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Bounded Solutions and Wavefronts for Discrete Dynamics

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Abstract—This paper deals with the second-order nonlinear difference equation

$$\Delta(r_k \Delta u_k) + q_k g(u_{k+1}) = 0,$$

where $\{r_k\}$ and $\{q_k\}$ are positive real sequences defined on $\mathbb{N} \cup \{0\}$, and the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is nonnegative and nontrivial. Sufficient and necessary conditions are given, for the existence of bounded solutions starting from a fixed initial condition u_0. The same dynamic, with f instead of g such that $uf(u) > 0$ for $u \neq 0$, was recently extensively investigated. On the contrary, our nonlinearity g is of a small appearance in the discrete case. Its introduction is motivated by the analysis of wavefront profiles in biological and chemical models. The paper emphasizes the many different dynamical behaviors caused by such a g with respect to the equation involving function f. Some applications in the study of wavefronts complete this work.

Keywords—Nonlinear difference equation, Bounded solution, Nonnegative nonlinearity, Discrete travelling wave solution.

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1. INTRODUCTION

Difference equations frequently arise in studying biological and chemical models, in the analysis of discrete systems, in the discretization methods for differential equations, etc. (for this purpose we refer to [1–5]). Therefore, the discussion on the existence, oscillatory, and asymptotic properties of their solutions has received considerable attention. Following this trend, in this paper, we investigate the second-order nonlinear difference equation

$$\Delta(r_k \Delta u_k) + q_k g(u_{k+1}) = 0, \tag{1}$$

where $\{r_k\}$ and $\{q_k\}$ are positive real sequences with $k \in \mathbb{N} \cup \{0\}$, Δ is the usual forward difference operator defined by $\Delta u_k = u_{k+1} - u_k$ and the nonlinearity $g : \mathbb{R} \to \mathbb{R}$ is nonnegative and nontrivial.

First note that equation (1) is in fact a recurrence relation, and hereby, the existence and uniqueness of its solution, for every initial value problem (IVP),

$$\begin{aligned} \Delta(r_k \Delta u_k) + q_k g(u_{k+1}) &= 0, \\ u_0 &= \alpha, \\ \Delta u_0 &= \beta, \end{aligned}$$

are guaranteed, for all $k \in \mathbb{N}$ and any real numbers α, β. We denote it by $u^{\alpha,\beta}$.

As it is known, several authors investigated the general difference equation

$$\Delta(r_k \varphi(\Delta u_k)) + f(k, u_{k+1}) = 0, \tag{2}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an homeomorphism such that $\varphi(0) = 0$ and $f : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ satisfies $f(n, \cdot)$ continuous, increasing and $f(n, 0) = 0$ for every $n \in \mathbb{N}$ (see, e.g., [6–15]). In these quoted papers, comparison results were obtained, as well as necessary and sufficient conditions, for the existence of solutions with a prescribed asymptotic behavior. Indeed, such a study was mainly devoted to the existence of positive monotone and bounded solutions of (2). We also recall [16], where a similar analysis was developed in the continuous case. Under some restrictions on φ and f, equation (2) becomes half-linear, i.e., its solution space is homogeneous. In [17], it was shown that its solutions behave, in many aspects, like those of linear difference equations.

A lot of further references can be found in [1], especially in Chapter VI. Note that our type of nonlinearity g differs from f and this brings quite different dynamical behaviors. Our term g derives its motivation from the study of wavefront profiles appearing in several biological and chemical models. As far as we know, this is the first investigation, in the discrete case, of such a nonlinearity.

The main aim of this work is to establish existence criteria of bounded increasing solutions of (1) with a fixed initial value. More precisely, we give sufficient conditions (and in some cases also necessary) in order that the IVP

$$\begin{aligned} \Delta(r_k \Delta u_k) + q_k g(u_{k+1}) &= 0, \\ u_0 &= \alpha, \end{aligned}$$

has a bounded increasing solution. According to our needs, we shall take

$$g \text{ bounded} \tag{3}$$

and/or

$$g \text{ continuous.} \tag{4}$$

We stress however that, for some results, even the continuity of g is not required. As we have already mentioned, this nonlinearity g makes our equation, although looking similarly, of a quite different type with respect to (2): in fact, no oscillatory solutions exist and the linear case is not included (see Section 2). It means that different kinds of approaches to its examination must be used. We employ, in particular, a shooting argument and Schauder's fixed-point theorem (see Theorem 7). Moreover, unlike in the quoted works concerning equation (2), where only the asymptotic behavior was investigated, our results take into account also the initial value of the solution.

In Section 2, we discuss all possible asymptotic behaviors of the solutions of (1). We also establish necessary conditions for the existence of bounded solutions in terms of the convergence/divergence of the series

$$\mathcal{S}_r = \sum_{j=0}^{\infty} \frac{1}{r_j}, \qquad \mathcal{S}_q = \sum_{j=0}^{\infty} q_j,$$

and

$$\mathcal{S}_{qr} = \sum_{j=0}^{\infty} q_j \sum_{i=0}^{j} \frac{1}{r_i}.$$

A similar approach was developed in many previous papers. In particular, the results in [8,9,14] were obtained according to the convergence or divergence of $\sum_{n=1}^{\infty} \varphi^{-1}(1/r_n)$. Similarly, under assumptions on the behavior of $\mathcal{S}_r$, in [10,12] the authors prove some comparison theorems and give necessary and sufficient conditions for the existence of a positive nondecreasing solution for a special form of (2). In Section 3, we consider the case when $\sum_{j=0}^{\infty} 1/r_j < \infty$, while Sections 4 and 5 treat the case $\sum_{j=0}^{\infty} 1/r_j = \infty$. In both situations, we obtain sufficient (and sometimes also necessary) conditions for the existence of bounded solutions of (1) with $u_0 = \alpha$ prescribed. When $\sum_{j=0}^{\infty} 1/r_j = \infty$ and $g(\alpha) = 0$, our existence criterion, contained in Section 5, is for the particular equation

$$\Delta^2 u_k - c_k \Delta u_k + b_k g(u_{k+1}) = 0, \tag{5}$$

where $\Delta^2 u_k = \Delta(\Delta u_k)$,

$$0 < c_k \le \bar{c}, \tag{6}$$

$$0 < \underline{b} \le b_k \le \bar{b}, \tag{7}$$

for all $k \in \mathbb{N} \cup \{0\}$, g is continuous and there exists $\theta \in (0,1)$ such that

$$g(t) > 0 \text{ for } t \in (\theta, 1) \text{ and } g \equiv 0 \text{ otherwise.} \tag{8}$$

