Homogenized elastic–viscoplastic behavior of anisotropic open-porous bodies with pore pressure

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A B S T R A C T

Constitutive modeling is studied for the homogenized elastic–viscoplastic behavior of pore-pressurized anisotropic open-porous bodies made of metallic base solids at small strains and rotations. For this purpose, by describing micro–macro relations relevant to periodic unit cells of anisotropic open-porous bodies subjected to pore pressure, constitutive features are discussed for the viscoplastic macrostrain rate in steady states. On the basis of the constitutive features found, the viscoplastic macrostrain rate is represented as an anisotropic function of Terzaghi’s effective stress, which is shown using Hill’s macrohomogeneity condition. The resulting viscoplastic equation is used to simulate the homogenized elastic–viscoplastic behavior of an ultrafine plate-fin structure subjected to uniaxial/biaxial loading in addition to pore pressure. The corresponding finite element homogenization analysis is also performed for comparison. It is demonstrated that the developed viscoplastic equation simulates well the anisotropic effect of pore pressure in the viscoplastic range in spite of there being no anisotropic factor and no fitting parameter in Terzaghi’s effective stress itself.

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1. Introduction

Ultrafine plate-fin cores have been fabricated for compact heat exchangers under development for high-temperature gas-cooled reactors (Kawashima et al., 2007). The fabricated cores have an anisotropic open-cellular structure consisting of alternately stacked plates and fins, which have considerably small thicknesses of 0.5 and 0.2 mm, respectively. This cellular structure is designed to be subjected to fairly high pore pressures in addition to thermal stresses. Perforated thick plates in fast reactor heat exchangers are also subjected to fairly high pore pressures in addition to thermal stresses (Ando et al., 2008). The plate-fin cores and perforated plates mentioned above can be regarded as macrobodies that have anisotropic open-porous microstructures in which pore pressure uniformly acts independently of thermal stress at small strains and rotations.

Full-scale finite element meshing of the plate-fin cores and perforated thick plates necessarily results in considerably high computational costs because of the large number of plate-fin layers and circular cylindrical holes. For example, more than 1000 layers of plates and fins need to be stacked for the compact heat exchanger cores (Kawashima et al., 2007). If a homogenized or macroscopic constitutive model is available, the high computational costs due to full-scale finite-element meshing can be drastically reduced. For this reason, Tsuda et al. (2010) developed a macroscopic elastic–viscoplastic constitutive model using quadratic equivalent stress that was employed for the homogenized, anisotropic rate-independent plastic behavior of cellular/lattice structures (Badicé et al., 2000; Deshpande et al., 2001; Xue and Hutchinson, 2004). The macroscopic constitutive model developed has been shown to be suitable for plate-fin structures and perforated thick plates in the absence of pore pressure (Tsuda et al., 2010; Tsuda and Ohno, 2011; Ikenoya et al., in press). It is worthwhile to include the effect of pore pressure in this constitutive model.

The effect of pore pressure can be significant in the homogenized behavior of porous solids and voided solids (e.g., Dormieux et al., 2002, 2006; Vincent et al., 2009a,b; Coussy, 2010; Julien et al., 2011). Terzaghi’s effective stress (Terzaghi, 1943) is well known in the mechanics of porous solids with pore pressure (Dormieux et al., 2006; Coussy, 2010). Although this effective stress was intuitively introduced, one can show the following (Coussy, 2010). If macrostress acts on a porous solid consisting of a

References

1 A general form of quadratic anisotropic yield functions was proposed by von Mises (1928).
2 Alkhader and Vural (2009, 2010) proposed an anisotropic yield criterion based on total elastic strain energy density for two-dimensional lattice structures.
volume-incompressible solid phase and a pressurized pore phase, the excess of macrostress over the pore pressure, called Terzaghi’s effective stress, becomes conjugate to the macrostrain rate for the mechanical work done in the solid phase. The work-conjugate pair mentioned above implies that the effect of pore pressure on the viscoplastic macrostress rate of porous solids/bodies appears through Terzaghi’s effective stress if the viscoplastic deformation of base solids is volume-incompressible. However, Terzaghi’s effective stress itself has no anisotropic factor no matter how anisotropic the microstructure of porous solids/bodies is. It is expected that the effect of pore pressure is more or less anisotropic if the microstructure is anisotropic. It is therefore worth investigating how the anisotropic effect of pore pressure on the homogenized viscoplastic behavior of anisotropic porous solids/bodies is represented in constitutive modeling.

