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Application of $(\frac{G'}{G})$-expansion method to Regularized Long Wave (RLW) equation

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ABSTRACT

In this paper, the $(\frac{G'}{G})$-expansion method with the aid of Maple is used to obtain a generalized soliton solution for the generalized Regularized Long Wave (RLW) equation. Each of the obtained solutions, namely hyperbolic function solutions and trigonometric function solutions contain an explicit linear function of the variables in the considered equation. It is shown that the proposed method provides a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering problems.

1. Introduction

The world around us has been inherently nonlinear. For instance, nonlinear evolution equations (NLEEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and biology. One of the basic physical problems for those models is to obtain their traveling wave solutions. Particularly, various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear PDEs. In the numerical methods [1], stability and convergence should be considered, so as to avoid divergent or inappropriate results. However, in recent years, a variety of effective analytical and semi analytical methods have been developed considerably to be used for nonlinear PDEs such as the homotopy perturbation method [2–4], the variational iteration method [5–7], the parameter-expansion method [8], the Exp-function method [9–17], the inverse scattering method [18], the sine–cosine method [19,20], the extended tanh-method [21,22], and others.

Recently, the $(\frac{G'}{G})$-expansion method, firstly introduced by Wang et al. [23], has become widely used to search for various exact solutions of NLEEs [23–30]. The value of the $(\frac{G'}{G})$-expansion method is that one treats nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second-order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation.

The aim of this paper is to apply the $(\frac{G'}{G})$-expansion method [30] to find new hyperbolic and trigonometric solutions of the Regularized Long wave (RLW) equation [16]:

$$u_t + u_x + a(u^2)_x - bu_{xxt} = 0 \quad a, b \in R. \tag{1}$$

2. Application of the $(\frac{G'}{G})$-expansion method for the RLW equation

In this section, we apply the proposed method to obtain new and more general exact solutions of Eq. (1), which arises in several physical applications including ion sound waves in a plasma.

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Let us assume the traveling wave solution of Eq. (1) in the form

$$u = U(\xi), \quad \xi = kx + \omega t, \tag{2}$$

where k, ω are arbitrary constants. Using the wave variable (2), the Eq. (1) is carried to ordinary differential equation (ODE)

$$(\omega + k)U' + 2akUU' - bk^2\omega U''' = 0. \tag{3}$$

Integrating Eq. (3) once with respect to ξ and setting the integration constant as zero, we obtain

$$(\omega + k)U + aU^2 - bk^2\omega^2 U'' = 0. \tag{4}$$

Suppose that the solution of the ODE (4) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U(\xi) = \sum_{i=1}^{m} \alpha_i \left(\frac{G'}{G}\right)^i + \alpha_0, \quad \alpha_m \neq 0, \tag{5}$$

where α_0, and α_i, are constants to be determined later, $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{\mathrm{d}^2 G(\xi)}{\mathrm{d}\xi^2} + \lambda \frac{\mathrm{d}G(\xi)}{\mathrm{d}\xi} + \mu G(\xi) = 0, \tag{6}$$

where λ and μ are arbitrary constants. The positive integer m can be determined by considering the homogeneous balance between the highest order derivative u'' and nonlinear term u^2 appearing in (4).

$$m + 2 = 2m, \tag{7}$$

so that

$$m = 2. \tag{8}$$

We then suppose that Eq. (4) has the following formal solutions:

$$U = \alpha_2 \left(\frac{G'}{G}\right)^2 + \alpha_1 \left(\frac{G'}{G}\right) + \alpha_0, \quad \alpha_2 \neq 0, \tag{9}$$

where α_2, α_1, and α_0, are positive integers which are unknown to be determined later.

Substituting Eq. (9) along with Eq. (6) into Eq. (4) and collecting all the terms with the same power of $(\frac{G'}{G})$ together, equating each coefficient to zero, yields a set of simultaneous algebraic equations for k, ω, α_0, α_1, and α_2, as follows:

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^0 : bk^2\omega\alpha_1\lambda^2\mu - \omega\alpha_1\mu - k\alpha_1\mu - 2ka\alpha_0\alpha_1\mu + 6bk^2\omega\alpha_2\lambda\mu^2 + 2bk^2\omega\alpha_1\mu^2 = 0,\\
&\left(\frac{G'}{G}\right)^1 : -2\omega\alpha_2\mu + 14bk^2\omega\alpha_2\lambda^2\mu + bk^2\omega\alpha_1\lambda^3 - k\alpha_1\lambda - \omega\alpha_1\lambda - 2k\alpha_2\mu - 2ka\alpha_1^2\mu\\
&\quad - 4ka\alpha_0\alpha_2\mu - 2ka\alpha_0\alpha_1\lambda + 16bk^2\omega\alpha_2\mu^2 + 8bk^2\omega\alpha_1\lambda\mu = 0,\\
&\left(\frac{G'}{G}\right)^2 : 7bk^2\omega\alpha_1\lambda^2 - 2k\alpha_2\lambda - \omega\alpha_1 - k\alpha_1 - 2\omega\alpha_2\lambda - 6ka\alpha_2\alpha_1\mu - 2ka\alpha_1^2\lambda - 4ka\alpha_0\alpha_2\lambda\\
&\quad - 2ka\alpha_0\alpha_1 + 8bk^2\omega\alpha_2\lambda^3 + 8bk^2\omega\alpha_1\mu + 52bk^2\omega\alpha_2\lambda\mu = 0,\\
&\left(\frac{G'}{G}\right)^3 : -2ka\alpha_1^2 - 2\omega\alpha_2 - 2k\alpha_2 - 4ka\alpha_2^2\mu - 6ka\alpha_2\alpha_1\lambda - 4ka\alpha_0\alpha_2 + 12bk^2\omega\alpha_1\lambda\\
&\quad + 40bk^2\omega\alpha_2\mu + 38bk^2\omega\alpha_2\lambda^2 = 0,\\
&\left(\frac{G'}{G}\right)^4 : -4ka\alpha_2^2\lambda - 6ka\alpha_2\alpha_1 + 54bk^2\omega\alpha_2\lambda + 6bk^2\omega\alpha_1 = 0,\\
&\left(\frac{G'}{G}\right)^5 : -4ka\alpha_2^2 + 24bk^2\omega\alpha_2 = 0.
\end{aligned} \tag{10}$$