Equation (5), but with coefficients $\{c_k\}_k$ and $\{b_k\}_k$ both constant, enables to study wavefront profiles of the reaction-diffusion equation

$$u_t = u_{xx} + g(u), \qquad t \ge 0, \quad x \in \mathbb{R} \tag{9}$$

(see, e.g., [4]). Of major interest, in this context, is to connect two stationary solutions of (9), typically $u \equiv 0$ and $u \equiv 1$, by means of a wave solution. In a discrete setting, this translates into the boundary value problem (BVP) on $\mathbb{Z}$

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k + g(u_{k+1}) &= 0, \\ \lim_{k\to-\infty} u_k &= 0, \\ \lim_{k\to+\infty} u_k &= 1. \end{aligned} \tag{10}$$

In Section 6 (see Theorem 7), we investigate the existence of increasing solutions of the equation in (10) when $k \in \mathbb{Z} \setminus \mathbb{N}$. By means of this result and of the techniques previously developed, we are able to discuss problem (10) both in the case when g satisfies conditions (8) as well as when there are $\varepsilon \in (0,1)$ and $L > 0$ such that

$$g/_{(0,\varepsilon)} \text{ is strictly increasing and } g(t) \leq Lt \text{ for } t \in [0,1], \tag{11}$$

$$g(t) > 0 \text{ for } t \in (0,1) \text{ and } g \equiv 0 \text{ otherwise.} \tag{12}$$

It is well known that there are striking similarities between qualitative theories of differential equations and difference equations. Our investigation, both of the general equation (1) and of its special type (5), was previously led in the continuous case, respectively, in [18–20]. We remark that the employed techniques in the discrete and continuous case are often quite different.

We will use the notation

$$\mathbb{N}_m = \{m, m+1, m+2, \dots\}, \qquad \text{for some } m \in \mathbb{Z},$$

and the convention that

$$\sum_{j=m}^{m-1} a_j = 0, \qquad \prod_{j=m}^{m-1} a_j = 1,$$

for any sequence $\{a_k\}_k$.

The following two equalities will frequently occur in the text: a summation of (1) from m to $k-1$ yields

$$\Delta u_k = r_m \Delta u_m \frac{1}{r_k} - \frac{1}{r_k} \sum_{j=m}^{k-1} q_j g(u_{j+1}), \tag{13}$$

a summation of (13) from m to $k-1$ gives

$$u_k = u_m + r_m \Delta u_m \sum_{j=m}^{k-1} \frac{1}{r_j} - \sum_{j=m}^{k-1} \frac{1}{r_j} \sum_{i=m}^{j-1} q_i g(u_{i+1}). \tag{14}$$

2. PRELIMINARY RESULTS

The following statement deals with monotonicity behaviors of all solutions of equation (1) when k varies in $\mathbb{N}_0$.

PROPOSITION 1. *Any solution of (1) is constant on $\mathbb{N}_0$, or strictly increasing on $\mathbb{N}_0$, or there exists exactly one $T \in \mathbb{N}_0$ such that $\Delta u_k > 0$ for $k \in \{0, 1, \dots, T-1\}$, $\Delta u_T \leq 0$, and $\Delta u_k < 0$ for $k \in \mathbb{N}_{T+1}$. Moreover, in the latter case, if $\mathcal{S}_r = \infty$, then $\lim_{k\to\infty} u_k = -\infty$.*

PROOF. Equation (1) is equivalent to the equation

$$\Delta u_{k+1} = \frac{r_k}{r_{k+1}} \Delta u_k - \frac{q_k}{r_{k+1}} g(u_{k+1}),$$

from which the first part of this statement, concerning the monotonicity properties, can be easily derived, because of the sign condition on g.

To prove the last part, using (14) with $m = T+1$, we get

$$u_k \leq u_{T+1} + r_{T+1} \Delta u_{T+1} \sum_{j=T+1}^{k-1} \frac{1}{r_j}. \tag{15}$$

Since $\Delta u_{T+1} < 0$, we obtain $\lim_{k\to\infty} u_k = -\infty$. ∎

The next proposition gives a necessary and sufficient condition for the existence of bounded solutions when $\mathcal{S}_r = \infty$ and $\mathcal{S}_{qr} < \infty$.

PROPOSITION 2. *Suppose that $\mathcal{S}_r = \infty$.*

(i) *If u is a bounded solution of (1), then*

$$r_m \Delta u_m - \sum_{j=m}^{\infty} q_j g(u_{j+1}) = 0, \tag{16}$$

for all $m \in \mathbb{N}_0$. Moreover, if $\mathcal{S}_{qr} < \infty$ and (3) holds, then $\lim_{k\to\infty} u_k \le u_0 + \sup_{t\in\mathbb{R}} g(t)\mathcal{S}_{qr}$.

(ii) *If a solution u of (1) satisfies (16) for some $m \in \mathbb{N}_0$, (3) holds and $\mathcal{S}_{qr} < \infty$, then u is bounded.*

PROOF.

(i) Let u be a bounded solution of (1). It is not difficult to verify that for any pair of sequences $\{a_k\}_k, \{b_k\}_k$ it holds

$$\sum_{j=h}^{k} \sum_{i=h}^{j-1} a_i b_j = \sum_{i=h}^{k-1} \sum_{j=i+1}^{k} a_i b_j,$$

where $h, k \in \mathbb{N}_0$ and $h < k$. Using this identity, from (14), with $m = 0$, we get

$$u_{k+1} = u_0 + \sum_{j=0}^{k} \frac{1}{r_j} \left[r_0 \Delta u_0 - \sum_{i=0}^{k-1} q_i g(u_{i+1}) \right] + \sum_{j=0}^{k-1} q_j g(u_{j+1}) \sum_{i=0}^{j} \frac{1}{r_i}. \tag{17}$$

Suppose, by a contradiction, that there exists $T \in \mathbb{N}_0$ such that

$$r_T \Delta u_T - \sum_{j=T}^{\infty} q_j g(u_{j+1}) \ne 0.$$

Then, in view of (13), one can find a constant $C > 0$ and $S \in \mathbb{N}_T$ satisfying $|\Delta u_k| \ge C/r_k$ for $k \in \mathbb{N}_S$. Hence, the condition $\mathcal{S}_r = \infty$ implies $\lim_{k\to\infty} u_k = \pm\infty$, a contradiction, and so (16) holds.