In this paper, constitutive modeling is studied for the homogenized elastic–viscoplastic behavior of pore-pressurized anisotropic open-porous bodies made of metallic base solids at small strains and rotations. To this end, micro–macro relations are described to introduce macroscopic variables, including Terzaghi’s effective stress, pertinent to periodic unit cells of pore-pressurized anisotropic open-porous bodies. Constitutive features are then discussed for the viscoplastic macrostrain rate in steady states. The constitutive features found are considered for extending the anisotropic viscoplastic constitutive equation developed by Tsuda et al. (2010). The viscoplastic macrostrain rate is thus represented as an anisotropic function of Terzaghi’s effective stress. The resulting viscoplastic equation is used to simulate the homogenized elastic–viscoplastic behavior of a pore-pressurized ultrafine plate-fin structure subjected to uniaxial/biaxial loading. Finite element homogenization analysis of the ultrafine plate-fin structure is also performed for comparison. It is demonstrated that the developed viscoplastic equation simulates well the anisotropic effect of pore pressure in the viscoplastic range, although Terzaghi’s effective stress itself has no anisotropic factor and no fitting parameter.

Throughout this paper, direct notations are used for vectors and tensors, and inner products between them are indicated by dots (e.g., $\mathbf{u} \cdot \mathbf{v} = u_i v_i$). In addition, the second-rank and fourth-rank unit tensors are denoted by $\mathbf{I}$ and $\mathbf{I}$, respectively.

### 2. Microscopic material properties

Let us consider a macrobody element that has an anisotropic open-porous microstructure undergoing small deformation. We assume that the microstructure is periodic. The macrobody element then has a periodic unit cell $Y$, which consists of a base solid region $V_s$ and a pore region $V_{po}$ as schematically illustrated in Fig. 1. This porous body can be considered as a two-phase composite. It is noted that $\partial Y$ is partitioned into $\partial Y_s$ and $\partial Y_{po}$, where $\partial Y_s$ and $\partial Y_{po}$ indicate the solid and pore parts of $\partial Y$. It is also noted that $\partial V_s$ consists of $\partial Y_s$ and $\partial V_{po}$, where $\partial V_{po}$ denotes the interfacial boundary between $V_s$ and $V_{po}$ (Fig. 1(b)).

We suppose that pore pressure uniformly acts in $V_{po}$, and the medium in $V_s$ has neither rigidity nor viscosity. Let $p$ be the pore pressure in $V_{po}$:

$$\mathbf{\sigma} = -p \mathbf{I} \text{ in } V_{po},$$

where $\mathbf{\sigma}$ denotes microstress in $V$.

For the region $V_s$, considering metallic base solids at small strains and rotations, we assume that microstrain $\mathbf{\varepsilon}$ is additively decomposed into elastic and viscoplastic parts:

$$\mathbf{\varepsilon} = \mathbf{\varepsilon}_e + \mathbf{\varepsilon}_{vp} \text{ in } V_s.$$ 

We further assume that the elastic part $\mathbf{\varepsilon}_e$ and the viscoplastic part $\mathbf{\varepsilon}_{vp}$ obey the simple constitutive relations expressed as:

$$\mathbf{\varepsilon}_e = \frac{1 + \nu}{E} \mathbf{\sigma} - \frac{\nu}{E} (\text{tr } \mathbf{\sigma}) \mathbf{I},$$

$$\mathbf{\varepsilon}_{vp} = \frac{3}{2} \left( \frac{\sigma_{eq}}{\sigma_0} \right)^{n-1} \frac{\mathbf{\sigma}_d}{\sigma_0},$$

where $E$ and $\nu$ are elastic constants, tr denotes the trace, the superscript dot indicates differentiation with respect to time $t$, $\sigma_0$, $\sigma_0$ and $n$ are the material parameters of viscoplasticity, $\mathbf{\sigma}_d$ denotes the deviatoric part of $\mathbf{\sigma}$, and $\sigma_{eq}$ expresses the von Mises equivalent stress defined as

$$\sigma_{eq} = \left( \frac{3}{2} \mathbf{\sigma}_d \cdot \mathbf{\sigma}_d \right)^{1/2}.$$

Eq. (4) is appropriate for metallic solids at high temperatures.

### 3. Micro–macro relations

This section describes micro–macro relations to define macrostress and macrostrain for the periodic unit cell $Y$ illustrated in Section 2. Terzaghi’s effective stress is also described using Hill’s macrohomogeneity condition (Hill, 1967), which is a micro–macro equation concerned with mechanical work in composites.

#### 3.1. Macrostress and macrostrain

The macrostress $\mathbf{\Sigma}$ of $Y$ is conventionally defined to be the volume average of microstress $\mathbf{\sigma}$ in $Y$:

$$\mathbf{\Sigma} = \frac{1}{|Y|} \int_Y \mathbf{\sigma} \, dV,$$

where $|Y|$ indicates the volume of $Y$.