Solving the set of algebraic Eq. (10) by use of Maple, we get the following results:

$$\alpha_2 = \frac{6bk\omega}{a}, \qquad \alpha_1 = \frac{6bk\omega\lambda}{a}, \qquad \alpha_0 = \frac{bk^2\omega\lambda^2 - \omega - k + 8bk^2\omega\mu}{2ka}, \tag{11}$$

where k, ω, λ and μ are arbitrary constants. Substituting (11) into (9), we obtain

$$U=\frac{6bk\omega}{a}\left(\frac{G'}{G}\right)^2+\frac{6bk\omega\lambda}{a}\left(\frac{G'}{G}\right)+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka}. \tag{12}$$

Substituting the general solutions of (6) into Eq. (12), we obtain two types of traveling wave solutions of Eq. (1):

When $\lambda^2-4\mu>0$, we obtain hyperbolic function solutions:

$$\begin{aligned}u_{hypr}(x,t)=&\frac{6bk\omega}{a}\left(\frac{\sqrt{\lambda^2-4\mu}\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)}{2\left(C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)}-\frac{\lambda}{2}\right)^2\\&+\frac{6bk\omega\lambda}{a}\left(\frac{\sqrt{\lambda^2-4\mu}\left(C_1\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_2\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)}{2\left(C_2\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)+C_1\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)\right)}-\frac{\lambda}{2}\right)\\&+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka},\end{aligned} \tag{13}$$

where $\xi=kx+\omega t$, and C_1, C_2, are arbitrary constants.

When $\lambda^2-4\mu<0$, we obtain trigonometric function solutions:

$$\begin{aligned}u_{trig}(x,t)=&\frac{6bk\omega}{a}\left(\frac{\sqrt{4\mu-\lambda^2}\left(-C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)\right)}{2\left(C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)\right)}-\frac{\lambda}{2}\right)^2\\&+\frac{6bk\omega\lambda}{a}\left(\frac{\sqrt{4\mu-\lambda^2}\left(-C_1\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_2\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)\right)}{2\left(C_2\sin\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)+C_1\cos\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)\right)}-\frac{\lambda}{2}\right)\\&+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka},\end{aligned} \tag{14}$$

where $\xi=kx+\omega t$, and C_1, C_2, are arbitrary constants.

Now, to obtain some special cases of the above solutions, we set $C_2=0$, then hyperbolic function solution (13) becomes

$$u(x,t)=\frac{3bk\omega}{2a}\left((4\mu-\lambda^2)\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)-4\mu\right)+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka}, \tag{15}$$

where $\xi=kx+\omega t$, and λ, μ are arbitrary constants.

If we set $C_1=0$, then hyperbolic type solution (13) becomes

$$u(x,t)=\frac{3bk\omega}{2a}\left((\lambda^2-4\mu)\operatorname{csch}^2\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi\right)-4\mu\right)+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka}, \tag{16}$$

where $\xi=kx+\omega t$, and λ, μ are arbitrary constants.

Similarly, setting $C_2=0$, and using trigonometric function solution of (14), we have

$$u(x,t)=\frac{3bk\omega}{2a}\left((4\mu-\lambda^2)\sec^2\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)-4\mu\right)+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka}, \tag{17}$$

where $\xi=kx+\omega t$, and if we get again $C_1=0$, then Eq. (14) becomes

$$u(x,t)=\frac{3bk\omega}{2a}\left((4\mu-\lambda^2)\csc^2\left(\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi\right)-4\mu\right)+\frac{bk^2\omega\lambda^2-\omega-k+8bk^2\omega\mu}{2ka}, \tag{18}$$

where $\xi=kx+\omega t$, and λ, μ are arbitrary constants.

3. Conclusions

This study shows that the $(\frac{G'}{G})$-expansion method is quite efficient and practically well suited for use in finding exact solutions for the RLW equation. The reliability of the method and the reduction in the size of computational domain give this method a wider applicability. Though the obtained solutions represent only a small part of the large variety of possible solutions for the equations considered, they might serve as seeding solutions for a class of localized structures existing in the physical phenomena. Furthermore, our solutions are in more general forms, and many known solutions to these equations are only special cases of them. With the aid of Maple, we have assured the correctness of the obtained solutions by putting them back into the original equation. We hope that they will be useful for further studies in applied sciences.

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