Assuming $\mathcal{S}_{qr} < \infty$ and g bounded and applying the discrete L'Hospital rule (see [1, Theorem 1.7.9]) to (17), we get

$$\lim_{k\to\infty} u_{k+1} = u_0 + \lim_{k\to\infty} \sum_{j=0}^{k} q_j g(u_{j+1}) \sum_{i=0}^{j} \frac{1}{r_i} \le u_0 + \sup_{t\in\mathbb{R}} g(t)\mathcal{S}_{qr}$$

and the statement is proved.

(ii) Let u be a solution of (1) satisfying (16) for some $m \in \mathbb{N}_0$, $\mathcal{S}_{qr} < \infty$, and (3). Then we can apply the discrete L'Hospital rule to (17) and in view of the assumptions we get $\lim_{k\to\infty} u_k \in \mathbb{R}$ and so u is bounded. ∎

We now consider the case when $\mathcal{S}_r = \mathcal{S}_{qr} = \infty$ and we give a necessary condition for the existence of bounded solutions.

PROPOSITION 3. *If $\mathcal{S}_{qr} = \mathcal{S}_r = \infty$ and (4) holds, then every bounded solution u of (1) satisfies $\lim_{k\to\infty} g(u_k) = 0$. Moreover, if u is not a constant, then there exists a sequence $\{\omega_n\}_{n\in\mathbb{N}}$, $\omega_n \to \lim_{k\to\infty} u_k$ as $n \to \infty$, such that $\omega_n < \lim_{k\to\infty} u_k$ and $g(\omega_n) > 0$ for $n \in \mathbb{N}$.*

PROOF. Let u be a bounded solution of (1). Then $\mathcal{S}_r = \infty$ implies $\Delta u_k \ge 0$ for $k \in \mathbb{N}_0$ and so there exists $\lim_{k\to\infty} u_k = L_u$. Suppose, by a contradiction, that $g(L_u) > 0$. Hence, we can choose $T \in \mathbb{N}_0$ such that $g(u_k) > g(L_u)/2$ for $k \in \mathbb{N}_T$, because g is continuous. By (16) and (17), we have

$$\begin{aligned} u_{k+1} &= u_0 + \sum_{j=0}^{k} \frac{1}{r_j} \sum_{i=k}^{\infty} q_i g(u_{i+1}) + \sum_{j=0}^{k-1} q_j g(u_{j+1}) \sum_{i=0}^{j} \frac{1}{r_i} \\ &\ge u_0 + \frac{g(L_u)}{2} \sum_{j=T-1}^{k-1} q_j \sum_{i=0}^{j} \frac{1}{r_i}, \end{aligned}$$

and so $u_k \to +\infty$ as $k \to \infty$, a contradiction.

Suppose that there exists $\varepsilon > 0$ such that $g(t) = 0$ for all $t \in [L_u - \varepsilon, L_u]$. Choose $S \in \mathbb{N}_0$ in such a way that $u_k \geq L_u - \varepsilon$ for $k \in \mathbb{N}_{S+1}$. From (16), we get $\Delta u_S = 0$, which implies $u_k = u_S$ for $k \geq S$, hence, Proposition 1 yields $u_k \equiv u_S$ which is a constant solution. ∎

The first part of the previous result holds also for bounded increasing solutions when $\mathcal{S}_r < \infty$ and $\mathcal{S}_{qr} = \infty$ as it is shown in the following statement.

PROPOSITION 4. *If $\mathcal{S}_r < \infty$, $\mathcal{S}_{qr} = \infty$, and (4) holds, then every bounded increasing solution u of (1) satisfies $\lim_{k\to\infty} g(u_k) = 0$.*

PROOF. First observe that $\mathcal{S}_r < \infty$ and $\mathcal{S}_{qr} = \infty$ imply $\mathcal{S}_q = \infty$. Let u be a bounded increasing solution of (1) with $\lim_{k\to\infty} u_k = L_u$. Suppose for a contradiction that $g(L_u) > 0$. Then one can find $T \in \mathbb{N}_0$ such that $g(u_k) \geq g(L_u)/2$ for $k \in \mathbb{N}_T$ since g is continuous. This yields $\sum_{j=T}^{\infty} q_j g(u_{j+1}) = \infty$ and according to (13), we obtain $\Delta u_k < 0$ for all sufficiently large k, which is false. ∎

3. EXISTENCE OF BOUNDED INCREASING SOLUTION IN THE CASE $\mathcal{S}_r < \infty$

This section deals with the case when $\mathcal{S}_r < \infty$. We start with the criterion concerning the case $\mathcal{S}_{qr} < \infty$, where the existence of a bounded increasing solution of (1) is guaranteed for every initial value α.

THEOREM 1. *Suppose that (3) holds. If $\mathcal{S}_r < \infty$ and $\mathcal{S}_{qr} < \infty$, then (1) has a bounded increasing solution u satisfying the initial condition $u_0 = \alpha$, where $\alpha \in \mathbb{R}$.*

PROOF. First observe that the only possibility in the case when $\mathcal{S}_r < \infty$ and $\mathcal{S}_{qr} < \infty$ is $\mathcal{S}_q < \infty$. Let $\beta > \mathcal{S}_q \sup_{t\in\mathbb{R}} g(t)/r_0$. We show that every solution $u^{\alpha,\beta}$ is increasing and bounded. Indeed, by (13), with $m = 0$, we have

$$\begin{aligned}\Delta u_k^{\alpha,\beta} &= \frac{r_0}{r_k}\beta - \frac{1}{r_k}\sum_{j=0}^{k-1} q_j g(u_{j+1}) \\ &\geq \frac{r_0}{r_k}\beta - \frac{1}{r_k}\sup_{t\in\mathbb{R}} g(t)\mathcal{S}_q > 0,\end{aligned}$$

which implies that $\{u_k^{\alpha,\beta}\}_k$ is increasing on $\mathbb{N}_0$. On the other hand, from (15) and $\mathcal{S}_r < \infty$, we conclude that u is bounded. ∎

When $\mathcal{S}_{qr} = \infty$, the following result, requiring additional conditions on g, can be applied. Two cases are considered, according to the positivity or not of g at the initial value of the solution. In the first case, we obtain a necessary and sufficient condition. In the second one, like in Section 4, only a sufficient condition is provided.