For the macrostrain $\mathbf{E}$, however, it is necessary to note that displacement is arbitrary and can be discontinuous in $V_{po}$ because there is neither rigidity nor viscosity in $V_{po}$. Hence, $\mathbf{E}$ cannot be the volume average of microstrain $\mathbf{\varepsilon}$ in $Y$ (Suquet, 1987; Michel et al., 1999). To represent $\mathbf{E}$ in terms of the displacement field
\[ u(x) \text{ in } V_s, \text{ we use the following relations that are valid for periodic solids (Suquet, 1987; Wu and Ohno, 1999):} \]

\[ u = H \cdot x + \ddot{u}, \quad \ddot{\epsilon} = E + \ddot{\varepsilon} \text{ in } V_s \]

\[ E = \frac{1}{2} \left( H + H^T \right), \quad \ddot{\varepsilon} = \frac{1}{2} \left( \frac{\partial \ddot{u}}{\partial x} + \left( \frac{\partial \ddot{u}}{\partial x} \right)^T \right). \]

Here \( H \) denotes the macroscopic displacement gradient, \( x \) is the position of a point, the superscript \( T \) indicates the transpose, and \( u \) denotes the perturbed part of \( u \) that satisfies the \( Y \)-periodic boundary condition:

\[ \ddot{u}(x^{(i)}) = \ddot{u}(x^{(i)}) \]

where \( x^{(i)} \) and \( x^{(i)} \) are a pair of points on the opposite boundary planes of \( Y \) (Fig. 1(b)).

Integrating \( u \otimes n \) on \( \partial Y_s \) using Eqs. (7) and (9), and noting that \( n(x^{(i)}) = -n(x^{(i)}) \), we derive

\[ \int_{\partial Y_s} u \otimes n \, ds = H \cdot G. \]

where \( n \) denotes the outward unit normal to \( \partial Y \), and

\[ G = \int_{\partial Y_s} x \otimes n \, ds. \]

Eqs. (8) and (10) then provide the following relation, in which \( E \) is expressed in terms of the displacement field \( u(x) \) on \( \partial Y_s \):

\[ E = \frac{1}{2} \int_{\partial Y_s} u \otimes n \, ds - G^{-1} + G^{-1} \cdot \int_{\partial Y_s} n \otimes u \, ds. \]

If \( V_{s0} \) has no intersection with \( \partial Y \) (i.e., \( \partial Y_s = \partial Y \)), the divergence theorem allows Eq. (11) to become \( G = |Y| \); consequently, Eq. (12) reduces to a conventional definition of \( E \):

\[ E = \frac{1}{2 |Y|} \int_{\partial Y} (u \otimes n + n \otimes u) \, ds. \]

If \( u \) is differentiable with respect to \( x \) everywhere in \( Y \), Eq. (13) further reduces to

\[ E = \frac{1}{|Y|} \int_{Y} \ddot{\varepsilon} \, dv. \]

3.2. Terzaghi's effective stress

As a consequence of Hill’s macrohomogeneity condition (Hill, 1967), one can state that the macroscopic work rate due to \( \Sigma \) and \( E \) is equal to the volume average of microscopic work rate in \( Y \). Here it is necessary to note that the microscopic work in \( V_{s0} \) is done by the pore pressure \( p \) and the volume change in \( V_{s0} \). Thus, Hill’s macrohomogeneity condition provides

\[ \Sigma : E = \frac{1}{|Y|} \int_{V_s} \ddot{\varepsilon} \, dv - \frac{d}{dt} \left( |Y| - |V_s| \right) \]

where the volume of \( V_{s0} \) is taken to be equal to \( |Y| - |V_s| \). Here, \( |V_s| \) denotes the volume of \( V_s \), and

\[ \frac{d}{dt} \left( |Y| - |V_s| \right) = \text{tr}(\dot{E}) |Y| - \int_{V_s} \ddot{\varepsilon} \, dv. \]

Substituting Eq. (16) into Eq. (15) leads to introducing Terzaghi’s effective stress \( \Sigma + pl \) as

\[ \Sigma + pl : E = \frac{1}{|Y|} \int_{V_s} (\sigma + pl) : \ddot{\varepsilon} \, dv. \]

The above micro–macro relation is proved similarly to that given for periodic composites by Suquet (1987), as described in the Appendix A.

Now we assume that the deformation in \( V_s \) is volume-incompressible; i.e., \( \ddot{\varepsilon} = \ddot{\varepsilon}_s \) in \( V_s \), where \( \ddot{\varepsilon}_s \) is the deviatoric part of \( \ddot{\varepsilon} \). Eq. (17) then reduces to

\[ (\Sigma + pl) : E = \frac{1}{|Y|} \int_{V_s} \sigma_{d} : \ddot{\varepsilon}_{d} \, dv. \]

Hence, Terzaghi’s effective stress \( \Sigma + pl \) and the macrostrain rate \( \dot{E} \) are a work-conjugate pair for the volume-incompressible deformation in \( V_s \). Coussy (2010) described this work-conjugate pair by taking account of a solid–fluid interaction due to pore pressure; however, he did not present such a proof as given in the Appendix A, though Hill’s macrohomogeneity condition requires a proof (Hill, 1967; Suquet, 1987).

