1. It has been shown (in settings different from ours) that higher differentiability can be guaranteed by more stringent conditions; see e.g. [4]. We will not go in this direction. Instead, we will show that (A1)–(A4) alone imply a weaker form of regularity, one which already has important implications.

Unlike Theorems 1 and 2, finite dimensionality of E^c is required for Theorem 3. First we introduce some needed definitions. For a $C^{1,\alpha}$ map $g : E^c \to E^s$ (where α is as in Theorems 1, 2) with $\|g\|_{C^0} < \infty$, $\|Dg\|_{C^0} \leqslant \mu_c$ and $\|Dg\|_{C^\alpha} < \infty$, we let $\Sigma_g \triangleq \mathrm{graph}(g)$ and call Σ_g a *transversal* to the stable foliation $\mathcal{F}^s$. All transversals considered in this paper are assumed to be of this form. The *holonomy map* $T_{g_1,g_2} : \Sigma_{g_1} \to \Sigma_{g_2}$ sends $x \in \Sigma_{g_1}$ to the unique point in $W^s_x \cap \Sigma_{g_2}$. We will confirm in Section 4 that this map is well defined. Next, we introduce a natural measure class on Σ_{g_i}: Assuming that $\dim(E^c) = k$, we fix (arbitrarily) a linear isomorphism between E^c and $\mathbb{R}^k$, and let m denote Lebesgue measure on E^c. The reference measure $m_{\Sigma_{g_i}}$ on Σ_{g_i} is then defined by letting $m_{\Sigma_{g_i}}(A) \triangleq m(\pi^c(A))$ for every Borel subset $A \subset \Sigma_{g_i}$. Finally, we say the foliation $\mathcal{F}^s$ is *absolutely continuous* if for every pair of transversals Σ_{g_1} and Σ_{g_2}, the holonomy map T_{g_1,g_2} carries $m_{\Sigma_{g_1}}$-zero measure sets to $m_{\Sigma_{g_2}}$-zero measure sets.

Theorem 3 *(Absolute continuity of the stable foliation). Assume* (A1)–(A4). *Then the foliation* $\mathcal{F}^s$ *is absolutely continuous. Moreover, for any admissible* $g_1, g_2 : E^c \to E^s$,

$$\frac{d(T_{g_1,g_2})_*(m_{\Sigma_{g_1}})}{dm_{\Sigma_{g_2}}} \leqslant C_2$$

where system constants aside, C_2 *depends only on* α, $\|Df\|_{C^\alpha}$, $\|Dg_{1,2}\|_{C^\alpha}$ *and* $\|g_1 - g_2\|_{C^0}$.

In finite dimensions, absolute continuity of the stable foliation has been proved a number of times, a fact which attests to the importance of this result. It was proved for Anosov diffeomorphisms in [1], and in nonuniform hyperbolic settings with or without singularities in [12,15,16,18]. Our proof of Theorem 3 follows the ideas outlined in [18]. To our knowledge this result is new in infinite dimensions, where it has the following interpretation:

Interpretation of our results. When a dynamical system defined on an infinite dimensional space X has a finite dimensional center manifold W^c which attracts all points in X, we think of its large-time dynamics as being captured by the finite dimensional system on W^c. The existence of a stable foliation says a little more, namely that it associates each initial condition $x_0 \in X$ to $y_0 \in W^s_{x_0} \cap W^c$, with the property that $\|f^k(x_0) - f^k(y_0)\| \to 0$ exponentially fast.

The absolute continuity of the stable foliation $\mathcal{F}^s$ permits us to introduce the following notion of "almost everywhere" on X: Let P be a property which either holds or does not hold at each $x \in X$.

We say P respects stable manifolds if for every x, P holds at x if and only if it holds for all $y \in W^s_x$. For such P, it makes sense to think of it as holding "almost everywhere" (a.e.) on X if it holds on a full m_Σ-measure set for some transversal Σ, for by Theorem 3 this condition is independent of Σ. Likewise, it makes sense to view P has holding on "a positive measure set" in X if it holds on a positive m_Σ-set on some Σ.

As an example, consider the following: Let φ be a uniformly continuous observable and $\bar\varphi$ a putative spatial average. Then the property that P holds at x if $\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) \to \bar\varphi$ as $n \to \infty$ clearly respects stable manifolds. It follows that if ν is an SRB measure on $f|_{W^c}$ with no zero Lyapunov exponents, and $\bar\varphi = \int \varphi d\nu$, then passing property P on W^c to all of X via the stable foliation, we have that P holds on a positive measure set (or possibly a.e.) in X in the sense above.

2. Verification of conditions for differential equations

In this section, we demonstrate that the setting in Section 1 is compatible with that for some differential equations to which our results apply. We first discuss our conditions in the framework of general C^0 semigroups, which includes ordinary differential equations and some partial differential equations, before giving concrete examples.

Specifically, we consider the differential equation

$$u_t = Au + g(u) \tag{1}$$

where X is a Banach space, $A : D(A) \to X$ is the generator of a C^0 semigroup, and $g \in C^{1,\alpha_0}(X, X)$, $\alpha_0 \in (0, 1]$. We assume:

(H1) There exist closed subspaces E^c and E^s of X such that $X = E^c \oplus E^s$ with $\|\pi^c\|, \|\pi^s\| = 1$.

(H2) The subspaces $E^{c,s}$ are invariant under A. Let $A^{c,s} = A|_{E^{c,s}}$. Then there exist a_0, $\lambda_\pm \in \mathbb{R}$ with $a_0 \geqslant 0$ and $\lambda_- < \min\{\lambda_+, 0\}$ such that
  (i) $\|e^{tA}\| \leqslant e^{a_0 t}$ and $\|e^{tA^s}\| \leqslant e^{\lambda_- t}$ $\forall t \geqslant 0$,
  (ii) $\|e^{tA^c}\| \leqslant e^{\lambda_+ t}$ $\forall t \leqslant 0$.

(H3) $\|g\|_{C^{1,\alpha_0}(X,X)} < \infty$.

In the hypotheses above, we omitted constants in front of the exponentials because they can be removed in a standard way by taking an equivalent norm of X. For the same reason we assumed $\|\pi^{c,s}\| = 1$ in this section. In practical problems, (H3) is often obtained through a cut-off which does not change the dynamics in a neighborhood of an absorbing ball.

Let $L = \|Dg\|_{C^0(X,\mathcal{L}(X))}$. Fix any $T > 0$, and let $f : X \to X$ be the time-T map of the semiflow generated by (1). Then $f \in C^{1,\alpha_0}(X, X)$ with $\|Df\|_{C^0} \leqslant e^{(a_0+L)T}$, see, for example [14]. We will show that with respect to the splitting $E^c \oplus E^s$ in (H1), the assumptions (A1)–(A4) in Section 1 are satisfied for some $\lambda_{c,s}$ and $\mu_{c,s}$ when L and $\lambda_\pm$ satisfy (2) below. (A3) can generally be verified in concrete problems, and (A4) is disposed of quickly since $\|Df\|_{C^0} \leqslant e^{(a_0+L)T}$. We now concentrate on (A1) and (A2):

Lemma 4. *Assume*

$$\frac{L}{\lambda_+ - \lambda_-} < \frac{1}{4} \quad \textit{and} \quad \lambda_- < -2L, \tag{2}$$

where $L = \|Dg\|_{C^0(X,\mathcal{L}(X))}$. Then there exist $\mu_c, \mu_s \in (0, 1)$ independent of T such that the invariant cones conditions (A1) *and* (A2) *are satisfied with*

$$\lambda_s = \big(\lambda_- + L(1+\mu_s)\big)T < 0 \quad \textit{and} \quad \lambda_c = \big(\lambda_+ - L(1+\mu_c)\big)T > \lambda_s.$$

Proof. We show the result for μ_c and λ_c; the case of μ_s and λ_s is treated similarly. Given a solution $u(t)$, $t \geqslant 0$, consider its linearized solution $u_1(t)$ and write $u_1^{c,s} = \pi^{c,s} u_1$, $g^{c,s} = \pi^{c,s} g$. Let $\beta > 1$ be a number to be chosen. For arbitrary t_1, we assume $|u_1^s(t_1)| \leqslant \mu_c |u_1^c(t_1)|$, and let $t_2 > t_1$ be such that

$$\left|u_1^s(t)\right| \leqslant \beta\mu_c \left|u_1^c(t)\right| \quad \forall t \in [t_1, t_2].$$

We will show that this in fact implies $|u_1^s(t_2)| \leqslant \mu_c |u_1^c(t_2)|$. From this we conclude

$$t_* = \sup\left\{t \geqslant t_1 \;\middle|\; \left|u_1^s(\tau)\right| \leqslant \beta\mu_c \left|u_1^c(\tau)\right| \; \forall \tau \in [t_1, t]\right\} = +\infty,$$

hence

$$\left|u_1^s(t)\right| \leqslant \mu_c \left|u_1^c(t)\right| \quad \forall t \geqslant t_1,$$

proving the first half of (A1).

For $t \in [t_1, t_2]$, we have

$$\begin{aligned}
\left|u_1^c(t)\right| &\leqslant \left|e^{(t-t_2)A^c} u_1^c(t_2)\right| + \int_t^{t_2} \left|e^{(t-\tau)A^c} Dg^c\big(u(\tau)\big) u_1(\tau)\right| d\tau \\
&\leqslant e^{\lambda_+(t-t_2)} \left|u_1^c(t_2)\right| + L(1+\beta\mu_c) \int_t^{t_2} e^{\lambda_+(t-\tau)} \left|u_1^c(\tau)\right| d\tau.
\end{aligned}$$

The Gronwall inequality yields

$$\left|u_1^c(t)\right| \leqslant e^{-(\lambda_+ - L(1+\beta\mu_c))(t_2-t)} \left|u_1^c(t_2)\right|, \quad t \in [t_1, t_2]. \tag{3}$$

Similarly (but going forward in time rather than backward) we estimate

$$\left|u_1^s(t_2)\right| \leqslant e^{\lambda_-(t_2-t_1)} \left|u_1^s(t_1)\right| + L(1+\beta\mu_c) \int_{t_1}^{t_2} e^{\lambda_-(t_2-\tau)} \left|u_1^c(\tau)\right| d\tau. \tag{4}$$

Substituting in (3), we obtain that the first term of (4) is

$$\leqslant \mu_c e^{(\lambda_- - \lambda_+ + L(1+\beta\mu_c))(t_2-t_1)} \left|u_1^c(t_2)\right|$$

and the second term is

$$\leqslant \frac{L(1+\beta\mu_c)}{\lambda_+ - \lambda_- - L(1+\beta\mu_c)} \cdot \left(1 - e^{(\lambda_- - \lambda_+ + L(1+\beta\mu_c))(t_2-t_1)}\right) \cdot \left|u_1^c(t_2)\right|.$$

Combining and letting $B = \frac{L(1+\beta\mu_c)}{\lambda_+ - \lambda_- - L(1+\beta\mu_c)}$, we obtain

$$\left|u_1^s(t_2)\right| \leqslant \left\{(\mu_c - B) e^{(\lambda_- - \lambda_+ + L(1+\beta\mu_c))(t_2-t_1)} + B\right\} \left|u_1^c(t_2)\right|,$$

which is $\leqslant \mu_c |u_1^c(t_2)|$ if we can guarantee that $0 \leqslant B \leqslant \mu_c$. With $\beta = 1$, $B \leqslant \mu_c$ is equivalent to

$$(1+\mu_c)^2 \leqslant \left(\frac{\lambda_+ - \lambda_-}{L}\right)\mu_c.$$

The assumption in the lemma guarantees that for μ_c close enough to 1, the inequality above is a strict inequality. Hence it remains valid for β slightly > 1. The second inequality in (A1) follows from (3). □

The computation above suggests that the larger $(\lambda_+ - \lambda_-)/L$, the smaller μ_c and μ_s can be, equivalently, the "narrower" the invariant cones centered at E^c and E^s. This is in agreement with intuition: $\lambda_+ - \lambda_-$ is the spectral gap, and L measures deviation from the linear map which preserves the splitting $E^c \oplus E^s$.