THEOREM 2. *Suppose that $\mathcal{S}_r < \infty$, $\mathcal{S}_{qr} = \infty$, and (4) holds. Then we have the following.*

(i) *Equation (1) has a bounded increasing solution u satisfying the initial condition $u_0 = \alpha$, where α is such that $g(\alpha) > 0$, if and only if*

$$\text{there exists } \omega > \alpha \text{ such that } g(\omega) = 0. \tag{18}$$

(ii) *If there exists $\omega > \alpha$ such that $g \equiv 0$ in $[\alpha, \omega]$, then (1) has a bounded increasing solution u satisfying the initial condition $u_0 = \alpha$.*

PROOF.

(i) Let α, ω be as in the theorem. Given $\beta \in \mathbb{R}$, denote $u^{\alpha,\beta}$ with u^β and $\Omega = \{\beta > 0 : u_k^\beta < \omega$ for all $k \in \mathbb{N}_0\}$. In view of Proposition 1, u^0 is nonincreasing. By virtue of the continuous dependence on initial data, we find out that $\Omega \neq \emptyset$, because no solutions of (1) admit a minimum point, by Proposition 1. Moreover, $\omega - \alpha \notin \Omega$. Hence, we can denote with $\bar{\beta} := \sup(\Omega \cap [0, \omega - \alpha]) \in \mathbb{R}$. Now we prove that $u^{\bar{\beta}}$ is a bounded increasing solution of (1) such that $u_k^{\bar{\beta}} < \omega$ for $k \in \mathbb{N}_0$. Clearly, one cannot have $T \in \mathbb{N}_0$ with $u_T^{\bar{\beta}} > \omega$. In fact, if such a T existed, by the continuous dependence on initial data, we would be able to find $\beta \in (0, \bar{\beta})$ with the property $u_T^{\beta} > \omega$. But this contradicts the definition of $\bar{\beta}$. Hence, $u_k^{\bar{\beta}} \le \omega$ for $k \in \mathbb{N}_0$. Further, if there existed $T \in \mathbb{N}_0$ such that $u_T^{\bar{\beta}} = \omega$, then $\Delta u_T^{\bar{\beta}} = 0$, by the previous argument. Hence, $\Delta u_T^{\bar{\beta}} = 0 = g(u_T^{\bar{\beta}})$ and the uniqueness of the solution of each IVP implies that $u^{\bar{\beta}}$ is a constant solution, which contradicts the fact that $u_0^{\bar{\beta}} = \alpha$ and $g(\alpha) > 0$. Consequently, $u_k^{\bar{\beta}} < \omega$ for $k \in \mathbb{N}_0$. It remains to show that $u^{\bar{\beta}}$ is increasing on $\mathbb{N}_0$. If there existed $T \in \mathbb{N}_0$ such that $\Delta u_T^{\bar{\beta}} \le 0$ (with $g(u_T^{\bar{\beta}}) > 0$ for $\Delta u_T^{\bar{\beta}} = 0$), without loss of generality we can suppose that $\Delta u_k^{\bar{\beta}} > 0$ for $k \in \{0, 1, \dots, T-1\}$. Then $u_k^{\bar{\beta}}$ is monotone decreasing for $k \in \mathbb{N}_T$, because of Proposition 1. Hence, the continuous dependence on initial data yields the existence of $\varepsilon > 0$ such that $\bar{\beta} + \varepsilon \in \Omega$, which contradicts the definition of $\bar{\beta}$.

The necessity of (18) follows from Proposition 4.

(ii) The sequence defined by $u_k = \alpha + (\omega - \alpha)/\mathcal{S}_r \sum_{j=0}^{k-1} 1/r_j$ is a desired solution. ∎

REMARK 1. Suppose that $\mathcal{S}_r < \infty$. According to Proposition 4, if $\mathcal{S}_{qr} = \infty$, the limit of any bounded solution of (1) is a zero of the function g, hence, the existence of $\omega \in \mathbb{R}$ such that $g(\omega) = 0$ is a necessary condition for (1) to have a bounded increasing solution. On the contrary, if $\mathcal{S}_{qr} < \infty$, Theorem 1 guarantees the existence of a bounded increasing solution of (1) even when $g(t) > 0$ for every t.

REMARK 2. Suppose that $\mathcal{S}_r < \infty$. It is not difficult to see that if u is a bounded increasing solution of (1), then $L_u - u_k$, where $L_u = \lim_{k\to\infty} u_k$, is asymptotic to or of higher order than $\sum_{j=k}^{\infty} 1/r_j$. Indeed, first observe that the sequence $r_k \Delta u_k$ is positive and nonincreasing, and hence, there exists the limit $\lim_{k\to\infty} r_k \Delta u_k = K_u$, where $K_u \in [0, \infty)$. Using the discrete L'Hospital rule, we get

$$\lim_{k\to\infty} \frac{L_u - u_k}{\sum\limits_{j=k}^{\infty} 1/r_j} = K_u$$

and so the statement holds.

4. EXISTENCE OF BOUNDED INCREASING SOLUTION IN THE CASE $\mathcal{S}_r = \infty$, $g(u_0) > 0$

This section deals with the case when $\mathcal{S}_r = \infty$ and $g(u_0) > 0$. We start with a criterion concerning the case $\mathcal{S}_{qr} < \infty$, which uses the result of Proposition 2.

THEOREM 3. *Suppose that (3) and (4) hold. If $\mathcal{S}_r = \infty$ and $\mathcal{S}_{qr} < \infty$, then (1) has a bounded increasing solution u satisfying the initial condition $u_0 = \alpha$, where $\alpha \in \mathbb{R}$ is such that $g(\alpha) > 0$.*

PROOF. Let α be as in the theorem. In view of our assumptions, we get $\mathcal{S}_q < \infty$. Using this fact together with the boundedness of g, we have the convergence of $\sum_{j=0}^{\infty} q_j g(u_{j+1})$ for every solution u of (1). Denote $u^{\alpha,\beta}$ simply by u^{β}, where $\beta \in \mathbb{R}$. Define the functional $\mathcal{T} : \mathbb{R} \to \mathbb{R}$ by

$$\mathcal{T}(\beta) = \beta r_0 - \sum_{j=0}^{\infty} q_j g\left(u_{j+1}^{\beta}\right).$$