4. Constitutive features for the viscoplastic macrostrain rate

In this section, using the micro–macro relations described in Section 3, constitutive features are discussed for the viscoplastic macrostrain rate of pore-pressurized anisotropic open-porous bodies. For this purpose, we consider a steady state in which \( \sigma = 0 \) everywhere in \( Y \) and consequently \( \Sigma = 0 \). Then, since \( \ddot{\varepsilon}_s = 0 \) and \( \ddot{\varepsilon} = \ddot{\varepsilon}_p \) owing to \( \sigma = 0 \) everywhere in \( V_s \), the macro-strain rate \( E \) is regarded as completely viscoplastic; i.e., \( E = E_{vp} \). This is convenient for discussing constitutive features of the viscoplastic macrostrain rate. Let the steady state have the microscopic fields, macrostress, and macrostrain rate expressed as

\[ \sigma = \sigma'(x), \quad \ddot{\varepsilon} = \ddot{\varepsilon}'(x), \quad \dot{u} = \dot{u}'(x) \text{ in } V_s, \]

\[ \sigma = -p' \text{ in } V_{s0}, \]

\[ \Sigma = \Sigma', \quad E = E'; \]

where the superscript * indicates the steady state.

Eq. (17) suggests superposing \( p' \text{I} \) uniformly on the above stress state:

\[ \sigma = \sigma'(x) + p' \text{I} \text{ in } V_s, \quad \sigma = 0 \text{ in } V_{s0}, \quad \Sigma = \Sigma' + p' \text{I}. \]

Since \( \sigma'(x) + p' \text{I} \) has the same deviatoric part as \( \sigma'(x) \), the uniform superposition of \( p' \text{I} \) does not affect \( \ddot{\varepsilon}'(x), \dot{u}'(x) \) and \( E' \) in the steady state if \( E_{vp} \) is insensitive to hydrostatic pressure. Here it is noted that \( \ddot{\varepsilon}'(x) = \dot{e}_{vp}(x) \) in \( V_s \), and that \( E' \) depends only on the velocity field \( \dot{u}'(x) \) in \( V_s \), as seen from Eq. (12). Therefore, one can conclude that the effect of \( -p' \text{I} \) on \( E' := E_{vp} \) appears through Terzaghi’s effective stress \( \Sigma' + p' \text{I} \) if \( E_{vp} \) does not depend on hydrostatic pressure.

Applying Eq. (18) to the steady state expressed as Eqs. (19a)–(19c), noting that \( \dot{e}_{vp}(x) = \dot{e}_{vp}(x) \) in \( V_s \) and bearing in mind that \( E' \) is regarded as completely viscoplastic (i.e., \( E' = E_{vp} \)), we obtain

\[ (\Sigma' + p' \text{I}) : E_{vp} = \frac{1}{|Y|} \int_{V_s} \sigma_{d} : \ddot{\varepsilon}_{d} \, dv. \]

Substituting Eq. (4) into Eq. (21) gives

\[ (\Sigma' + p' \text{I}) : E_{vp} = \frac{1}{|Y|} \int_{V_s} \dot{w}_{vp} \, dv. \]

\[ \dot{w}_{vp} = \sigma_{d}(\sigma_{eq} \sigma_0)^{n+1}, \]

where \( \dot{w}_{vp} \) denotes the viscoplastic work rate in \( V_s \). Since \( \dot{w}_{vp} \geq 0 \), we have

\[ (\Sigma' + p' \text{I}) : E_{vp} \geq 0. \]

Let us consider another steady state that has

\[ \sigma = \sigma'(x) \text{ in } V_s, \quad \sigma = -cp' \text{I} \text{ in } V_{s0}, \quad \Sigma = \Sigma', \]

where \( c \) is an arbitrary constant. Then, since \( \dot{w}_{vp} \) is multiplied by \( c^{n+1} \), Eq. (22) is satisfied irrespective of the value of \( c \) if \( E_{vp} \) is replaced by \( c \dot{E}_{vp} \). This means that \( E_{vp} \) becomes \( c \dot{E}_{vp} \) if Terzaghi’s effective stress \( \Sigma' + p' \text{I} \) is multiplied by \( c \).
5. Macroscopic constitutive model

Tsuda et al. (2010) developed a macroscopic constitutive model for the homogenized elastic-viscoplastic behavior of anisotropic porous bodies at small strains and rotations in the absence of pore pressure, as stated in Section 1. In the present section, their constitutive model is extended on the basis of the constitutive features found in Section 4.