Remark 1. The arguments above remain valid if A is the generator of an analytic semigroup and $g : X^\beta \to X$ is smooth for some $\beta \in [0, 1)$.

Assumptions (H1)–(H3) and (2) are easily checked for ODEs. To demonstrate PDE applications, consider a reaction–diffusion equation

$$u_t = \Delta u + g(u), \quad x \in \Omega \Subset \mathbb{R}^n, \quad u|_{\partial\Omega} = 0 \tag{5}$$

or a damped Klein–Gordon equation

$$u_{tt} - \Delta u + \gamma u_t + g(u) = 0, \quad x \in \Omega \Subset \mathbb{R}^n, \quad u|_{\partial\Omega} = 0, \quad \gamma > 0, \tag{6}$$

where g is a C^{1,α_0} function. Suppose $g(0) = 0$. Let $A = \Delta + g'(0)$ in the first example and $A = \begin{pmatrix} 0 & 1 \\ -\Delta - g'(0) & -\gamma \end{pmatrix}$ in the second example. It is clear that in both cases, A has discrete spectrum, so $\dim(E^c) < \infty$. To study the local dynamics near $u = 0$, one may replace $g(u)$ by $g'(0)u + G(u)$ where $G(u) = (g(u) - g'(0)u)\eta(\frac{u}{\delta})$ and $\eta(u)$ is a cut-off function: $\eta|_{[-1,1]} = 1$ and $\eta|_{|u|>2} = 0$. Clearly, the dynamics of the modified equation is identical to that of the original equation in a δ-neighborhood of 0 in the Sobolev spaces $H_0^1 \cap H^k$ for (5) and $(H_0^1 \cap H^k) \times H^{k-1}$ for (6), with $k > 1 + \frac{n}{2}$. When $\delta \ll 1$, it is easy to see that $L \leqslant C\,\mathrm{Lip}(G) = O(\delta)$ where C depends only on n and k and all the above assumptions are satisfied.

When $n = 1$, one does not need to assume $g(0) = 0$. The eigenvalues of $A = \Delta$ of (5) are $\lambda_1 > \lambda_2 > \cdots \to -\infty$ with $|\lambda_k| = O(k^2)$. Under certain conditions such as those given in [17], (5) has an absorbing ball of radius $R > 0$ which contains the attractor of (5). Let $G(u) = g(u)\eta(\frac{u}{R})$, where η is defined in the above, and $L = C\,\mathrm{Lip}(G)$. Then there exists $m \geqslant 1$ such that $\lambda_+ = \lambda_m$ and $\lambda_- = \lambda_{m+1}$ satisfy (2). Hence the theorems of this paper apply.

3. Proofs of Theorems 1 and 2

The setting is as in Section 1, and (A1), (A2), (A3′) and (A4) are assumed throughout. In Sections 3.1–3.3, we prove Theorems 1 and 2 under the additional assumption

(A5) $\frac{e^{\lambda_s - \lambda_c^-}}{1-\mu_c\mu_s} < 1$.

This provisional assumption is used only to show that graph transforms are contractions, and is removed in Section 3.4.

While the basic outline of the proofs follow standard graph transform ideas, a number of technical issues arise as a result of (i) the infinite dimensionality, (ii) the global nature of our results, and (iii) the fact that our invariant cones conditions are assumed only for Df. For example, passing (A2) to the nonlinear map f is not at all immediate as we will see in Section 3.2.

Notation: In the proofs to follow, "K" will be used as a generic constant, i.e., it is used multiple times with different meaning in different contexts. With no exception, however, it is allowed to depend only on system constants and on the value of α in question.

3.1. Existence of a Lipschitz center manifold W^c

The aim of this subsection is to prove the existence of a center manifold W^c that is the graph of a function h^c with $\mathrm{Lip}(h^c) \leqslant \mu_c$. Let

$$\mathcal{W}^c = \{h \in C^0(E^c, E^s) \mid \|h\|_{C^0} < \infty,\ \mathrm{Lip}(h) \leqslant \mu_c\}.$$

We seek to define $\Gamma : \mathcal{W}^c \to \mathcal{W}^c$ such that if $\tilde{h} = \Gamma(h)$, then

$$\mathrm{graph}(\tilde{h}) = f(\mathrm{graph}(h)).$$

For $h \in \mathcal{W}^c$, we use the notation

$$H = I + h : E^c \to X \quad \text{and} \quad F_h \triangleq \pi^c f H : E^c \to E^c. \tag{7}$$

Lemma 5.

(a) F_h *is a homeomorphism satisfying* $\mathrm{Lip}(F_h^{-1}) \leqslant e^{-\lambda_c}$.
(b) *For* $h_{1,2} \in \mathcal{W}^c$, $|F_{h_2}^{-1} - F_{h_1}^{-1}|_{C^0} \leqslant M e^{a-\lambda_c} |h_2 - h_1|_{C^0}$.

We postpone the proof of this lemma so as not to disrupt the flow of the main argument. Continuing, we let

$$\tilde{h} = \pi^s f H F_h^{-1}. \tag{8}$$

Then clearly $\mathrm{graph}(\tilde{h}) = f(\mathrm{graph}(h))$. We check next that $\tilde{h} \in \mathcal{W}^c$: (A4) together with $\|h\|_{C^0} < \infty$ implies $\|\tilde{h}\|_{C^0} < \infty$. To see the Lipschitz property of $\tilde{h}$, consider any $\tilde{x}^c_{1,2} = F_h(x^c_{1,2})$ with $|\tilde{x}^c_2 - \tilde{x}^c_1| \ll 1$. Lemma 5(a) tells us that $|x^c_2 - x^c_1| \leqslant e^{-\lambda_c} |\tilde{x}^c_2 - \tilde{x}^c_1|$, permitting us to write

$$\begin{aligned} |\tilde{x}^c_2 - \tilde{x}^c_1| &= |\pi^c(f(H(x^c_2)) - f(H(x^c_1)))| \\ &\geqslant |\pi^c (Df)_{H(x^c_1)}(H(x^c_2) - H(x^c_1))| - o(|x^c_2 - x^c_1|). \end{aligned}$$

Similarly

$$\begin{aligned} |\tilde{h}(\tilde{x}^c_2) - \tilde{h}(\tilde{x}^c_1)| &\leqslant |\pi^s (Df)_{H(x^c_1)}(H(x^c_2) - H(x^c_1))| + o(|x^c_2 - x^c_1|) \\ &\leqslant \mu_c |\pi^c (Df)_{H(x^c_1)}(H(x^c_2) - H(x^c_1))| + o(|x^c_2 - x^c_1|). \end{aligned}$$

Due to the definition of $\mathcal{W}^c$ and the fact that $\|Df\|_{C^{\alpha_0}} < \infty$, the $o(\cdot)$ above is uniform in $x^c_{1,2}$ and $h \in \mathcal{W}^c$. These inequalities together imply that on any ball of radius δ, $\tilde{h}$ has local Lipschitz constant $\mu_c + o(\delta)$, which yields the same global Lipschitz constant. Taking $\delta \to 0$ we obtain $\mathrm{Lip}(\tilde{h}) \leqslant \mu_c$. Thus $\tilde{h} \in \mathcal{W}^c$.

We record the following analog of (A1) for f (as opposed to Df) that will be useful later:

Lemma 6. *Given $x_{1,2} \in X$ with $|\pi^s(x_2 - x_1)| \leqslant \mu_c|\pi^c(x_2 - x_1)|$, let $\tilde{x}_{1,2} = f(x_{1,2})$. Then*

$$\left|\pi^s(\tilde{x}_2 - \tilde{x}_1)\right| \leqslant \mu_c\left|\pi^c(\tilde{x}_2 - \tilde{x}_1)\right| \quad \textit{and} \quad \left|\pi^c(x_2 - x_1)\right| \leqslant e^{-\lambda_c}\left|\pi^c(\tilde{x}_2 - \tilde{x}_1)\right|.$$

To prove the lemma, one notes that, by the Hahn–Banach Theorem, there exists $\tilde{L} \in \mathcal{L}(E^c, E^s)$ such that $\tilde{L}(\pi^c(x_2 - x_1)) = \pi^s(x_2 - x_1)$ and $|\tilde{L}| \leqslant \mu^c$. Let $h \in \mathcal{W}^c$ be such that $\operatorname{graph}(h) = x_1 + (I + \tilde{L})(E^c)$ (with cut-off at infinity). The lemma follows from the estimates above.

Returning to the main proof, note that with respect to the C^0 metric, $\mathcal{W}^c$ is a complete metric space. We will show that Γ is a contraction mapping. Our center manifold W^c is then given by $W^c = \operatorname{graph}(\hat{h})$ where $\hat{h}$ is the unique fixed point of Γ.

Specifically, we will show

$$\|\tilde{h}_2 - \tilde{h}_1\|_{C_0} \leqslant \frac{e^{\lambda_s}}{1 - \mu_c\mu_s}\|h_2 - h_1\|_{C^0} \tag{9}$$

for all $h_{1,2} \in \mathcal{W}^c$. By (A5), this Lipschitz constant is < 1. It suffices to consider $h_{1,2}$ where $\|h_1 - h_2\|_{C^0}$ is arbitrarily small, for by letting $h^{(i)} = h_1 + \frac{i}{N}(h_2 - h_1)$, $i = 0, \dots, N$, for arbitrarily large N and comparing $h^{(i)}$ and $h^{(i+1)}$ (which are in $\mathcal{W}^c$), we obtain the desired result for arbitrary $h_{1,2}$. To derive a contradiction, then, we assume there exists δ_0 such that for arbitrarily small $\epsilon > 0$, there exist $h_{1,2}$ and $x^c \in E^c$ such that $\|h_2 - h_1\|_{C^0} \leqslant \epsilon$ and

$$\left|\tilde{h}_2(x^c) - \tilde{h}_1(x^c)\right| > \left(\delta_0 + \frac{e^{\lambda_s}}{1 - \mu_c\mu_s}\right)\|h_2 - h_1\|_{C^0}. \tag{10}$$

Let $x^c_{1,2}$ be such that $f(H_{1,2}(x^c_{1,2})) = \tilde{H}_{1,2}(x^c)$; from the discussion above we know $x^c_{1,2}$ exist and are unique.

Our first step is to show $H_2(x^c_2) - H_1(x^c_1)$ lies in the stable cone. Let $\tilde{x}^c = \pi^c f(H_2(x^c_1))$. Then

$$\left|\tilde{H}_2(\tilde{x}^c) - \tilde{H}_1(x^c)\right| = \left|f\left(H_2(x^c_1)\right) - f\left(H_1(x^c_1)\right)\right| \leqslant e^a\|h_2 - h_1\|_{C^0}$$

which, after applying π^c, implies

$$\left|\tilde{x}^c - x^c\right| \leqslant Me^a\|h_2 - h_1\|_{C^0}.$$

This along with Lemma 5 and (A1) applied to the points $H_2(x^c_1)$ and $H_2(x^c_2)$ gives

$$\left|x^c_2 - x^c_1\right| \leqslant e^{-\lambda_c}\left|\tilde{x}^c - x^c\right| \leqslant Me^{a-\lambda_c}\|h_2 - h_1\|_{C^0}. \tag{11}$$

By Taylor's expansion,

$$\begin{aligned}
\tilde{h}_2(x^c) - \tilde{h}_1(x^c) &= f\left(H_2(x^c_2)\right) - f\left(H_1(x^c_1)\right) \\
&= (Df)_{H_1(x^c_1)}\left(H_2(x^c_2) - H_1(x^c_1)\right) + o\left(\|h_2 - h_1\|_{C_0}\right).
\end{aligned}$$

We see that $o(\cdot)$ is uniform in h_1, h_2 and x^c due to the C^{α_0} bound on Df and (11). Observe that $\tilde{h}_2(x^c) - \tilde{h}_1(x^c) \in E^s$ and has norm $\geqslant \text{const} \cdot \|h_2 - h_1\|_{C^0}$ by (10). For $o(\cdot)$ small enough, we have that $(Df)_{H_1(x^c_1)}(H_2(x^c_2) - H_1(x^c_1))$ is in the stable cone. That together with (A2) puts $H_2(x^c_2) - H_1(x^c_1)$ in the stable cone.