Since $g(u_1^0) = g(u_0^0) = g(\alpha) > 0$, and g is nonnegative, $\mathcal{T}(0) < 0$. If $\beta > \sup_{t\in\mathbb{R}} g(t)\mathcal{S}_q/r_0$, then $\mathcal{T}(\beta) > 0$. We prove now that $\mathcal{T}$ is continuous. Let $\varepsilon > 0$ and $\bar{\beta} \in \mathbb{R}$ be given and $J \subseteq \mathbb{R}$ be an open interval containing $\bar{\beta}$. Take $n = n(\varepsilon) \in \mathbb{N}_0$ in such a way that

$$\sum_{j=n}^{\infty} q_j < \frac{\varepsilon}{6 \sup\limits_{t\in\mathbb{R}} g(t)}.$$

The uniform continuity of g in any compact interval implies the existence of $\sigma = \sigma(\varepsilon)$ satisfying $|g(x_1) - g(x_2)| < \varepsilon/\{3n(\max_{j\in\{0,1,\dots,n-1\}} q_j)\}$ whenever $x_1, x_2 \in [\min\{u_i^\beta : \beta \in \bar{J},\, 0 \le i \le n\}, \max\{u_i^\beta : \beta \in \bar{J},\, 0 \le i \le n\}]$ and $|x_1 - x_2| < \sigma$. By the continuous dependence on initial data, it is then possible to find $\delta = \delta(\sigma) > 0$ such that, whenever $|\beta - \bar{\beta}| < \delta$, it follows $|u_j^{\bar{\beta}} - u_j^\beta| < \sigma$ for all $j = 0, 1, \dots, n$. Further, set $\rho = \rho(\varepsilon) = \min\{\varepsilon/(3r_0), \delta(\sigma(\varepsilon))\}$ and take $\beta \in J$ in order that $|\beta - \bar{\beta}| < \rho$. Then we have

$$\begin{aligned}
\left|\mathcal{T}(\beta) - \mathcal{T}(\bar{\beta})\right| &\le \left|\beta - \bar{\beta}\right| r_0 + \sum_{j=0}^{n-1} q_j \left| g\left(u_{j+1}^\beta\right) - g\left(u_{j+1}^{\bar{\beta}}\right)\right| \\
&\quad + \sum_{j=n}^{\infty} q_j \left| g\left(u_{j+1}^\beta\right) - g\left(u_{j+1}^{\bar{\beta}}\right)\right| \\
&\le \frac{\varepsilon}{3} + n\left(\max_{j\in\{0,1,\dots,n-1\}} q_j\right)\left(\max_{j\in\{1,2,\dots,n\}} \left| g\left(u_j^\beta\right) - g\left(u_j^{\bar{\beta}}\right)\right|\right) \\
&\quad + 2\sup_{t\in\mathbb{R}} g(t) \sum_{j=n}^{\infty} q_j \\
&\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence, $\mathcal{T}$ is continuous and there exists $\tilde{\beta}$ with $0 < \tilde{\beta} < \sup_{t\in\mathbb{R}} g(t)\mathcal{S}_q/r_0$ such that $\mathcal{T}(\tilde{\beta}) = 0$. By Proposition 2, we get the statement. ∎

When $\mathcal{S}_{qr} = \infty$, we can use the following criterion requiring an additional condition on the nonlinearity g, which, however, is also necessary.

THEOREM 4. *Suppose that (4) holds. If $\mathcal{S}_r = \mathcal{S}_{qr} = \infty$, then (1) has a bounded increasing solution u satisfying the initial condition $u_0 = \alpha$, where $\alpha \in \mathbb{R}$ is such that $g(\alpha) > 0$, if and only if (18) holds.*

PROOF. The sufficient part can be proved like in Theorem 2. The necessity of (18) follows from Proposition 3. ∎

REMARK 3. Suppose that $\mathcal{S}_r = \infty$. In the case $\mathcal{S}_{qr} = \infty$, the necessity of (18) comes from Proposition 3 (see also Remark 1). On the contrary, if $\mathcal{S}_{qr} < \infty$, Theorem 3 and Proposition 2 guarantee the existence of a bounded increasing solution u of (1) satisfying $u_0 = \alpha$ such that $\lim_{k\to\infty} u_k \le \alpha + \mathcal{S}_{qr}\sup_{t\in\mathbb{R}} g(t)$ also in the case when the function g is positive in $[\alpha, \alpha + \mathcal{S}_{qr}\sup_{t\in\mathbb{R}} g(t)]$.

5. EXISTENCE OF BOUNDED INCREASING SOLUTION IN THE CASE $\mathcal{S}_r = \infty$, $g(u_0) = 0$

In this section, we consider equation (5) where (6) and (7) hold and $g : \mathbb{R} \to \mathbb{R}$ is a nonnegative and nontrivial function satisfying (4) and (8).

Notice that (5) is a particular case of (1) with $\mathcal{S}_r = \mathcal{S}_{qr} = \infty$ given by $r_k = \prod_{j=0}^{k-1}(1 + c_j)^{-1}$ and $q_k = b_k \prod_{j=0}^{k}(1 + c_j)^{-1}$.

Since for the arguments of g which are outside $[\theta, 1]$, (5) reduces to an (explicitly solvable) equation, outside $[\theta, 1]$, which can be explicitly solved, Proposition 3 and (8) imply that the only possible limit at infinity for a bounded increasing solution of (5) is 1. Hence, the problem of the existence of solutions for the boundary value problem on $\mathbb{N}_0$

$$\begin{aligned}
\Delta^2 u_k - c_k \Delta u_k + b_k g(u_{k+1}) &= 0, \\
u_0 &= \theta, \\
\lim_{k\to\infty} u_k &= 1,
\end{aligned} \tag{19}$$

clearly, Proposition 1 guarantees that the solution of (19) is monotone increasing turns out to be interesting.