We assume that the macrostrain \( \mathbf{E} \) of \( Y \) is additively decomposed into elastic and viscoplastic parts:

\[
\mathbf{E} = \mathbf{E}_e + \mathbf{E}_{vp}.
\]

(25)

We further assume that the elastic part \( \mathbf{E}_e \) is related to the macrostress \( \Sigma \) by the Hooke–Biot law (Biot, 1941; Dormieux et al. 2006):

\[
\Sigma = \mathbb{D}_{el} : \mathbf{E}_e - p\mathbf{B}.
\]

(26)

where \( \mathbb{D}_{el} \) is the fourth-rank tensor standing for the homogenized elastic stiffness of \( Y \), and \( \mathbf{B} \) is the second-rank tensor called Biot’s coefficient. Dormieux et al. (2002, 2006) and Vincent et al. (2009a) showed that \( \mathbf{B} \) is analytically expressed as

\[
\mathbf{B} = (1 - \mathbb{D}_{el} : \mathbb{D}_{el}^T) : \mathbf{I}.
\]

(27)

where \( \mathbb{D}_{el} \) indicates the elastic stiffness of base solids.

To develop a constitutive relation for \( \mathbf{E}_{vp} \), let us remember that the effect of pore pressure \( p \) on \( \mathbf{E}_{vp} \) appears through Terzaghi’s effective stress \( \Sigma + pl \) in steady states (Section 4). Taking this into account, we consider the following type of quadratic anisotropic elastic stiffness of \( \mathbb{R} \), and Xue and Hutchinson (2004):

\[
\Sigma_{eq} = \left[ \frac{3}{2} (\Sigma + pl) : \mathbf{M} : (\Sigma + pl) \right]^{1/2},
\]

(28)

where \( \mathbf{M} \) is a positive-definite symmetric fourth-rank tensor. Then, assuming the normality of \( \mathbf{E}_{vp} \) to a viscoplastic potential \( f = \Sigma_{eq}^2 \) leads to

\[
\mathbf{E}_{vp} = \frac{\partial f}{\partial \Sigma_{eq}} = 3j_{eq} \mathbf{M} : (\Sigma + pl),
\]

(29)

where \( j \) is a scalar function of \( \Sigma + pl \), and

\[
\frac{\partial f}{\partial \Sigma} = \frac{\partial f}{\partial \Sigma_{eq}}.
\]

(30)

To specify the scalar function \( j \), we note that, in steady states, \( \mathbf{E}_{vp} \) becomes \( c^e \mathbf{E}_{0p} \) if Terzaghi’s effective stress \( \Sigma + pl \) is multiplied by an arbitrary constant \( c \) (Section 4). This constitutive feature is satisfied if \( \mathbf{E}_{vp} \) expressed by Eq. (29) has the same stress exponent \( n \) as that in Eq. (4). Eq. (29) can thus have the following form, which is an extension of that derived by Tsuda et al. (2010):

\[
\mathbf{E}_{vp} = \mathbf{E}_0 \left( \frac{\Sigma_{eq}}{\Sigma_0} \right)^{n-1} \mathbf{M} : (\Sigma + pl),
\]

(31)

where \( \mathbf{E}_0 \) and \( \Sigma_0 \) are a reference macrostrain and a reference macrostress. In Eq. (31), \( \mathbf{E}_0 \) and \( \Sigma_0 \) can be arbitrarily chosen without loss of generality, because \( \mathbf{M} \) has an arbitrary proportional coefficient. Needless to say, Eqs. (31) and (28) must reduce to Eqs. (4) and (5), respectively, in the absence of the pore region \( V_{vp} \). It is seen that Eq. (28) becomes identical to Eq. (5) if \( M = I_4 \), where \( I_4 \) is the deviatoric operator defined as \( I_4 = I - 1/3 \mathbf{I} \otimes \mathbf{I} \). Therefore, Eq. (31) with \( M \) replaced by \( I_4 \) must reduce to Eq. (4). Eq. (31) thus becomes

\[
\mathbf{E}_{vp} = \frac{3}{2} \mathbf{E}_0 \left( \frac{\Sigma_{eq}}{\Sigma_0} \right)^{n-1} \mathbf{I} : (\Sigma + pl),
\]

(32)

where \( \mathbf{I} \) and \( \Sigma_0 \) are the same material parameters as those in Eq. (4).

Condition (23) is always satisfied because the mechanical work rate due to \( \Sigma + pl \) and \( \mathbf{E}_{vp} \) is shown to be nonnegative using Eqs. (32) and (28):

\[
(\Sigma + pl) : \mathbf{E}_{vp} = \sigma_0 \mathbf{I} : \left( \frac{\Sigma_{eq}}{\Sigma_0} \right)^{n-1} \geq 0.
\]

(33)

As seen from Eq. (32), \( \mathbf{E}_{vp} \) vanishes if \( \Sigma = -pl \). This feature was discussed to briefly show Eq. (32) in a preliminary short paper (Ikenoya and Ohno, 2011), which did not describe at all the theoretical considerations given in Sections 3 and 4.