Thus we have, by (A2),

$$\left|x_2^c - x_1^c\right| \leqslant \mu_s \left|h_2\left(x_2^c\right) - h_1\left(x_1^c\right)\right| \leqslant \mu_s \|h_2 - h_1\|_{C^0} + \mu_c \mu_s \left|x_2^c - x_1^c\right|, \tag{12}$$

which yields

$$\left|x_2^c - x_1^c\right| \leqslant \frac{\mu_s}{1 - \mu_c \mu_s} \|h_2 - h_1\|_{C^0}. \tag{13}$$

Applying again (A2) followed by the second half of (12) and then (13), we obtain

$$\begin{aligned} \left|\tilde{h}_2\left(x^c\right) - \tilde{h}_1\left(x^c\right)\right| &\leqslant e^{\lambda_s} \left|h_2\left(x_2^c\right) - h_1\left(x_1^c\right)\right| + o\left(\|h_2 - h_1\|_{C_0}\right) \\ &\leqslant e^{\lambda_s} \left(\|h_2 - h_1\|_{C^0} + \frac{\mu_c \mu_s}{1 - \mu_c \mu_s} \|h_2 - h_1\|_{C^0}\right) + o\left(\|h_2 - h_1\|_{C_0}\right) \\ &= \left(o(1) + \frac{e^{\lambda_s}}{1 - \mu_c \mu_s}\right) \|h_2 - h_1\|_{C^0}, \end{aligned}$$

contradicting (10) and completing the proof. □

Proof of Lemma 5. (a) Let $h \in \mathcal{W}^c$ be fixed. For $x_0^c \in E^c$, it is an easy exercise to show that

$$\tilde{F}\left(x^c\right) \triangleq \pi^c f H\left(x_0^c\right) + \pi^c (Df)_{H(x_0^c)} \left(H\left(x^c\right) - H\left(x_0^c\right)\right)$$

defines a homeomorphism from E^c to itself with

$$\mathrm{Lip}\left(\tilde{F}^{-1}\right) \leqslant e^{-\lambda_c}.$$

Since $\tilde{F}(x^c)$ is the principal part of the Taylor expansion of $F_h(x^c)$ for x^c close to x_0^c, using our assumption that $\|Df\|_{C^{0,\alpha}} < \infty$, we obtain the following uniform estimate: given $\delta > 0$, there exists $\epsilon > 0$ such that for every $x_0^c \in E^c$, F_h maps $\{|x^c - x_0^c| < \epsilon\}$ homeomorphically onto its image, on which $\mathrm{Lip}(F_h^{-1}) \leqslant (1 + \delta) e^{-\lambda_c}$.

To prove that F_h is in fact a global homeomorphism, it remains to prove that F_h maps E^c bijectively onto E^c, which follows from a topological argument. From the fact that F_h is a local homeomorphism the inverse of which has a uniform Lipschitz bound, we deduce that $F_h(E^c)$ is both open and closed; hence $F_h(E^c) = E^c$, and it remains to prove that the map is one to one.

We assume $F_h(x_1) = F_h(x_2) \triangleq x_0$, and show that $x_1 = x_2$. Let $L = \{t x_1 + (1 - t) x_2 \mid t \in [0, 1]\}$. Since E^c is simply connected, there exists a homotopy (continuous) map $\Psi : [0, 1] \times L \to E^c$ such that

(1) $\Psi(0, \cdot) = F_h|_L$;
(2) $\Psi(t, x_1) = \Psi(t, x_2) = x_0$, $t \in [0, 1]$;
(3) $\Psi(1, x) = x_0$, $x \in L$.

We claim there is a continuous map $G : [0, 1] \times L \to E^c$ such that

(a) $G|_{\{0\} \times L} = id$;
(b) $\Psi = F_h \circ G$.

This is true because F_h is locally invertible, and we can use $G = F_h^{-1} \circ \Psi$ to continue G from $\{0\} \times L$ to $[0, \epsilon] \times L$, then to $[\epsilon, 2\epsilon] \times L$, and so on until its domain includes all of $[0, 1] \times L$. The invertibility of F_h in neighborhoods of x_1 and x_2 is crucial in the next argument. That together with properties of Ψ and the continuity of G forces $G \equiv x_1$ on $(\{x_1\} \times [0, 1]) \cup (L \times \{1\})$, and $G \equiv x_2$ on $(\{x_2\} \times [0, 1]) \cup (L \times \{1\})$, implying $x_1 = x_2$. Therefore F_h is a homeomorphism. Finally, noting that the local Lipschitz constant of $(1 + \delta)e^{-\lambda_c}$ for F_h^{-1} above is in fact a global Lipschitz constant, and we finish the proof of (a) by letting $\delta \to 0$.

(b) For any $h_{1,2} \in \mathcal{W}^c$, we have from (7) that

$$|F_{h_2} - F_{h_1}|_{C^0} \leqslant Me^a |h_2 - h_1|_{C^0}.$$

Now fix $x^c \in E^c$, and let $x^c_{1,2} = F^{-1}_{h_{1,2}}(x^c)$. Then

$$\left|x_2^c - x_1^c\right| \leqslant e^{-\lambda_c}\left|F_{h_2}\left(x_2^c\right) - F_{h_2}\left(x_1^c\right)\right| = e^{-\lambda_c}\left|F_{h_1}\left(x_1^c\right) - F_{h_2}\left(x_1^c\right)\right|$$

which is estimated above. □

3.2. The stable foliation

In this subsection we present the proof of Theorem 2 under assumption (A5). For clarify, we divide the proof into Parts A, B and C.

A. *Invariant stable subspaces.* We prove here the existence of a Df-invariant bundle of stable subspaces defined everywhere on X. Recall that $E^{c,s}$ are not Df-invariant.

Lemma 7. *There exist $\alpha > 0$ depending only on system constants, $K \geqslant 1$, and mappings $\hat{L} : X \to \mathcal{L}(E^s, E^c)$ and $\hat{F} : X \to \mathcal{L}(E^s, E^s)$ such that if we write $\hat{L}_x = \hat{L}(x)$, then:*

(1) $Df_x(\mathrm{graph}(\hat{L}_x)) \subset \mathrm{graph}(\hat{L}_{fx})$ $\forall x$;
(2) $(Df)_x(I + \hat{L}_x) = (I + \hat{L}_{fx})\hat{F}_x$ $\forall x$;
(3) $\|\hat{L}\|_{C^0} \leqslant \mu^s$, $\|\hat{L}\|_{C^\alpha} \leqslant K\|Df\|_{C^\alpha}$, *and* $\|\hat{F}\|_{C^0} \leqslant e^{\lambda_s}$, $\|\hat{F}\|_{C^\alpha} \leqslant K(1 + \|Df\|_{C^0})\|Df\|_{C^\alpha}$.

Viewing $X^s_x \triangleq \mathrm{graph}(\hat{L}_x) = (I + \hat{L}_x)E^s$ as a subspace of the tangent space at x, (1) above says that $\{X^s_x,\ x \in X\}$ is a Df-invariant stable subbundle of the tangent bundle over X.

Proof of Lemma 7 assuming (A5). Let

$$\Sigma = \left\{L \in C^0\left(X, \mathcal{L}\left(E^s, E^c\right)\right) \,\middle|\, \|L\|_{C^0} \leqslant \mu_s\right\}.$$

Our first task is to define a graph transform Γ from Σ to itself such that if $\tilde{L} = \Gamma(L)$, then for each $x \in X$, $(Df)_x(\mathrm{graph}(\tilde{L}_x)) \subset \mathrm{graph}(L_{fx})$. This relation is equivalent to

$$\pi^c(Df)_x\big|_{E^s} + \pi^c(Df)_x\big|_{E^c}\tilde{L}_x = L_{fx}\pi^s(Df)_x\big|_{E^s} + L_{fx}\pi^s(Df)_x\big|_{E^c}\tilde{L}_x,$$

from which we deduce

$$\left(\pi^c - L_{fx}\pi^s\right)(Df)_x\big|_{E^c}\tilde{L}_x = \left(L_{fx}\pi^s - \pi^c\right)(Df)_x\big|_{E^s}. \tag{14}$$

We claim the operator $(\pi^c - L_{fx}\pi^s)(Df)_x|_{E^c} : E^c \to E^c$ is an isomorphism satisfying

$$\left\| \left(\left(\pi^c - L_{f(x)}\pi^s\right)(Df)_x\big|_{E^c} \right)^{-1} \right\| \leqslant \frac{e^{-\lambda_c}}{1-\mu_s\mu_c}. \tag{15}$$

To see that, consider the family of operators

$$P(\tau) \triangleq \left(\pi^c - \tau L_{fx}\pi^s\right)(Df)_x\big|_{E^c} : E^c \to E^c, \quad \tau \in [0,1].$$

Injectivity of $P(\tau)$ is assured by (A1) and the definition of Σ:

$$\left|\left(\pi^c - \tau L_{fx}\pi^s\right)(Df)_x z\right| \geqslant e^{\lambda_c}(1-\tau\mu_c\mu_s)|z| \geqslant e^{\lambda_c}(1-\mu_c\mu_s)|z|. \tag{16}$$

By (A3′), $P(0)$ is invertible. Now for any $\tau_0 \in [0,1]$, if $P(\tau_0)$ is invertible, then $\|(P(\tau_0))^{-1}\|$ is given by (16), and for $\tau \in (\tau_0-\delta, \tau_0+\delta)$ with $\delta = \frac{1}{M\mu_s}e^{\lambda_c - a}(1-\mu_c\mu_s)$, $\|P(\tau)-P(\tau_0)\| < \|(P(\tau_0))^{-1}\|^{-1}$ so $P(\tau)$ is invertible as well. It follows that $P(1)$ is invertible and thus (14) implies

$$\tilde{L}_x \triangleq \left(\left(\pi^c - L_{fx}\pi^s\right)(Df)_x\big|_{E^c} \right)^{-1} \left(L_{fx}\pi^s - \pi^c\right)(Df)_x\big|_{E^s}.$$

This together with $\|\tilde{L}\|_{C^0} \leqslant \mu_s$ (by (A2)) guarantees that $\Gamma(\Sigma) \subset \Sigma$ since the continuity of f, Df, and L in x implies the continuity of $\tilde{L}$ in x. Given $L^{1,2} \in \Sigma$, a straightforward computation using (14) yields

$$\tilde{L}_x^2 - \tilde{L}_x^1 = \left(\left(\pi^c - L_{fx}^2\pi^s\right)(Df)_x\big|_{E^c} \right)^{-1} \left(L_{fx}^2 - L_{fx}^1\right)\pi^s(Df)_x\left(I+\tilde{L}_x^1\right).$$

The bound in (15) together with (A2) and (A5) imply

$$\mathrm{Lip}(\Gamma) \leqslant \frac{e^{\lambda_s-\lambda_c}}{1-\mu_c\mu_s} < 1,$$

giving, by the contraction mapping theorem, the existence of a unique fixed point $\hat{L}$ of Γ with the desired estimate on its C^0 norm. This is the $\hat{L}$ in the theorem.