THEOREM 5. *Suppose that (4), (6)–(8) hold. Then there exists $c^* > 0$ such that (19) has a solution for all $\bar{c} < c^*$.*

PROOF. Given $\beta \in \mathbb{R}$, denote $u^{\theta,\beta}$ with u^β and set $\Omega = \{\beta > 0 : \lim_{k\to\infty} u_k^\beta = +\infty\}$. Clearly, $\Omega \neq \emptyset$, because $[1-\theta, +\infty) \subset \Omega$. Take $\beta \in \Omega$ and $T > 0$ such that $u_k^\beta < 1$ for $k \in \{0, \dots, T\}$ and $u_k^\beta \geq 1$ for $k \in \mathbb{N}_{T+1}$. Summing (5) from 0 to $T-1$, with the substitution of u^β for u, we get

$$\Delta u_T^\beta - \beta - \sum_{j=0}^{T-1} c_j \Delta u_j^\beta + \sum_{j=0}^{T-1} b_j g\left(u_{j+1}^\beta\right) = 0. \tag{20}$$

Multiplying (5) by u_k^β, with the substitution of u^β for u, we get

$$u_k^\beta \Delta^2 u_k^\beta - c_k u_k^\beta \Delta u_k^\beta + b_k u_k^\beta g\left(u_{k+1}^\beta\right) = 0.$$

It is not difficult to verify that for any sequences $\{a_k\}_k, \{b_k\}_k$ it holds the formula of summation by parts

$$\sum_{j=0}^{k-1} u_j \Delta v_j = u_k v_k - u_0 v_0 - \sum_{j=0}^{k-1} \Delta u_j v_{j+1}. \tag{21}$$

Hence, summing the last identity from 0 to $T-1$ and using the summation by parts, we get

$$u_T^\beta \Delta u_T^\beta - \theta\beta - \sum_{j=0}^{T-1} c_j u_j^\beta \Delta u_j^\beta + \sum_{j=0}^{T-1} b_j u_j^\beta g\left(u_{j+1}^\beta\right) = \sum_{j=0}^{T-1} \Delta u_j^\beta \Delta u_{j+1}^\beta. \tag{22}$$

Similarly, multiplying (5) by u_{k+1}^β, with the substitution of u^β for u, and summing we obtain

$$u_T^\beta \Delta u_T^\beta - \theta\beta - \sum_{j=0}^{T-1} c_j u_{j+1}^\beta \Delta u_j^\beta + \sum_{j=0}^{T-1} b_j u_{j+1}^\beta g\left(u_{j+1}^\beta\right) = \sum_{j=0}^{T-1} \left(\Delta u_j^\beta\right)^2. \tag{23}$$

Hence, subtracting twice (20) from the sum of (22) and (23) and recalling (6)–(8), we have

$$\begin{aligned}
\sum_{j=0}^{T-1} \Delta u_j^\beta \left(\Delta u_j^\beta + \Delta u_{j+1}^\beta\right) &= 2\Delta u_T^\beta \left(u_T^\beta - 1\right) + 2\beta(1-\theta) \\
&\quad + \sum_{j=0}^{T-1} c_j \Delta u_j^\beta \left(2 - u_{j+1}^\beta - u_j^\beta\right) \\
&\quad - \sum_{j=0}^{T-1} b_j g\left(u_{j+1}^\beta\right)\left(2 - u_{j+1}^\beta - u_j^\beta\right) \\
&\leq 2\beta(1-\theta) + \bar{c} \sum_{j=0}^{T-1} \Delta u_j^\beta \left(2 - u_{j+1}^\beta - u_j^\beta\right) \\
&= 2\beta(1-\theta) + \bar{c}\left\{2\left(u_T^\beta - \theta\right) - \left[\left(u_T^\beta\right)^2 - \theta^2\right]\right\} \\
&\leq 2\beta(1-\theta) + 2\bar{c}(1-\theta)^2,
\end{aligned} \tag{24}$$

because $\theta < u_T^\beta < 1$. In the same way, a multiplication of (5) by Δu_k^β and Δu_{k+1}^β, with the substitution of u^β for u, and a summation between 0 and $T-1$, respectively, yield (noting that in the latter equation we have used the formula of summation by parts (21))

$$\sum_{j=0}^{T-1} b_j g\left(u_{j+1}^\beta\right) \Delta u_j^\beta = -\left(\Delta u_T^\beta\right)^2 + \beta^2 + \sum_{j=0}^{T-1} \Delta^2 u_j^\beta \Delta u_{j+1}^\beta + \sum_{j=0}^{T-1} c_j \left(\Delta u_j^\beta\right)^2 \tag{25}$$

and

$$\sum_{j=0}^{T-1} b_j g\left(u_{j+1}^{\beta}\right)\Delta u_{j+1}^{\beta} = -\sum_{j=0}^{T-1}\Delta^2 u_j^{\beta}\Delta u_{j+1}^{\beta} + \sum_{j=0}^{T-1} c_j \Delta u_j^{\beta}\Delta u_{j+1}^{\beta}. \tag{26}$$

Therefore, summing (25) and (26), by (7), (6), and (24), it follows

$$\begin{aligned}\underline{b}\sum_{j=0}^{T-1} g\left(u_{j+1}^{\beta}\right)\left(\Delta u_j^{\beta}+\Delta u_{j+1}^{\beta}\right) &\le \sum_{j=0}^{T-1} b_j g\left(u_{j+1}^{\beta}\right)\left(\Delta u_j^{\beta}+\Delta u_{j+1}^{\beta}\right)\\ &= \beta^2 - \left(\Delta u_T^{\beta}\right)^2 + \sum_{j=0}^{T-1} c_j \Delta u_j^{\beta}\left(\Delta u_j^{\beta}+\Delta u_{j+1}^{\beta}\right)\\ &\le \beta^2 + 2\beta\bar{c}(1-\theta) + 2\bar{c}^2(1-\theta)^2\\ &\le 2\left[\beta+\bar{c}(1-\theta)\right]^2.\end{aligned}$$

Thus,

$$\beta \ge -\bar{c}(1-\theta) + \sqrt{\frac{\underline{b}}{2}\sum_{j=0}^{T-1} g\left(u_{j+1}^{\beta}\right)\left(\Delta u_j^{\beta}+\Delta u_{j+1}^{\beta}\right)}. \tag{27}$$