6. Plate-fin simulation and analysis

The macroscopic viscoplastic constitutive relation (32) includes the effect of pore pressure \( p \) acting through Terzaghi’s effective stress \( \Sigma + pl \), which has no anisotropic factor and no fitting parameter. As a result, the anisotropic effect of \( p \) on \( \mathbf{E}_{vp} \) appears only through the fourth-rank anisotropic tensor \( \mathbf{M} \) in Eq. (32). To verify this consequence, the macroscopic constitutive model developed in Section 5 is examined by simulating the homogenized elastic-viscoplastic behavior of a pore-pressurized ultrafine plate-fin structure subjected to uniaxial/biaxial loading (Fig. 2). For comparison, the homogenized behavior is also analyzed by performing a detailed finite element analysis.

The macroscopic constitutive model developed has no fitting parameter to simulate the effect of pore pressure \( p \), as stated above. Therefore, all material parameters including \( \mathbb{D}_{el} \) and \( \mathbf{M} \) in the constitutive model are first determined in the absence of \( p \), and then the constitutive model is predictively validated in the presence of \( p \).

6.1. Periodic unit cell

Fig. 2 illustrates the plate-fin unit cell \( Y \) considered in this study. The base solid is Hasselot X at 900 °C, which has the material parameters listed in Table 1. The unit cell \( Y \) is subjected to pore pressure of \( p = 5 \) MPa (Kawashima et al., 2007). Tsuda et al. (2010) computationally analyzed the homogenized elastic-viscoplastic behavior of this unit cell in the absence of \( p \) using a fully implicit homogenization method (Asada and Ohno, 2007). They thus obtained the components of \( \mathbb{D}_{el} \) and \( \mathbf{M} \) given in Table 2, where \( \mathbb{D}_{el} \) and \( \mathbf{M} \) are expressed as 6 \times 6 matrices using Voigt’s notation. Here it is noted that Tsuda et al. (2010) used Eq. (28) with 3/2 replaced by 1/2 to define \( \Sigma_{eq} \) in the case of \( p = 0 \); as a result, the components of \( \mathbf{M} \) in Table 2 differ by a factor of 3 from those reported by Tsuda et al. (2010). Hereafter, Cartesian coordinates \( x, y \) and \( z \) are set for \( Y \), as shown in Fig. 3.

Buckling of plates and fins is not considered in accordance with the assumption of small deformation in this paper, though
solid region loading history using Abaqus. The history is simulated by subjecting one cubic finite element to this elastic–viscoplastic behavior of Yulu derived by Tsuda et al. (2010). As a result, the homogenized based on Voigt’s notation with component order of $\Gamma_{xx}$.

6.2. Methods of simulation and analysis

The homogenized elastic–viscoplastic behavior of the plate-fin unit cell Y is simulated using the macroscopic constitutive model described in Section 5. For this purpose, the constitutive model has been implemented in a finite element code Abaqus with a user subroutine UMAT by taking into account pore pressure p in the implicit stress integration algorithm and the consistent tangent modulus derived by Tsuda et al. (2010). As a result, the homogenized elastic–viscoplastic behavior of Y under any macroscopic loading history is simulated by subjecting one cubic finite element to this loading history using Abaqus.

To perform a finite element homogenization analysis, the base solid region $V_o$ in the periodic unit cell Y is divided into 20-node quadratic brick, reduced integration finite elements (Fig. 3). To compute the components of macrostress easily, thin flexible films are fictitiously placed at the lateral boundary of $V_o$ (Fig. 4). The thin flexible films are modeled using elastic shell elements with a very small thickness of 0.01 mm and an extremely low Young’s modulus of 1.0 MPa. The pore region $V_o$ inside the fictitious thin films does not need any constitutive model, and is not divided into finite elements. This is because $V_o$ is assumed to have only pore pressure $p$ because of neither rigidity nor viscosity in $V_o$ (Section 2). The inner surfaces of fictitious thin films and the interfacial boundary surfaces of $V_o$ are then subjected to $p$, as shown in Fig. 4(a). The finite element homogenization analysis is performed using Abaqus: the following periodic boundary condition, which is derived from Eqs. (7) and (9) (Feyel and Chaboche, 2000; Xia et al., 2003), is imposed using the Equation command available in Abaqus:

$$u(x^{(+)}) - u(x^{(-)}) = H(\mathbf{x}^{(+)}) - H(\mathbf{x}^{(-)})$$ on $\partial V_o$.

where $\mathbf{x}^{(+)}$ and $\mathbf{x}^{(-)}$ indicate a pair of points on the opposite boundary planes of Y (Fig. 4(a)). The above periodic boundary condition is applied to the fictitious thin films as well as to the solid part $\partial V_o$ of $\partial Y$. The fictitious thin films then allow the components of macrostress to be easily computed from the resultant forces acting on the rectangular boundary planes of Y.