Associated with each $L \in \Sigma$, there is an associated mapping F where F_x maps E^s in the tangent space of x to E^s in the tangent space of fx. This mapping is defined by

$$F_x = \pi^s(Df)_x(I+\tilde{L}_x),$$

and satisfies the equation

$$(Df)_x(I+\tilde{L}_x) = (I+L_{fx})F_x.$$

The asserted C^0 norm of F follows immediately from (A2).

To obtain the Hölder estimates, for any $L \in \Sigma$ and $x_{1,2} \in X$, another straightforward computation using (14) gives

$$\begin{aligned}\tilde{L}_{x_2} - \tilde{L}_{x_1} = {} & \left(\left(\pi^c - L_{fx_2}\pi^s\right)(Df)_{x_2}\big|_{E^c} \right)^{-1} \big((L_{fx_2} - L_{fx_1})\pi^s(Df)_{x_1} \\ & + \left(L_{fx_2}\pi^s - \pi^c\right)\left((Df)_{x_2} - (Df)_{x_1}\right)\big)(I+\tilde{L}_{x_1}).\end{aligned}$$

Assumptions (A1)–(A4) along with (15) imply

$$\|\tilde{L}\|_{C^\alpha} \leqslant \frac{e^{\lambda_s-\lambda_c+\alpha a}}{1-\mu_c\mu_s}\|L\|_{C^\alpha} + \frac{Me^{-\lambda_c}(1+\mu_s)^2}{1-\mu_c\mu_s}\|Df\|_{C^\alpha}.$$

Choosing $0<\alpha\leqslant\alpha_0$ so that $\frac{e^{\lambda_s-\lambda_c+\alpha a}}{1-\mu_c\mu_s}<1$, we have that the iteration sequence $\Gamma^n L$ for $L\in C^\alpha$ is bounded in C^α and thus the limit fixed point of Γ has the same C^α bound. The C^α estimate on F follows from its representation above in terms of L. □

Lemma 7 provides another splitting, an alternative to $X=E^c\oplus E^s$, namely

$$X=E^c\oplus X_x^s, \quad \text{where } X_x^s=\big((I+\hat{L}_x)E^s\big),\ x\in X.$$

Let $P_x^c\in\mathcal{L}(X,E^c)$ and $P_x^s\in\mathcal{L}(X,X^s)$ denote the associated projections, i.e.

$$P_x^s=(I+\hat{L}_x)\pi^s, \qquad P_x^c=I-P_x^s=\pi^c-\hat{L}_x\pi^s. \tag{17}$$

This splitting will be useful later.

B. *Invariant stable foliation with Lipschitz leaves.* Let

$$\mathcal{W}^s=\big\{\phi\in C^0\big(X\times E^s,E^c\big)\ \big|\ \phi_x(0)=0,\ \mathrm{Lip}(\phi_x)\leqslant\mu_s\big\},$$

where $\phi_x\triangleq\phi(x,\cdot)\in C^0(E^s,E^c)$. Define the norm $\|\cdot\|_s$ on $\mathcal{W}^s$ as

$$\|\phi\|_s\triangleq\sup_{x\in X}\big\|\phi(x,\cdot)\big\|_s \quad \text{where } \big\|\phi(x,\cdot)\big\|_s\triangleq\sup_{x^s\in E^s}\frac{|\phi(x,x^s)|}{|x^s|}.$$

For $\phi\in\mathcal{W}^s$, we use the notation

$$\Phi_x(\cdot)\triangleq x+\cdot+\phi_x(\cdot)\in C^0\big(E^s,X\big),$$

so that $\Phi_x(E^s)=\mathrm{graph}(\phi_x)+x$.

Before proceeding further we record the following simple lemma that will be used a number of times:

Lemma 8. *For any $h\in\mathcal{W}^c$, $\phi\in\mathcal{W}^s$, and $y\in X$, let $g:E^s\to E^c$ be defined by*

$$g\big(x^s\big)=\pi^c y+\phi\big(y,x^s-\pi^s y\big).$$

Then $\mathrm{graph}(h)$ *meets* $\mathrm{graph}(g)$ *in a unique point $\hat{x}$. Moreover*:

(a) *With ϕ fixed, $\hat{x}$ varies continuously with y and is Lipschitz in h, satisfying, for fixed y,*

$$\big|\hat{x}_2^c-\hat{x}_1^c\big|\leqslant\frac{\mu_s}{1-\mu_c\mu_s}|h_2-h_1|_{C^0}.$$

Here $\hat{x}_i$ is the point of intersection for $\mathrm{graph}(h_i)$ *and $\hat{x}_i^c=\pi^c\hat{x}_i$.*

(b) *If g is C^1, and $h_\tau\in\mathcal{W}^c$ is a 1-parameter family such that $h_\tau(x^c)$ is C^1 in both x^c and τ, then $\hat{x}$ is also C^1 in τ.*

This lemma follows from the observation that given h, ϕ and y, $\hat{x}^c$ is the fixed point of the mapping $x^c \mapsto g(h(x^c))$. Since this mapping has Lipschitz constant $\mu_c\mu_s < 1$, the assertions follow immediately from the contraction mapping theorem.

We now define a graph transform Γ^s on $\mathcal{W}^s$ with the property that if $\tilde{\phi} = \Gamma^s(\phi)$ and $\tilde{\Phi}$ has the obvious meaning, then

$$f\big(\tilde{\Phi}_x\big(E^s\big)\big) \subset \Phi_{fx}\big(E^s\big).$$

Fix $x \in X$ and $\tilde{x}^s \in E^s$, we define $\tilde{\phi}_x(\tilde{x}^s)$ as follows: Let $h \in \mathcal{W}^c$ be the constant function $h \equiv \pi^s x + \tilde{x}^s$. From Section 3.1, $f(\text{graph}(h))$ is the graph of a function $\tilde{h} \in \mathcal{W}^c$. By Lemma 8, graph$(\tilde{h})$ and $\Phi_{fx}(E^s)$ meet in a unique point $\hat{x}$. The graph transform property forces us to take $\tilde{\phi}_x(\tilde{x}^s) = F_h^{-1}(\pi^c\hat{x}) - \pi^c x$.

Lemma 5(b) and Lemma 8 imply that $\tilde{\phi}$ is continuous in both x and $\tilde{x}^s$ and is Lipschitz in $\tilde{x}^s$; this Lipschitz constant can be written explicitly in terms of M, λ_c, a, and $\mu_{c,s}$, uniformly in x and ϕ. However, these lemmas do not guarantee $\text{Lip}(\tilde{\phi}_x) \leqslant \mu_s$, which we now prove.

Lemma 9. *Let $\varphi, \tilde{\varphi} : E^s \to E^c$ be Lipschitz. Assume* $\text{Lip}(\varphi) \leqslant \mu_s$ *and* $f(\text{graph}(\tilde{\varphi})) \subset \text{graph}(\varphi)$. *Then* $\text{Lip}(\tilde{\varphi}) \leqslant \mu_s$.

Proof. This lemma hinges on the following geometric facts about the linear maps Df_x:

(a) Given $\epsilon_0 > 0$, there exists $\delta_0 > 0$ such that if $|v^c| \geqslant (\mu_s + \epsilon_0)|v^s|$, then $|Df_x(v)| \geqslant \delta_0|v|$.
(b) Given $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that if $|\pi^c Df_x v| \leqslant (\mu_s + \delta_1)|\pi^s Df_x v|$, then $|v^c| \leqslant (\mu_s + \epsilon_1)|v^s|$.

We first prove the lemma assuming (a) and (b). Suppose $\text{Lip}(\tilde{\varphi}) \geqslant \mu_s + c$ for some $c > 0$. Then there exist $x_1, x_2 \in \text{graph}(\tilde{\varphi})$ with $|x_2 - x_1|$ arbitrarily small such that $|\pi^c(x_2 - x_1)| \geqslant (\mu_s + c/2)|\pi^s(x_2 - x_1)|$. (a) ensures that for such x_1, x_2,

$$f(x_2) - f(x_1) = Df_{x_1}(x_2 - x_1) + \text{error}$$

where $|\text{error}|/|Df_{x_1}(x_2 - x_1)| \to 0$ as $|x_2 - x_1| \to 0$. This together with $f(x_{1,2}) \in \text{graph}(\varphi)$ and $\text{Lip}(\varphi) \leqslant \mu_s$ tells us that by taking $|x_2 - x_1| \to 0$, $Df_{x_1}(x_2 - x_1)$ is arbitrarily close to the stable cone in the sense of (b). We then conclude, by (b), that the same is true for $x_2 - x_1$, contradicting our initial assumption.

The proof of (a) is easy and left as an exercise (hint: use $P_x^s + P_x^c$).

Proof of (b): For any v, let $w^c = w^c(v) \in E^c$ be such that $\pi^c Df_x w^c = \pi^c Df_x v$; w^c exists since $\pi^c Df_x|_{E^c}$ is a bijection. It suffices to prove

(∗) given any $\epsilon > 0$, $\exists \delta_1 > 0$ s.t. if $|\pi^c Df_x v| \leqslant (\mu_s + \delta_1)|\pi^s Df_x v|$, then $v - \epsilon w^c$ lies in the stable cone.

This is because

$$\big|w^c\big| \leqslant e^{-\lambda_c}\big|\pi^c Df_x w^c\big| = e^{-\lambda_c}\big|\pi^c Df_x v\big| \leqslant Me^{a-\lambda_c}|v|.$$

Combined with (∗), this gives

$$\big|v^c\big| \leqslant \big|v^c - \epsilon w^c\big| + \epsilon\big|w^c\big| \leqslant \mu_s\big|v^s\big| + \epsilon Me^{a-\lambda_c}\big(\big|v^c\big| + \big|v^s\big|\big)$$

which implies (b) if ϵ is small enough.

To prove $(*)$, it suffices to show, by (A2), that

$$\left|\pi^c\left(Df_x v - \epsilon Df_x w^c\right)\right| \leqslant \mu_s \left|\pi^s\left(Df_x v - \epsilon Df_x w^c\right)\right|. \tag{18}$$

The two sides are estimated by

$$\begin{aligned}
&\left|\pi^c\left(Df_x v - \epsilon Df_x w^c\right)\right| \leqslant (1-\epsilon)\left|\pi^c Df_x v\right| \leqslant (1-\epsilon)(\mu_s + \delta_1)\left|\pi^s Df_x v\right|;\\
&\left|\pi^s\left(Df_x v - \epsilon Df_x w^c\right)\right| \geqslant \left|\pi^s Df_x v\right| - \mu_c \epsilon \left|\pi^c Df_x v\right| \geqslant \left|\pi^s Df_x v\right|\left(1 - \mu_c \epsilon(\delta_1 + \mu_s)\right).
\end{aligned}$$

To obtain (18), one requires that

$$(1-\epsilon)(\mu_s + \delta_1) \leqslant \mu_s\left(1 - \mu_c \epsilon(\delta_1 + \mu_s)\right),$$

which is true for $\delta_1 < \frac{\mu_s \epsilon (1-\mu_c \mu_s)}{1-(1-\mu_c\mu_s)\epsilon}$. $\square$

This completes the proof that $\Gamma^s(\mathcal{W}^s) \subset \mathcal{W}^s$. The last lemma says that f (as opposed to Df) satisfies the invariant cones property in the stable direction. In the same spirit, we now prove the contractive property for f in the stable direction.