In view of (20), we get that $\Delta u_j^{\beta} \le \beta + \bar{c}(1-\theta)$ for every $j = 0,\dots,T$, hence, according to the definition of the Riemann integral,

$$\begin{aligned}&\forall\,\varepsilon \in \left(0, 2\int_{\theta}^{1} g(t)\,dt\right) \exists\,\delta = \delta(\varepsilon) : \text{if } \beta + \bar{c}(1-\theta) < \delta \text{ and } \beta \in \Omega,\\ &\quad\text{then } \sum_{j=0}^{T-1} g\left(u_{j+1}^{\beta}\right)\left(\Delta u_j^{\beta}+\Delta u_{j+1}^{\beta}\right) > 2\int_{\theta}^{1} g(t)\,dt - \varepsilon,\end{aligned} \tag{28}$$

with T defined at the beginning of this proof. Let $\rho(\varepsilon) = \sqrt{(\underline{b}/2)\,(2\int_\theta^1 g(t)\,dt - \varepsilon)}$. Since we can take, with no loss of generality, the function $\delta(\varepsilon)$ increasing, it is possible to find $\bar{\varepsilon} \in (0, 2\int_\theta^1 g(t)\,dt)$ such that $\rho(\bar{\varepsilon}) < \delta(\bar{\varepsilon})$. Therefore, given $\bar{c} < \rho(\bar{\varepsilon})/(1-\theta)$ and assuming the existence of $\beta \in \Omega \cap (0, \rho(\bar{\varepsilon}) - \bar{c}(1-\theta))$, we obtain $\beta + \bar{c}(1-\theta) < \rho(\bar{\varepsilon}) < \delta(\bar{\varepsilon})$ implying

$$\sqrt{\frac{\underline{b}}{2}\sum_{j=0}^{T-1} g\left(u_{j+1}^{\beta}\right)\left(\Delta u_j^{\beta}+\Delta u_{j+1}^{\beta}\right)} > \sqrt{\frac{\underline{b}}{2}\left(2\int_{\theta}^{1} g(t)\,dt - \bar{\varepsilon}\right)} = \rho\left(\bar{\varepsilon}\right) > \beta + \bar{c}(1-\theta),$$

which yields a contradiction with (27). Therefore, by Propositions 1 and 3, for every $\beta \in (0, \rho(\bar{\varepsilon}) - \bar{c}(1-\theta)]$ the solution u^{β} of (5) is either bounded increasing to 1, or eventually decreasing to $-\infty$. In conclusion, the continuous dependence by initial data, applied like in Theorem 2, enables us to find $c^* > 0$ such that (19) is solvable for any $\bar{c} < c^*$ and the proof is complete. ∎

REMARK 4. Recalling the definitions of ρ and c^* given in the previous theorem, for every $\varepsilon \in (0, 2\int_\theta^1 g(t)\,dt)$, one has $\rho(\varepsilon) \le \sqrt{\underline{b}\int_\theta^1 g(t)\,dt}$, hence, $c^* \in (0, \sqrt{\underline{b}\int_\theta^1 g(t)\,dt}/(1-\theta))$.

6. APPLICATIONS TO DISCRETE BOUNDARY VALUE PROBLEMS ON $\mathbb{Z}$

This part is mainly devoted to some applications of the results in Sections 4 and 5. We investigate the equation

$$\Delta^2 u_k - c\Delta u_k + g(u_{k+1}) = 0 \tag{29}$$

on the set $\mathbb{Z}$, whose solutions correspond to discrete wave profiles of the reaction-diffusion equation (9). Here c stands for the wave speed and it is a positive constant, while g is continuous and behaves like in (8) or in (11),(12). Notice that (29) is a special type of equation (5) having all its coefficients constant. We are interested, in particular, in wavefront profiles connecting the two stationary states $u \equiv 0$ and $u \equiv 1$ of (9). This problem translates into the BVP (10) associated to equation (29). We remark that, in both cases for the nonlinearity g, Propositions 1 and 3 imply that the only possible limit at $+\infty$ for a bounded solution of (29) is, in fact, the value 1.

We start our investigation with g satisfying condition (8). Since (29) is an autonomous equation, it is not restrictive, in this case, to look for solutions of (10) such that $u_0 = \theta$. Consequently, problem (10) on $\mathbb{Z} \setminus \mathbb{N}$ with $u_0 = \theta$ can be reduced to the second-order linear BVP

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k &= 0, \\ \lim_{k\to-\infty} u_k &= 0, \\ u_0 &= \theta, \end{aligned} \tag{30}$$

which is solvable if and only if $c > 0$. Therefore, it is natural to require the positivity of c. The solution of (30), $u_k = \theta(c+1)^k$, has the velocity $\Delta u_0 = c\theta$. Hence, problem (10), up to a translation of the origin, is equivalent to

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k + g(u_{k+1}) &= 0, \\ u_0 &= \theta, \\ \Delta u_0 &= c\theta, \\ \lim_{k\to+\infty} u_k &= 1. \end{aligned} \tag{31}$$

The following result holds, which is essentially an application of Theorem 5.

THEOREM 6. *If (4) and (8) hold, then there exists $c^* > 0$ such that for $c = c^*$ (10) has a solution.*

PROOF. In view of the above observation, it is sufficient to prove the existence of $c^* > 0$ such that (31) has a solution. We reason as in Theorem 5. Given $c > 0$, denote by u^c the solution of the IVP associated to (29), when $u_0 = \theta$ and $\Delta u_0 = c\theta$ and set $\Omega = \{c > 0 : \lim_{k\to+\infty} u_k^c = +\infty\}$. Clearly, $\Omega \neq \emptyset$, because $[(1-\theta)/\theta, +\infty) \subset \Omega$. Taken $c \in \Omega$, let $T > 0$ be such that $u_k^c < 1$ for $k \in \{0, \dots, T\}$ and $u_k^c \geq 1$ for $k \in \mathbb{N}_{T+1}$. By (27), we get

$$c \geq \sqrt{\frac{1}{2}\sum_{j=0}^{T-1} g\left(u_{j+1}^c\right)\left(\Delta u_j^c + \Delta u_{j+1}^c\right)} \tag{32}$$

and, in view of (20), $\Delta u_j^c \leq c$ for every $j = 0, \dots, T$. Hence,

$$\forall \varepsilon \in \left(0, \int_\theta^1 g(t)\,dt\right) \exists\, \delta = \delta(\varepsilon) : \text{if } c < \delta, \text{ then}$$
$$\sqrt{\frac{1}{2}\sum_{j=0}^{T-1} g\left(u_{j+1}^c\right)\left(\Delta u_j^c + \Delta u_{j+1}^c\right)} > \sqrt{\int_\theta^1 g(t)\,dt - \varepsilon}.$$