6.3. Comparison of simulation and analysis results

Figs. 5 and 6 show the effects of pore pressure $p$ on the macrostress versus macrostrain relations under uniaxial loading at a macrostress rate of $10^{-4}$ s$^{-1}$. The periodic unit cell Y is subjected to pore pressurization before uniaxial loading when $p = 5$ MPa. The pore pressurization is performed almost instantaneously under the condition of free macrostress, resulting in a macrostress $\mathbf{E}^{(p)}$ before uniaxial loading. The variation in macrostrain $\mathbf{E}$ from $\mathbf{E}^{(p)}$ (i.e., $\mathbf{E} - \mathbf{E}^{(p)}$) is used for the abscissa axes in Figs. 5 and 6.

The macrostress $\mathbf{E}^{(p)}$ is almost equal to the macrostrain evaluated from Eq. (26) with $\Sigma = 0$:

$$\mathbf{E}^{(p)} = p\mathbf{D}_{eff}^{-1} : \mathbf{B}.$$  

It is observed from Fig. 5 that the effect of $p = 5$ MPa on the homogenized behavior is considerably anisotropic: the pore pressure noticeably affects the homogenized behavior under uniaxial tensile loading in the y-direction (Fig. 5(b)), whereas the effect is negligible with respect to the x-direction (Fig. 5(a)). Tension-compression asymmetry is also noticeable in the homogenized behavior under uniaxial loading in the y-direction (Fig. 6(b)), whereas the asymmetry in the x-direction is very small (Fig. 6(a)). Here it is emphasized that the macroscopic constitutive model developed in Section 5 simulates very well the above-mentioned anisotropic/asymmetric effects of $p$ that have been revealed in the present finite element homogenization analysis (Figs. 5 and 6).

Fig. 7 illustrates the steady-state macrostress surfaces attained under the following biaxial loading condition in the presence of $p = 0$ and $p = 5$ MPa:

$$\mathbf{E}_{yy} = \mathbf{E}_{zz} \cos \theta, \quad \mathbf{E}_{yy} = \mathbf{E}_{zz} \sin \theta, \quad \Sigma_{xy} = 0.$$  

where $\mathbf{E}_c$ and $\theta$ are loading parameters, and $\Sigma_{xx} = \Sigma_{zz} = \Sigma_{yy} = 0$. As seen from the figure, the steady-state macrostress surfaces at $\mathbf{E}_c = 10^{-4}$ and $10^{-5}$ s$^{-1}$ are almost equally translated by the pore pressure of $p = 5$ MPa, and the translation occurs mainly along the $\Sigma_{yy}$-axis. These effects of $p$ are simulated well by the macroscopic constitutive model described in Section 5.

Fig. 8 compares the macrostress trajectories provided by the homogenization analysis and the macroscopic constitutive model under the biaxial loading condition (36) combined with $p = 5$ MPa. It is observed from the figure that the macroscopic constitutive model is able to reproduce the macrostress trajectories predicted by the homogenization analysis. However, when $\theta = 0$, $\pi/4$ and $5\pi/4$, there are some deviations as macrostress approaches the steady-state surfaces. This is because the steady-state surfaces

---

**Table 1**

<table>
<thead>
<tr>
<th>Material parameters of Hastelloy X at 900 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Young’s modulus $E$ (GPa)</strong></td>
</tr>
<tr>
<td><strong>Poisson’s ratio $\nu$</strong></td>
</tr>
<tr>
<td><strong>Reference strain rate $\dot{e}_0$ (s$^{-1}$)</strong></td>
</tr>
<tr>
<td><strong>Reference stress $\sigma_0$ (MPa)</strong></td>
</tr>
<tr>
<td><strong>Stress exponent $n$</strong></td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>Components of $D_{eff}$ and $M$ for a plate-fin unit cell (Tsuda et al., 2010): $6 \times 6$ matrices based on Voigt’s notation with component order of $xx, yy, zz, xy, yz$ and $zx$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_{eff}$ = $E$</td>
</tr>
<tr>
<td>[0.480 0.088 0.170 0.0 0.0 0.0]</td>
</tr>
<tr>
<td>[0.088 0.159 0.074 0.0 0.0 0.0]</td>
</tr>
<tr>
<td>[0.170 0.074 0.606 0.0 0.0 0.0]</td>
</tr>
<tr>
<td>[0.0 0.0 0.0 0.036 0.0 0.0]</td>
</tr>
<tr>
<td>[0.0 0.0 0.0 0.0 0.079 0.0]</td>
</tr>
<tr>
<td>[0.0 0.0 0.0 0.0 0.0 0.178]</td>
</tr>
<tr>
<td>$M$ =</td>
</tr>
<tr>
<td>[2.81 -1.82 -0.97 0.0 0.0 0.0]</td>
</tr>
<tr>
<td>[-1.82 13.23 -0.97 0.0 0.0 0.0]</td>
</tr>
<tr>
<td>[-0.97 -0.97 1.94 0.0 0.0 0.0]</td>
</tr>
<tr>
<td>[0.0 0.0 0.0 78.05 0.0 0.0]</td>
</tr>
<tr>
<td>[0.0 0.0 0.0 0.0 44.28 0.0]</td>
</tr>
<tr>
<td>[0.0 0.0 0.0 0.0 0.0 7.08]</td>
</tr>
</tbody>
</table>
predicted by the homogenization analysis have ‘noses’ at \( \theta = \pi/4 \) and \( 5\pi/4 \), while those simulated by the macroscopic constitutive model are elliptic as a consequence of the quadratic equivalent stress represented as Eq. (28). Nevertheless, the homogenization analysis and the macroscopic constitutive model have less than 10% difference with respect to the magnitude of macrostress.
We have seen in Figs. 5–8 that the anisotropic effects of $p$ revealed in the homogenization analysis are simulated well using the macroscopic constitutive model developed in Section 5. Here it is restated that this constitutive model represents the anisotropic effects of $p$ on $E_p$ only through the fourth-rank anisotropic tensor $\mathbf{M}$ in Eq. (32) because Terzaghi’s effective stress $\Sigma + p\mathbf{I}$ has no anisotropic factor. It is also restated that the constitutive model has no fitting material parameter to simulate the effects of $p$. Hence, we can say that the macroscopic constitutive model has been predictively verified with respect to the anisotropic effects of $p$ found in the finite element homogenization analysis presented in this section.