Lemma 10. *For $\tilde{x}_{1,2} \in X$, let $x_{1,2} = f(\tilde{x}_{1,2})$. Assume these 4 points satisfy*

$$\left|\pi^c(x_2 - x_1)\right| \leqslant \mu_s \left|\pi^s(x_2 - x_1)\right| \quad \text{and} \quad \left|\pi^c(\tilde{x}_2 - \tilde{x}_1)\right| \leqslant \mu_s \left|\pi^s(\tilde{x}_2 - \tilde{x}_1)\right|.$$

Then

$$\left|\pi^s(x_2 - x_1)\right| \leqslant e^{\lambda_s}\left|\pi^s(\tilde{x}_2 - \tilde{x}_1)\right|.$$

Proof. By the Hahn–Banach Theorem, there exists $T \in \mathcal{L}(E^s, E^c)$ such that $|T| \leqslant \mu_s$ and $T\pi^s(x_2 - x_1) = \pi^c(x_2 - x_1)$. Let $\phi \in \mathcal{W}^s$ be such that $\phi(y, x^s) = Tx^s$ for all y, and let $\tilde{\phi} = \Gamma^s(\phi)$. From the uniqueness given by Lemma 8, we have $\tilde{\Phi}_{\tilde{x}_1}(\pi^s(\tilde{x}_2 - \tilde{x}_1)) = \tilde{x}_2$. Taking $\tilde{x}^s$ in the construction of Γ^s to be $\tau\pi^s(\tilde{x}_2 - \tilde{x}_1)$, $\tau \in [0,1]$, it follows from Lemmas 8 and 9(b), together with the fact that F_h^{-1} is smooth if h is smooth, that $\tilde{\phi}_{\tilde{x}_1}$ is C^1 in x^s with $|D\tilde{\phi}_{\tilde{x}_1}| \leqslant \mu_s$. Therefore (A2) implies

$$\begin{aligned}
\left|\pi^s(x_2 - x_1)\right| &\leqslant \int_0^1 \left|\pi^s(Df)_{\tilde{\Phi}_{\tilde{x}_1}(\tau\pi^s(\tilde{x}_2 - \tilde{x}_1))}(D\tilde{\Phi}_{\tilde{x}_1})_{\tau\pi^s(\tilde{x}_2 - \tilde{x}_1)}\pi^s(\tilde{x}_2 - \tilde{x}_1)\right| d\tau\\
&\leqslant e^{\lambda_s}\left|\pi^s(\tilde{x}_2 - \tilde{x}_1)\right|.
\end{aligned}$$

Here we have used the fact that $(Df)(D\tilde{\Phi})\pi^s(\tilde{x}_2 - \tilde{x}_1)$ belongs to the stable cone, which follows from $f(\tilde{\Phi}_{\tilde{x}_1}(E^s)) = \Phi_{x_1}(E^s)$. $\square$

Returning to the main argument, we prove next that Γ^s is a contraction with respect to the norm $\|\cdot\|_s$, with contraction constant $(e^{\lambda_s - \lambda_c})/(1 - \mu_s\mu_c)$, which is < 1 by (A5). The unique fixed point $\hat{\phi} \in \mathcal{W}^s$ of Γ^s is our candidate for invariant stable foliation.

Let $\phi_1, \phi_2 \in \mathcal{W}^s$, $x \in X$ and $\tilde{x}^s \in E^s$. We need to show

$$\frac{|\tilde{x}_2^c - \tilde{x}_1^c|}{|\tilde{x}^s|} \leqslant \frac{e^{\lambda_s - \lambda_c}}{1 - \mu_s\mu_c} \cdot \left\|(\phi_2)_{fx} - (\phi_1)_{fx}\right\|_s \tag{19}$$

where $\tilde{x}^c_{1,2} = (\tilde{\phi}_{1,2})_x(\tilde{x}^s)$. Let $x^s_{1,2}$ be given by $f((\tilde{\Phi}_{1,2})_x(\tilde{x}^s)) = (\Phi_{1,2})_{fx}(x^s_{1,2})$. Applying Lemma 6, we get

$$\left|\tilde{x}^c_2 - \tilde{x}^c_1\right| \leqslant e^{-\lambda_c}\left|(\phi_2)_{fx}\left(x^s_2\right) - (\phi_1)_{fx}\left(x^s_1\right)\right| \tag{20}$$

and

$$\begin{aligned}\left|x^s_2 - x^s_1\right| &\leqslant \mu_c\left|(\phi_2)_{fx}\left(x^s_2\right) - (\phi_1)_{fx}\left(x^s_1\right)\right| \\ &\leqslant \mu_c\left|(\phi_2)_{fx}\left(x^s_1\right) - (\phi_1)_{fx}\left(x^s_1\right)\right| + \mu_c\mu_s\left|x^s_2 - x^s_1\right|.\end{aligned}$$

These inequalities yield

$$\left|x^s_2 - x^s_1\right| \leqslant \frac{\mu_c}{1-\mu_c\mu_s}\left|(\phi_2)_{fx}\left(x^s_1\right) - (\phi_1)_{fx}\left(x^s_1\right)\right|.$$

Plugging all this back into (20), and using Lemma 10, we obtain

$$\frac{|\tilde{x}^c_2 - \tilde{x}^c_1|}{|\tilde{x}^s|} \leqslant \frac{e^{\lambda_s - \lambda_c}}{1-\mu_c\mu_s} \cdot \frac{|\phi_{2fx}(x^s_1) - \phi_{1fx}(x^s_1)|}{|x^s_1|} \tag{21}$$

proving (19).

We finish this subsection by checking that $\hat{\phi}$, the fixed point of Γ^s, has the following properties: Let $\{\hat{\Phi}_x\}$ be the associated maps. Then

(i) each $\hat{\Phi}_x(E^s)$ is a stable manifold, i.e. for all $x \in X$ and $\tilde{x}^s_{1,2} \in E^s$, $|\pi^s f^n(\Phi_x(\tilde{x}^s_2)) - \pi^s f^n(\Phi_x(\tilde{x}^s_1))| \leqslant e^{n\lambda_s}|\tilde{x}^s_2 - \tilde{x}^s_1|$; and
(ii) $\hat{\phi}$ defines a foliation, i.e. for all $x, y \in X$, either $\hat{\Phi}_x(E^s) = \hat{\Phi}_y(E^s)$ or $\hat{\Phi}_x(E^s) \cap \hat{\Phi}_y(E^s) = \emptyset$.

(i) follows from (A2) and Lemma 10.

With regard to (ii), notice that this is not true for arbitrary $\phi \in \mathcal{W}^s$. (Had we built this property into the definition of $\mathcal{W}^s$, the space would not be complete under the metric $\|\cdot\|_s$.) Suppose, to derive a contradiction, that there exist $x, y \in X$ and $\eta_{1,2} \in \hat{\Phi}_x(E^s)$, $\xi_{1,2} \in \hat{\Phi}_y(E^s)$, $i = 1, 2$, such that

$$\eta_1 = \xi_1, \qquad \pi^s\eta_2 = \pi^s\xi_2 \quad \text{and} \quad \eta_2 \neq \xi_2.$$

Then by (i) above, $|\pi^s(f^n(\eta_2)) - \pi^s(f^n(\eta_1))| \leqslant e^{n\lambda_s}|\pi^s\eta_2 - \pi^s\eta_1|$, and an analogous estimate holds for $\xi_{1,2}$. This implies

$$\left|f^n(\eta_2) - f^n(\xi_2)\right| \leqslant \left|f^n(\eta_2) - f^n(\eta_1)\right| + \left|f^n(\xi_1) - f^n(\xi_2)\right| \leqslant 4e^{n\lambda_s}\left|\pi^s\eta_2 - \pi^s\xi_2\right|,$$

but by Lemma 6, these two points are $\geqslant e^{n\lambda_c}|\eta_2 - \xi_1|$ apart, contradicting the previous estimate for large n.

To summarize, we have shown that through each $x \in X$, there is a unique Lipschitz stable manifold $W^s_x \triangleq \hat{\Phi}_x(E^s)$, and the family $\{W^s_x\}$ defines a continuous foliation which is the foliation $\mathcal{F}^s$ in Theorem 2.

C. *Regularity of stable manifolds.* Let $\hat{\phi}^s \in \mathcal{W}^s$ be the fixed point of Γ^s. We will show that $\hat{\phi}_x$ is differentiable with $D\hat{\phi}^s_x(0) = \hat{L}_x$. The $C^{1,\alpha}$ property of $\hat{\phi}_x$ will then follow immediately from Lemma 7, as will the $C^{0,\alpha}$ property of the mapping H^s in the statement of Theorem 2.

Let

$$R(\delta) \triangleq \sup\left\{\frac{|\hat{\phi}_x^s(x^s) - \hat{L}_x x^s|}{|x^s|}:\ x \in X,\ |x^s| \in (0,\delta)\right\} \leqslant 2\mu_s.$$

For $x \in X$ and $x^s \in E^s$ with $|x^s| < \delta$, let $\bar{x}^s \in E^s$ be such that $\hat{\Phi}^s_{f(x)}(\bar{x}^s) = f(\hat{\Phi}^s_x(x^s))$. We will show that

$$\frac{|\hat{\phi}_x^s(x^s) - \hat{L}_x x^s|}{|x^s|} \leqslant o(1) + \left(\frac{e^{\lambda_s - \lambda_c}}{1 - \mu_c\mu_s}\right)\frac{|\hat{\phi}^s_{f_x}(\bar{x}^s) - \hat{L}_{f_x}\bar{x}^s|}{|\bar{x}^s|} \tag{22}$$

where $o(1) \to 0$ as $\delta \to 0$ uniformly in x and x^s. Since the left side of (22) can be chosen arbitrarily close to $R(\delta)$ and the quantity in parenthesis is < 1 by (A5), (22) gives immediately $R(\delta) = o(1)$ and thus the uniform differentiability of $\hat{\phi}^s_x$ in x^s with the desired derivative.

To prove (22), observe first that by Taylor's expansion, we have

$$\bar{x}^s = \pi^s\big((Df)_x\big(x^s + \hat{\phi}^s_x(x^s)\big) + f_1\big)$$

where $f_1(x, \hat{\Phi}^s_x(x^s)) = o(|x - \hat{\Phi}^s_x(x^s)|) = o(|\bar{x}^s|)$. From (14), we obtain

$$\hat{\phi}^s_{f(x)}(\bar{x}^s) - \hat{L}_{f(x)}\bar{x}^s = \big(\pi^c - \hat{L}_{f(x)}\pi^s\big)\big((Df)_x\big(\hat{\phi}^s_x(x^s) - \hat{L}_x x^s\big) + f_1\big).$$

Note also that $|\bar{x}^s| \leqslant e^{\lambda_s}|x^s| < \delta$ (Lemma 10). The desired inequality then follows from (15).

3.3. *Regularity of W^c and related estimates*

This subsection discusses the regularity of $\Gamma^n(h)$ where Γ is the graph transform in Section 3.1 and $h \in \mathcal{W}^c \cap C^{1,\alpha}$ where $\alpha > 0$ is a sufficiently small number to be determined. In addition to completing the proof of Theorem 1 under assumption (A5), this subsection contains a number of estimates that will be useful in the proof of Theorem 3.

All notation is as in Section 3.1. Since much of the discussion here is about derivatives, it is notationally clearer to distinguish between T_zX, the tangent space of $z \in X$, and X (no distinction was made previously). Similarly, $E_z^{c,s}$ denote subspaces of T_zX, and $T_z\,\mathrm{graph}(h)$ has the obvious meaning for $z \in \mathrm{graph}(h)$.

A. A priori *$C^{1,\alpha}$ bounds.* For $h \in \mathcal{W}^c$, we let $h^{(n)} \triangleq \Gamma^n(h)$. The main result of Part A is

Proposition 11. *The following holds for all sufficiently small α:*

- (i) *Given $C_1 > 0$, there exists $C_2 \geqslant C_1$ such that for all $h \in \mathcal{W}^c \cap C^{1,\alpha}$, if $\|Dh\|_{C^\alpha} \leqslant C_1$, then $\|Dh^{(n)}\|_{C^\alpha} \leqslant C_2$ for all $n \geqslant 1$.*
- (ii) *It is in fact true that* $\limsup_{n\to\infty} \|Dh^{(n)}\|_{C^\alpha} \leqslant K\|Df\|_{C^\alpha}$.

Recall that at each $z \in X$, there are two ways to represent $x \in T_zX$:

$$x = \pi^c(x) + \pi^s(x) \in E^c_z \oplus E^s_z \quad \text{and} \quad x = P^c_z(x) + P^s_z(x) \in E^c_z \oplus X^s_z.$$

It will turn out that the second representation yields better estimates due to the invariance of X^s. Let $h \in \mathcal{W}^c \cap C^1$ and $z \in \mathrm{graph}(h)$; z will always be a point in $\mathrm{graph}(h)$ in the discussion to follow. Since P^c_z maps $T_z\,\mathrm{graph}(h)$ isomorphically onto E^c_z, we may define $S_{h,z} \in \mathcal{L}(E^c_z, E^s_z)$ by

$$S_{h,z}\big(P^c_z x\big) = \pi^s P^s_z x \quad \text{for } x \in T_z \operatorname{graph}(h). \tag{23}$$

Suppose $z = H(z^c)$. Since $\pi^s P^s_z x = (Dh)_{z^c}(P^c_z x + \hat{L}_z \pi^s P^s_z x)$, one obtains by a straightforward computation the following relations between $S_{h,z}$ and $(Dh)_{z^c}$:

$$S_{h,z} = (Dh)_{z^c}(I + \hat{L}_z S_{h,z}); \tag{24}$$

$$S_{h,z} = \big(I - (Dh)_{z^c}\hat{L}_z\big)^{-1}(Dh)_{z^c}. \tag{25}$$

From (25), it follows immediately that

$$\|S_{h,z}\| \leqslant \frac{\mu_c}{1-\mu_c\mu_s}. \tag{26}$$

Instead of estimating $\|Dh^{(n)}\|_{C^\alpha}$ directly, we will work with $\|S_{h^{(n)}}\|_{C^\alpha}$ where $S_h : E^c \to \mathcal{L}(E^c, E^s)$ is defined by $S_h(z^c) = S_{h,z}$. Specifically, we will show that there exist $\beta \in (0,1)$ and a constant C such that the following hold for all $\alpha > 0$ sufficiently small and all $h \in \mathcal{W}^c \cap C^{1,\alpha}$:

(B1) $\|Dh\|_{C^\alpha} \leqslant C\|S_h\|_{C^\alpha} + C$;
(B2) $\|S_{\tilde{h}}\|_{C^\alpha} \leqslant \beta\|S_h\|_{C^\alpha} + C$, $\tilde{h} = \Gamma(h)$;
(B3) $\|S_h\|_{C^\alpha} \leqslant C\|Dh\|_{C^\alpha} + C$.

Assuming $\|S_h\|_{C^\alpha} < \infty$, it is easy to see that a repeated application of (B2) gives a C' with $\|S_{h^{(n)}}\|_{C^\alpha} \leqslant C'$ for all $n \geqslant 0$. That together with (B3) applied to h and (B1) applied to $h^{(n)}$ imply immediately the assertion in item (i) of Proposition 11. Item (ii) follows from the nature of the "C" in the proofs below.

Proofs of (B1) *and* (B3): Let $h \in \mathcal{W}^c \cap C^{1,\alpha}$; we will omit the "$h$" in $S_{h,\cdot}$ since it is the only graph in question. Let $z_1, z_2 \in \operatorname{graph}(h)$. From (24), we obtain

$$\begin{aligned}
&(Dh_{z_2^c} - Dh_{z_1^c})(I + \hat{L}_{z_1}S_{z_1}) \\
&\quad = Dh_{z_2^c}(I + \hat{L}_{z_1}S_{z_1}) - Dh_{z_2^c}(I + \hat{L}_{z_2}S_{z_2}) + S_{z_2} - S_{z_1} \\
&\quad = (I - Dh_{z_2^c}\hat{L}_{z_2})(S_{z_2} - S_{z_1}) - Dh_{z_2^c}(\hat{L}_{z_2} - \hat{L}_{z_1})S_{z_1}.
\end{aligned}$$

Note that for any $z \in \operatorname{graph}(h)$ and $x \in T_z \operatorname{graph}(h)$,

$$(I + \hat{L}_z S_z)P^c_z x = x^c \quad \text{and} \quad P^c_z x = (I - \hat{L}_z Dh_z)x^c,$$

so that

$$(I + \hat{L}_z S_z)^{-1} = I - \hat{L}_z Dh_z.$$

Using $\|I - Dh\hat{L}\|, \|I - \hat{L}Dh\| \leqslant 1 + \mu_c\mu_s$, we obtain

$$\|Dh_{z_2^c} - Dh_{z_1^c}\| \leqslant \big[(1+\mu_c\mu_s)\|S_{z_2} - S_{z_1}\| + \mu_c\|\hat{L}_{z_2} - \hat{L}_{z_1}\|\|S_{z_1}\|\big] \cdot (1+\mu_c\mu_s).$$

This together with the bound on $\|\hat{L}\|_{C\alpha}$ in Lemma 7 and (26) gives

$$\|Dh\|_{C^\alpha} \leqslant (1+\mu_s\mu_c)^2\|S\|_{C^\alpha} + K\|Df\|_{C^\alpha}$$

proving (B1).

The equations at the beginning of the proof also give

$$\|S\|_{C^\alpha} \leqslant (1-\mu_c\mu_s)^{-2}\|Dh\|_{C^\alpha} + K\|Df\|_{C^\alpha},$$

proving (B3).

Proof of (B2): Let h, S be as above. Let $\tilde{h} = \Gamma(h)$, and let $\tilde{S}$ be the corresponding operator. We will prove

$$\|\tilde{S}\|_{C^\alpha} \leqslant \left(\frac{e^{\lambda_s-(1+\alpha)\lambda_c}}{1-\mu_c\mu_s}\right)\|S\|_{C^\alpha} + K\|Df\|_{C^\alpha}. \tag{27}$$

For α small enough, the quantity in parenthesis is < 1 by (A5). The α in Proposition 11 is the smaller of this α and the one in Lemma 7.

Let $z \in \operatorname{graph}(h)$, $x \in T_z \operatorname{graph}(h)$, $\tilde{z} = f(z)$ and $\tilde{x} = Df_z x$. By the definition of S, we have

$$x = \big(I + (I+\hat{L}_z)S_z\big)P_z^c x \quad \text{and} \quad \tilde{x} = \big(I + (I+\hat{L}_z\tilde{z})\tilde{S}_{\tilde{z}}\big)P_{\tilde{z}}^c Df_z x.$$

Also, by the invariance of X^s, we have

$$P_{\tilde{z}}^c Df_z x = P_{\tilde{z}}^c Df_z\big(P_z^c x\big).$$

Substituting $Df_z x = \tilde{x}$ into the equations above, we obtain

$$Df_z\big(I + (I+\hat{L}_z)S_z\big) = \big(I + (I+\hat{L}_{\tilde{z}})\tilde{S}_{\tilde{z}}\big)\bar{F}_z, \tag{28}$$

where $\bar{F}_z = P_{\tilde{z}}^c Df_z|_{E_z^c}$. From (17) and (15), it follows that

$$\big\|(\bar{F}_z)^{-1}\big\| \leqslant \frac{e^{-\lambda_c}}{1-\mu_c\mu_s}. \tag{29}$$

Applying $\pi^s P_z^s = \pi^s$ to (28), we obtain from the invariance of X^s

$$\tilde{S}_{\tilde{z}}\bar{F}_z = \pi^s(Df)_z(I+\hat{L}_z)S_z + \pi^s(Df)_z\big|_{E_z^c}. \tag{30}$$

We now use (30) to estimate $\|\tilde{S}\|_{C^\alpha}$: For $z_i \in \operatorname{graph}(h)$ and $\tilde{z}_i = f(z_i) \in \operatorname{graph}(\tilde{h})$, $i = 1, 2$, we have

$$\begin{aligned}(\tilde{S}_{\tilde{z}_2} - \tilde{S}_{\tilde{z}_1})\bar{F}_{z_1} = {} & \pi^s(Df)_{z_2}(I+\hat{L}_{z_2})(S_{z_2} - S_{z_1}) \\ & + \pi^s\big((Df)_{z_2}(I+\hat{L}_{z_2}) - (Df)_{z_1}(I+\hat{L}_{z_1})\big)S_{z_1} \\ & + \pi^s\big((Df)_{z_2} - (Df)_{z_1}\big) - \tilde{S}_{\tilde{z}_2}(\bar{F}_{z_2} - \bar{F}_{z_1}).\end{aligned}$$

From (A2) and Lemma 7, we obtain

$$\big\|\pi^s(Df)_{z_2}(I+\hat{L}_{z_2})\big\|_{\mathcal{L}(E_{z_2}^s)} \leqslant e^{\lambda_s}.$$

The bound in (27) follows from the inequality above, Lemma 5, (29), (26), and the C^α-property of Df, $\hat{L}$ and S, the factor involving α in (27) coming from

$$\|S_{z_1} - S_{z_2}\| \leqslant \|S\|_{C^\alpha} \cdot \left|z_1^c - z_2^c\right|^\alpha \leqslant \|S\|_{C^\alpha} \cdot \left(e^{-\lambda_c}\left|\tilde{z}_1^c - \tilde{z}_2^c\right|\right)^\alpha.$$

The proof of Proposition 11 is now complete.

B. $C^{1,\alpha}$ *property of* W^c. Let $h^c \in \mathcal{W}^c$ be the fixed point of Γ, and let α be as in Proposition 11. We will show that for any $h \in \mathcal{W}^c \cap C^{1,\alpha}$, the sequence $\{Dh^{(n)}\}_{n\geqslant 0}$ is Cauchy in the C^0-norm. It will then follow that h^c is differentiable with $\|Dh^{(n)} - Dh^c\|_{C^0} \to 0$. Proposition 11(ii) further implies that $\|Dh^c\|_{C^\alpha} \leqslant K\|Df\|_{C^\alpha}$.

Let $h_{1,2} \in \mathcal{W}^c \cap C^1$. From Lemma 8, any stable manifold $W^s = W^s_z$ has unique intersection points with graph$(h_{1,2})$; we denote them by $z_{1,2}$, and let $\tilde{z}_{1,2} = f(z_{1,2})$; equivalently, $\tilde{z}_{1,2}$ are the points of intersection of $W^s_{f(z)}$ with graph$(\tilde{h}_{1,2})$.