7. Concluding remarks

In this paper, constitutive modeling was studied for the homogenized elastic–viscoplastic behavior of pore-pressurized anisotropic open-porous bodies made of metallic materials at small strains and rotations. For this purpose, micro–macro relations relevant to periodic unit cells of anisotropic open-porous bodies subjected to pore pressure were described: macrostress was newly represented in terms of base solid displacements, and Terzaghi’s effective stress was simply introduced using Hill’s macrohomogeneity condition. Constitutive features were then discussed for the viscoplastic macrostress rate in steady states. It was thus plainly shown that the viscoplastic macrostress rate depends on the pore pressure acting through Terzaghi’s effective stress if the viscoplastic strain rate of base solids is insensitive to hydrostatic pressure. The viscoplastic macrostress rate was then represented as an anisotropic function of Terzaghi’s effective stress, resulting in the macroscopic viscoplastic constitutive relation (32).

It was found that the anisotropic effect of pore pressure on the viscoplastic macrostress rate appears only through a fourth-rank relation (32) because Terzaghi’s effective stress $\Sigma + p\mathbf{I}$ has no anisotropic factor and no fitting parameter. To verify this consequence, anisotropic function of Terzaghi’s effective stress, resulting in the macroscopic constitutive model developed in Section 5. Here it is restated that this constitutive model represents the anisotropic effects of $p$ on $E_p$ only through the fourth-rank anisotropic tensor $\mathbf{M}$ in Eq. (32) because Terzaghi’s effective stress $\Sigma + p\mathbf{I}$ has no anisotropic factor. It is also restated that the constitutive model has no fitting material parameter to simulate the effects of $p$. Hence, we can say that the macroscopic constitutive model has been predictively verified with respect to the anisotropic effects of $p$ found in the finite element homogenization analysis presented in this section.

Appendix A. Proof of Eq. (17)

Substituting Eqs. (7) and (8) into Eq. (17) leads to

$$\mathbf{(\Sigma + p\mathbf{I}): E} = \frac{1}{V} \int_{V_s} (\sigma + p\mathbf{I}) : \left( \mathbf{E} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right) \, dV.$$  \hfill (A1)

Since $\text{div}(\sigma + p\mathbf{I}) = 0$ in $V_s$, we have

$$\int_{V_s} (\sigma + p\mathbf{I}) : \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \, dV = \int_{V_s} \mathbf{v} \cdot (\sigma + p\mathbf{I}) \cdot \mathbf{u} \, dS.$$  \hfill (A2)

where $\mathbf{v}$ indicates the outward unit normal to $\partial V_s$. It is recalled that the boundary of $V_s$ consists of $\partial V_s$ and $\partial V_{leo}$ (Fig. 1b). The right-hand side in Eq. (A2) vanishes, because Eq. (9) and $(\sigma + p\mathbf{I}) \cdot \mathbf{v} = 0$ are satisfied on $\partial V_s$ and $\partial V_{leo}$ respectively. Hence, Eq. (A1) becomes

$$\mathbf{(\Sigma + p\mathbf{I}): E} = \frac{1}{V} \int_{V_s} (\sigma + p\mathbf{I}) dV : \mathbf{E}.$$  \hfill (A3)

Since $\sigma + p\mathbf{I} = 0$ in $V_{leo}$ (Eq. (1)), Eq. (A3) is satisfied if $\Sigma$ is defined as Eq. (6).

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References