Proceeding as in Part A, we let $S^{1,2} = S_{h_{1,2}}$. From (30), we obtain

$$\begin{aligned}\left(\tilde{S}^2_{\tilde{z}_2} - \tilde{S}^1_{\tilde{z}_1}\right)\bar{F}_{z_1} &= \pi^s(Df)_{z_2}(I + \hat{L}_{z_2})\left(S^2_{z_2} - S^1_{z_1}\right)\\ &\quad + \pi^s\left((Df)_{z_2}(I + \hat{L}_{z_2}) - (Df)_{z_1}(I + \hat{L}_{z_1})\right)S^1_{z_1}\\ &\quad + \pi^s\left((Df)_{z_2} - (Df)_{z_1}\right) - \tilde{S}^2_{\tilde{z}_2}(\bar{F}_{z_2} - \bar{F}_{z_1}).\end{aligned}$$

Using (A2) and (29), we obtain

$$\left\|\tilde{S}^2_{\tilde{z}_2} - \tilde{S}^1_{\tilde{z}_1}\right\| \leqslant \frac{e^{\lambda_s - \lambda_c}}{1 - \mu_c\mu_s}\left\|S^2_{z_2} - S^1_{z_1}\right\| + K\|Df\|_{C^\alpha}|z_2 - z_1|^\alpha. \tag{31}$$

Since $z_{1,2}$ are on the same stable fiber, one also has

$$|z_2 - z_1| \leqslant (1 + \mu_s)\left|z_2^s - z_1^s\right|. \tag{32}$$

This leads to the following Inclination or λ-lemma:

Lemma 12. *We let $h_{1,2} \in \mathcal{W}^c \cap C^1$, fix an arbitrary $W^s = W^s_x$, and let $z_{1,2}$ be the unique points in* graph$(h_{1,2}) \cap W^s$. *For $n \geqslant 1$, we let $h^{(n)}_{1,2} = \Gamma^n(h_{1,2})$, and $z^n_{1,2} \in$* graph$(h^{(n)}_{1,2}) \cap W^s_{f^n(x)}$. *Then*

$$\begin{aligned}\left\|\left(Dh^{(n)}_2\right)_{(z^n_2)^c} - \left(Dh^{(n)}_1\right)_{(z^n_1)^c}\right\| &\leqslant K\left(\frac{e^{\lambda_s - \lambda_c^-}}{1 - \mu_c\mu_s}\right)^n \left\|(Dh_2)_{z_2^c} - (Dh_1)_{z_1^c}\right\|\\ &\quad + nK\|Df\|_{C^\alpha}\left(\frac{e^{\lambda_s - \lambda_c^-}}{1 - \mu_c\mu_s}\right)^{\alpha n}\left|z_2^s - z_1^s\right|^\alpha.\end{aligned} \tag{33}$$

Proof. Letting $S^{n,1}$ and $S^{n,2}$ be the operators associated to $Dh^{(n)}_{1,2}$, we obtain by (31), (32) and Lemma 10

$$\begin{aligned}\left\|S^{n,2}_{z^n_2} - S^{n,1}_{z^n_1}\right\| &\leqslant \left(\frac{e^{\lambda_s - \lambda_c}}{1 - \mu_c\mu_s}\right)^n \left\|S^2_{z_2} - S^1_{z_1}\right\|\\ &\quad + nK\|Df\|_{C^\alpha}\left(\frac{e^{\lambda_s - \lambda_c^-}}{1 - \mu_c\mu_s}\right)^{\alpha n}\left|z_2^s - z_1^s\right|^\alpha.\end{aligned}$$

Following the proof of (B1) of Proposition 11, we obtain that

$$\left\|\left(Dh_2^{(n)}\right)_{(z_2^n)^c} - \left(Dh_1^{(n)}\right)_{(z_1^n)^c}\right\| \leqslant (1+\mu_s\mu_c)^2\left\|S^{n,2}_{z_2^n} - S^{n,1}_{z_1^n}\right\| + K\|Df\|_{C^\alpha}\left|z_2^n - z_1^n\right|^\alpha.$$

Also, $|z_2^n - z_1^n| \leqslant (1+\mu^s)|(z_2^n)^s - (z_1^n)^s|$. Thus by applying Lemma 10 and the above inequalities, we complete the proof. □

To prove the assertion at the beginning of Part B, we now assume $h_{1,2} \in \mathcal{W}^c \cap C^{1,\alpha}$, and write

$$\begin{aligned}
&\left\|\left(Dh_2^{(n)}\right)_{(z_1^n)^c} - \left(Dh_1^{(n)}\right)_{(z_1^n)^c}\right\| \\
&\quad\leqslant \left\|\left(Dh_2^{(n)}\right)_{(z_2^n)^c} - \left(Dh_1^{(n)}\right)_{(z_1^n)^c}\right\| + \left\|Dh_2^{(n)}\right\|_{C^\alpha}\left|\left(z_2^n\right)^c - \left(z_1^n\right)^c\right|^\alpha.
\end{aligned}$$

The first term is estimated in Lemma 12, and the second by

$$\left|\left(z_2^n\right)^c - \left(z_1^n\right)^c\right| \leqslant \mu_s\left|\left(z_2^n\right)^s - \left(z_1^n\right)^s\right| \leqslant \mu_s e^{n\lambda_s}\left|z_2^s - z_1^s\right|.$$

3.4. Removing assumption (A5)

Proof of Lemma 7, Theorem 1, Theorem 2. Notice that for all $n \in \mathbb{Z}^+$, assumptions (A1)–(A4), with (A3) replaced by (A3′), are satisfied by f^n with the same splitting $X = E^c \oplus E^s$ and $\mu_{c,s}$, but with $\lambda_{c,s}$ replaced by $\lambda_{c,s}^{(n)} = n\lambda_{c,s}$ and with a replaced by $a^{(n)} = na$. Since (A5) is satisfied by f^n for some n (depending only on $\mu_{c,s}$ and $\lambda_{c,s}$), it follows from Sections 3.1–3.3 that for the system defined by f^n, there is

(i) a unique invariant center manifold W^c,
(ii) an invariant stable subbundle X^s, and
(iii) an equivariant stable foliation $\mathcal{F}^s$.

Moreover, there exists $\alpha > 0$ such that all relevant C^α-norms are as described in Theorems 1 and 2. As $\|Df^n\|_{C^\alpha} \leqslant \text{const}\|Df\|_{C^\alpha}$ for a constant depending on $\|Df\|$ and n, it remains only to show that the objects in (i)–(iii) are invariant under f.

Observe first that (A5) was not used in the construction of the graph transforms in the proofs of Theorem 1, Lemma 7, and Theorem 2. Thus these graph transforms are well defined for f without (A5).

What gives the desired results is that the uniqueness of the fixed points of the graph transforms for f^n imply that they are also fixed points for the corresponding graph transforms for f. As an illustration, let $h \in \mathcal{W}^c$ be the unique fixed point of Γ_{f^n}, the graph transform defined by f^n, and let $\tilde{h} = \Gamma_f(h)$ where Γ_f is the graph transform for f. Since

$$f^n\left(f\left(\text{graph}(h)\right)\right) = f\left(f^n\left(\text{graph}(h)\right)\right) = f\left(\text{graph}(h)\right),$$

i.e., $\Gamma_{f^n}(\tilde{h}) = \tilde{h}$, it follows that $\tilde{h} = h$. □

4. Absolute continuity of stable foliations

Assumptions (A1)–(A4) are in effect throughout this section. Without loss of generality, we may assume (A5) also by considering a power of f. Let $\Theta \triangleq \frac{e^{\lambda_s - \lambda_c^-}}{1-\mu_c\mu_s}$ be the constant in (A5), and let α be as in Theorems 1 and 2.

4.1. Preliminary estimates

For $g : E^c \to E^s$, we will use the notation $\Sigma_g = \mathrm{graph}(g)$. We begin with a lemma that is an immediate consequence of Lemma 8:

Lemma 13. *The following hold for all $g \in \mathcal{W}^c$:*

(i) *for every $x \in X$, W^s_x meets Σ_g in exactly one point*;
(ii) *there is a homeomorphism $T_g : E^c \to \Sigma_g$ such that for each $x^c \in E^c$, $T_g(x^c)$ is the unique point in $W^s(x^c) \cap \Sigma_g$.*

Item (i) above asserts that Σ_g is a genuine global transversal to the stable foliation $\mathcal{F}^s$ in Theorem 2. Item (ii) defines the holonomy map between E^c and Σ_g. More generally, for any $g_{1,2} \in \mathcal{W}^c$ the *holonomy map* $T_{g_1,g_2} : \Sigma_{g_1} \to \Sigma_{g_2}$ is given by

$$T_{g_1,g_2} = T_{g_2} \circ T_{g_1}^{-1}.$$

As in Section 3.3, for $g \in \mathcal{W}^c$, let $g^{(n)} = \Gamma^n(g)$, $n = 1, 2, \dots$. By the invariance of stable manifolds (Theorem 2), we have, for $g_{1,2} \in \mathcal{W}^c$ and $n \in \mathbb{Z}^+$, the relation

$$f^n\big|_{\Sigma_{g_2}} \circ T_{g_1,g_2} = T_{g_1^{(n)},g_2^{(n)}} \circ f^n\big|_{\Sigma_{g_1}}. \tag{34}$$

Lemma 14. *The following hold for all $g_{1,2} \in \mathcal{W}^c$, $x \in \Sigma_{g_1}$, $y = T_{g_1,g_2}(x)$, and $n \geqslant 1$:*

(i) $|\pi^c(x) - \pi^c(y)| \leqslant \mu_s |\pi^s(x) - \pi^s(y)|$;
(ii) $|\pi^s(f^n x) - \pi^s(f^n y)| \leqslant e^{n\lambda_s} |\pi^s(x) - \pi^s(y)|$;
(iii) *assuming additionally that $g_{1,2}$ are C^1, we have*

$$\big\| \big(Dg_1^{(n)}\big)_{\pi^c(f^n x)} - \big(Dg_2^{(n)}\big)_{\pi^c(f^n y)} \big\| \leqslant 2K\mu_c\Theta^n + nK\|Df\|_{C^\alpha}\Theta^{n\alpha}\big|\pi^s(x) - \pi^s(y)\big|^\alpha$$

where Θ is the constant in (A5) *and K is as in* (33)*.*

Item (i) is a property of stable manifolds; (ii) follows from Lemma 10, and (iii) follows from (33).

We consider next how the measures m_{Σ_g} on Σ_g (see Section 1 for notation) are transformed by the composite maps in (34). If $g \in \mathcal{W}^c$ is C^1, then $m_{\Sigma_{g(n)}}$ is related to m_{Σ_g} via the Jacobian of $(Df^n)|_{\Sigma_g}$. Since the m_{Σ_g}-measures are induced from Lebesgue measure on E^c, this is equivalent to studying $|\det(DF^n_g)|$ where $F^n_g : E^c \to E^c$ is defined by

$$F^n_g\big(x^c\big) = \pi^c f^n\big(x^c + g\big(x^c\big)\big), \quad n = 1, 2, \dots.$$

Notice that $F_g = F^1_g$ is invertible with $\mathrm{Lip}((F_g)^{-1}) \leqslant e^{-\lambda_c}$.

Lemma 15. *Given $Q_0 > 0$, there exists $C_0 \geqslant 1$ such that for all $g_1, g_2 \in \mathcal{W}^c \cap C^1(E^c, E^s)$ with $\|g_1 - g_2\|_{C^0} \leqslant Q_0$, $x \in \Sigma_{g_1}$, $y = T_{g_1,g_2}x$ and $n \geqslant 0$,*

$$C_0^{-1} \leqslant \frac{|\det(DF^n_{g_1})_{\pi^c x}|}{|\det(DF^n_{g_2})_{\pi^c y}|} \leqslant C_0.$$

Here C_0 depends only on system constants, α, $\|Df\|_{C^\alpha}$ and Q_0.

Proof. Taking logarithm of the quotient in question, we are led to terms of the form

$$\Big|\log\big|\det(DF_{g_1^{(i)}})_{\pi^c(f^i x)}\big| - \log\big|\det(DF_{g_2^{(i)}})_{\pi^c(f^i y)}\big|\Big|,$$

which we claim to be

$$\leqslant C\big\|(DF_{g_1^{(i)}})_{\pi^c(f^i x)} - (DF_{g_2^{(i)}})_{\pi^c(f^i y)}\big\|$$

for some C independent of $g_{1,2}, i$ or x. This is because by (A4),

$$\big\|(DF_g)_{x^c}\big\| \leqslant M(1+\mu_c)e^a$$

for all $g \in \mathcal{W}^c$ that are C^1 and $x^c \in E^c$. This together with $\mathrm{Lip}((F_g)^{-1}) \leqslant e^{-\lambda_c}$ implies that $(DF_g)_{x^c}$ lies in a compact region $\mathcal{U} \subset GL(k,\mathbb{R})$ where $k = \dim(E^c)$. The constant C above is the Lipschitz constant of the mapping $A \mapsto \log|\det(A)|$ for $A \in \mathcal{U}$.

Since

$$(DF_g)_{\pi^c x} = \pi^c(D_1 f)_x + \pi^c(D_2 f)_x Dg_{\pi^c x},$$

we have, for every $i \geqslant 1$,

$$\begin{aligned}
&\big\|(DF_{g_1^{(i)}})_{\pi^c(f^i x)} - (DF_{g_2^{(i)}})_{\pi^c(f^i y)}\big\| \\
&\quad \leqslant \big\|\pi^c(D_1 f)_{f^i x} - \pi^c(D_1 f)_{f^i y}\big\| + \big\|\pi^c(D_2 f)_{f^i x} - \pi^c(D_2 f)_{f^i y}\big\|\big\|\big(Dg_1^{(i)}\big)_{\pi^c(f^i x)}\big\| \\
&\qquad + \big\|\pi^c(D_2 f)_{f^i y}\big\|\big\|\big(Dg_1^{(i)}\big)_{\pi^c(f^i x)} - \big(Dg_2^{(i)}\big)_{\pi^c(f^i y)}\big\| \\
&\quad \leqslant M\|Df\|_{C^\alpha}(1+\mu_c)(1+\mu_s)^\alpha\big|\pi^s\big(f^i x\big) - \pi^s\big(f^i y\big)\big|^\alpha \\
&\qquad + Me^a\big(2K\mu_c\Theta^i + iK\|Df\|_{C^\alpha}\Theta^{i\alpha}\big|\pi^s(x) - \pi^s(y)\big|^\alpha\big) \\
&\quad \leqslant C'\hat{\Theta}^{i\alpha}\max\big\{\|g_1 - g_2\|_{C^0}^\alpha, 1\big\} \quad \text{for some } \Theta < \hat{\Theta} < 1.
\end{aligned}$$

To obtain the last line we have used

(i) $|\pi^s(f^i x) - \pi^s(f^i y)| \leqslant e^{i\lambda_s}|\pi^s(x) - \pi^s(y)|$ (Lemma 10), and
(ii) $|\pi^s(x) - \pi^s(y)| \leqslant \frac{1}{1-\mu_c\mu_s}\|g_1 - g_2\|_{C^0}$.

Notice that C' does not depend on $g_{1,2}$ or i.

Finally,

$$\begin{aligned}
\log\frac{|\det(DF_{g_1}^n)_{\pi^c x}|}{|\det(DF_{g_2}^n)_{\pi^c y}|} &\leqslant \sum_{i=0}^{n-1}\Big|\log\big|\det(DF_{g_1^{(i)}})_{\pi^c(f^i x)}\big| - \log\big|\det(DF_{g_2^{(i)}})_{\pi^c(f^i y)}\big|\Big| \\
&\leqslant C\sum_{i=0}^{n-1}\big\|(DF_{g_1^{(i)}})_{\pi^c(f^i x)} - (DF_{g_2^{(i)}})_{\pi^c(f^i y)}\big\| \\
&\leqslant C\sum_{i=0}^{\infty}C'\hat{\Theta}^{i\alpha}\max\big\{\|g_1 - g_2\|_{C^0}^\alpha, 1\big\} < \infty,
\end{aligned}$$

which completes the proof. □

Lemma 16. *Fix arbitrary $\lambda < \lambda_c^-$. Given $Q_1 > 0$, there exists $C_1 \geqslant 1$ such that for all $g \in \mathcal{W}^c \cap C^{1,\alpha}$ with $\|Dg\|_{C^\alpha} \leqslant Q_1$ and $n \in \mathbb{Z}^+$, if $x, y \in \operatorname{graph}(g)$ are such that $|f^n(x) - f^n(y)| \leqslant e^{n\lambda}$, then*

$$C_1^{-1} \leqslant \frac{|\det(DF_g^n)_{\pi^c x}|}{|\det(DF_g^n)_{\pi^c y}|} \leqslant C_1.$$

Here C_1 depends only on system constants, α, $\|Df\|_{C^\alpha}$, λ and Q_1.

Proof. The idea is similar to that in the proof of Lemma 15: we need to show that $|(DF_{g^{(i)}})_{\pi^c(f^i x)} - (DF_{g^{(i)}})_{\pi^c(f^i y)}|$ decays at a rate independent of g, x, y or n. Proceeding as before, we have

$$\begin{aligned} &\bigl|(DF_{g^{(i)}})_{\pi^c(f^i x)} - (DF_{g^{(i)}})_{\pi^c(f^i y)}\bigr| \\ &\quad\leqslant \bigl\|\pi^c(D_1 f)_{f^i x} - \pi^c(D_1 f)_{f^i y}\bigr\| \\ &\qquad + \bigl\|\pi^c(D_2 f)_{f^i x}\bigl(Dg^{(i)}\bigr)_{\pi^c(f^i x)} - \pi^c(D_2 f)_{f^i y}\bigl(Dg^{(i)}\bigr)_{\pi^c(f^i y)}\bigr\| \\ &\quad\leqslant M\bigl(\|Df\|_{C^\alpha}\bigl(1+\mu^c\bigr)^{(1+\alpha)} + e^a\bigl\|Dg^{(i)}\bigr\|_{C^\alpha}\bigr)\cdot\bigl|\pi^c\bigl(f^i x\bigr) - \pi^c\bigl(f^i y\bigr)\bigr|^\alpha. \end{aligned}$$

Using Proposition 11 to bound $\|Dg^{(i)}\|_{C^\alpha}$ and noticing that

$$\bigl|\pi^c\bigl(f^i x\bigr) - \pi^c\bigl(f^i y\bigr)\bigr| \leqslant Me^{n\lambda-(n-i)\lambda_c} \leqslant Me^{i\lambda}$$

by Lemma 5, we complete the argument. □

Remark 2. Lemma 16 holds, in fact, if g is only C^1, in which case the constant C_1 depends on $\|g - h^c\|_{C^0}$ (instead of $\|Dg\|_{C^\alpha}$). This is because $\|Dg^i - Dh^c\| \to 0$ exponentially fast by the Inclination Lemma (Lemma 12) and $h^c \in C^{1,\alpha}$. All the results in the next subsection remain valid under this modified assumption.

4.2. Proof of absolute continuity

We will show that given $Q_0, Q_1 > 0$, there exists $C_2 > 1$ such that the following holds for all $g_{1,2} \in \mathcal{W}^c \cap C^{1,\alpha}$ with $\|g_1 - g_2\|_{C^0} \leqslant Q_0$ and $\|Dg_{1,2}\|_{C^\alpha} \leqslant Q_1$: Let $\tilde{A} \subset \Sigma_{g_1}$ be an arbitrary Borel subset. Then

$$C_2^{-1} m_{\Sigma_{g_1}}(\tilde{A}) \leqslant m_{\Sigma_{g_2}}\bigl(T_{g_1,g_2}(\tilde{A})\bigr) \leqslant C_2 m_{\Sigma_{g_1}}(\tilde{A}). \tag{35}$$

Equivalently, for every Borel subset $A \subset E^c$,

$$C_2^{-1} m(A) \leqslant m\bigl(\check{T}_{g_1,g_2}(A)\bigr) \leqslant C_2 m(A), \tag{36}$$

where $\check{T}_{g_1,g_2}(x^c) \triangleq \pi^c T_{g_1,g_2}(x^c + g_1(x^c))$. System constants aside, we permit C_2 to depend only on α, $\|Df\|_{C^\alpha}$, Q_0 and Q_1.

Clearly, it suffices to prove (36) for bounded sets A, and since every bounded Borel set can be approximated from the inside by a compact subset that is arbitrarily close to it in measure, it suffices to prove (36) for compact sets. Finally, since the choice of g_1 and g_2 is arbitrary, it suffices to prove the second inequality in (36).

We will use the notation $B^c(x^c, r) = \{z \in E^c\colon |x^c - z| < r\}$.

Let a compact set $A \subset E^c$ be fixed, and let U be a small neighborhood of A such that $m(U) \leqslant 2m(A)$. Since $\operatorname{dist}(A, \partial U) > 0$, there exists N_1 such that for any $n \geqslant N_1$ and any $x^c \in A$,

$$\left(F_{g_1}^n\right)^{-1}\left(B^c\left(F_{g_1}^n\left(x^c\right), e^{n\lambda}\right)\right) \subset U$$

where $\lambda = \frac{\lambda_c^- + \lambda_s}{2}$. Lemma 5 was used in this last step.

Next we choose $N_2 > N_1$ such that for any $n > N_2$,

$$\check{T}_{g_1^n, g_2^n}\left(B^c\left(F_{g_1}^n\left(x^c\right), e^{n\lambda}\right)\right) \subset B^c\left(F_{g_2}^n\left(\check{T}_{g_1,g_2}\left(x^c\right)\right), 2e^{n\lambda}\right).$$

This can be done because if $x = x^c + g_1(x^c)$ and $y = T_{g_1,g_2}(x)$, then by Lemma 10,

$$\begin{aligned}\left|F_{g_1}^n\left(x^c\right) - F_{g_2}^n\left(\check{T}_{g_1,g_2}\left(x^c\right)\right)\right| &= \left|\pi^c\left(f^n x\right) - \pi^c\left(f^n y\right)\right| \\ &\leqslant \mu_s\left|\pi^s\left(f^n x\right) - \pi^s\left(f^n y\right)\right| \\ &\leqslant \mu_s e^{n\lambda^s}\left|\pi^s(x) - \pi^s(y)\right|.\end{aligned}$$

Now for $n \geqslant N_2$, we let $A_1^n = F_{g_1}^n(A)$, and let $\{B_1, \ldots, B_\ell\}$ be a Besicovitch cover of A_1^n by $e^{n\lambda}$-balls centered at points in A_1^n, i.e., $A_1^n \subset \bigcup_i B_i$ and no point in A_1^n is contained in more than C^* of these balls where $C^* = C^*(\dim(E^c))$ is given by the Besicovitch Covering Lemma (see e.g. [9]).

For each i, we let x_i^c be the center of B_i, $y_i^c = \check{T}_{g_1,g_2}(x_i^c)$, and $B_i' = B^c(y_i^c, 2e^{n\lambda})$. Since $U \supset \bigcup_i (F_{g_1}^n)^{-1}(B_i)$, we have

$$m(A) \geqslant \frac{1}{2} m(U) \geqslant \frac{1}{2C^*} \sum_i m\left(\left(F_{g_1}^n\right)^{-1}(B_i)\right); \tag{37}$$

we have divided the right side by C^* to compensate for potential overcount due to overlaps. Similarly, $\bigcup_i (F_{g_1}^n)^{-1}(B_i) \supset A$ implies

$$m\left(\check{T}_{g_1^n, g_2^n}(A)\right) \leqslant \sum_i m\left(\left(F_{g_2}^n\right)^{-1}\left(B_i'\right)\right). \tag{38}$$

Comparing (37) and (38), we see that to complete the proof, it remains to produce a constant C which is permitted to depend on the same quantities as C_2 such that for each i,

$$m\left(\left(F_{g_2}^n\right)^{-1}\left(B_i'\right)\right) \leqslant C m\left(\left(F_{g_1}^n\right)^{-1}(B_i)\right).$$

This inequality follows from the following comparisons: By Lemma 16, we have

$$\det\left(DF_{g_1}^n\right)\left(x_1^c\right) \approx \det\left(DF_{g_1}^n\right)\left(x_2^c\right) \quad \text{for all } x_1^c, x_2^c \in \left(F_{g_1}^n\right)^{-1}(B_i),$$

as well as the analogous conclusion for $y_1^c, y_2^c \in (F_{g_2}^n)^{-1}(B_i')$. The problem is therefore reduced to showing that for $x^c \in E^c$ and $y^c = \check{T}_{g_1,g_2}(x^c)$,

$$\left|\det\left(DF_{g_2}^n\right)\left(y^c\right)\right| \approx \left|\det\left(DF_{g_1}^n\right)\left(x^c\right)\right|,$$

and that is guaranteed by Lemma 15.

The proof of Theorem 3 is now complete. □

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