Since we can assume $\delta(\varepsilon)$ increasing, it is possible to find $\bar{\varepsilon} \in (0, \int_\theta^1 g(t)\,dt)$ satisfying $\delta(\bar{\varepsilon}) > \sqrt{\int_\theta^1 g(t)\,dt - \bar{\varepsilon}}$. Therefore, taken $c \leq \sqrt{\int_\theta^1 g(t)\,dt - \bar{\varepsilon}}$, we obtain $c < \delta(\bar{\varepsilon})$ implying

$$\sqrt{\frac{1}{2}\sum_{j=0}^{T-1} g\left(u_{j+1}^c\right)\left(\Delta u_j^c + \Delta u_{j+1}^c\right)} > \sqrt{\int_\theta^1 g(t)\,dt - \bar{\varepsilon}} \geq c,$$

which is in a contradiction with (32). Therefore, u^c is bounded increasing to 1 or eventually decreasing to $-\infty$ for every $c \in (0, \sqrt{\int_\theta^1 g(t)\,dt - \bar{\varepsilon}}]$ and the existence of a solution for (10) follows by the continuous dependence on initial data. ∎

REMARK 5. According to the proof of the previous theorem, it also follows that

$$0 < c^* < \sqrt{\int_\theta^1 g(t)\,dt}.$$

We investigate now the case when g satisfies (11) and (12). The following result states the existence of solutions $\{w_k\}_k$ of (29) which strictly decrease to zero when $k \to -\infty$. Precisely such that $\Delta w_k > 0$ for all $k \in \mathbb{Z} \setminus \mathbb{N}$. By means of it and Theorem 4, we discuss (see Remark 6), also in this case, the existence of wavefront profiles for the reaction-diffusion equation (9).

THEOREM 7. *Suppose that (4), (11), and (12) hold. Then, for all $c > L + 2\sqrt{L}$, equation (29) has an increasing solution $\{w_k\}_k$, with $k \in \mathbb{Z} \setminus \mathbb{N}$, satisfying $\lim_{k\to-\infty} w_k = 0$.*

PROOF. The condition $c > L + 2\sqrt{L}$ implies that both the solutions λ_1 and λ_2 of $\lambda^2 + (L - c)\lambda + L = 0$ are positive real numbers and they are distinct. Take $\gamma \in (0, g(\varepsilon)/2)$ and $\alpha = \gamma(c - L)/cL \in (0,1)$. According to (11), $g(\varepsilon) \le L < cL/(c - L)$ implying $\alpha \in (0, \varepsilon)$. For $k \in \mathbb{Z} \setminus \mathbb{N}$, put $\varphi_k = \gamma[(\lambda_1 + 1)^k + (\lambda_2 + 1)^k]$ and consider the closed and convex subset of the Banach space $\ell^\infty = \{\{v_k\}_k : \sup_{k\in\mathbb{Z}\setminus\mathbb{N}} |v_k| < +\infty\}$ defined by

$$Q = \left\{\{v_k\}_{k\in\mathbb{Z}\setminus\mathbb{N}} : 0 \le v_k \le g^{-1}(\varphi_k),\ \forall\, k \in \mathbb{Z} \setminus \mathbb{N}\right\},$$

which is well defined, because $\varphi_k \le 2\gamma < g(\varepsilon)$. According to the definition of $\{\varphi_k\}_k$ and Q, for every $\{v_k\}_k \in Q$, the operator

$$\{T(v_k)\}_k = \left\{ \frac{c\alpha - \sum\limits_{j=-\infty}^{0} g(v_j)}{c(c+1)^{-k}} + \frac{\sum\limits_{j=k+1}^{0} g(v_j)(c+1)^{-j}}{c(c+1)^{-k}} + \frac{\sum\limits_{j=-\infty}^{k} g(v_j)}{c} \right\}_k$$

is well defined. Denoting by $u_k = T(v_k)$, for all $k \in \mathbb{Z} \setminus \mathbb{N}$, it holds

$$\Delta u_k = \frac{c\alpha - \sum\limits_{j=-\infty}^{0} g(v_j)}{(c+1)^{-k}} + \frac{\sum\limits_{j=k+1}^{0} g(v_j)(c+1)^{-j}}{(c+1)^{-k}} \tag{33}$$

and this easily implies $\Delta u_{k+1} = (1 + c)\Delta u_k - g(v_{k+1})$. Therefore, the sequence $\{T(v_k)\}_k$ is a solution of $\Delta^2 u_k - c\Delta u_k + g(v_{k+1}) = 0$ for $k \in \mathbb{Z} \setminus \mathbb{N}$ and it satisfies $u_0 = T(v_0) = \alpha$. Moreover, (4) and the definition of Q yield

$$\frac{1}{c} \lim_{k\to-\infty} \frac{-g(v_{k+1})(c+1)^{-k}}{-c(c+1)^{-k}} = \frac{1}{c^2} \lim_{k\to-\infty} g(v_{k+1}) = 0.$$

Hence, possibly applying the discrete L'Hospital rule, we obtain

$$\lim_{k\to-\infty} T(v_k) = \frac{1}{c} \lim_{k\to-\infty} \frac{\sum\limits_{j=k+1}^{0} g(v_j)(c+1)^{-j}}{(c+1)^{-k}} = 0.$$

Consequently, $\{T(v_k)\}_k$ is a solution of the BVP

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k + g(v_{k+1}) &= 0, \qquad k \in \mathbb{Z} \setminus \mathbb{N}, \\ u_0 &= \alpha, \\ \lim_{k\to-\infty} u_k &= 0. \end{aligned}$$

Our plan now, in order to complete the proof, is to apply Schauder's fixed-point theorem to the operator T on the set Q. For this purpose, we need to show that T is a completely continuous operator and it maps Q into itself. First, we show that $T(Q) \subseteq Q$. Since, for all $k \in \mathbb{Z} \setminus \mathbb{N}$, $(c+1)^{-k} \geq 1$, we get

$$\begin{aligned} T(v_k) &= \frac{\alpha}{(c+1)^{-k}} + \frac{\sum_{j=k+1}^{0} g(v_j)\left[(c+1)^{-j} - 1\right]}{c(c+1)^{-k}} \\ &\quad + \frac{\left[(c+1)^{-k} - 1\right] \sum_{j=-\infty}^{k} g(v_j)}{c(c+1)^{-k}} \\ &\geq \frac{\alpha}{(c+1)^{-k}} > 0, \end{aligned}$$

for all $k \in \mathbb{Z} \setminus \mathbb{N}$. On the other hand, according to the definition of $\{\varphi_k\}_k$, for all $k < 0$, we get

$$\begin{aligned} \frac{\sum_{j=k+1}^{0} \varphi_j (c+1)^{-j}}{c(c+1)^{-k}} &= \frac{\gamma}{c(c+1)^{-k}} \left[\frac{1-((c+1)/(\lambda_1+1))^{-k}}{1-(c+1)/(\lambda_1+1)} + \frac{1-((c+1)/(\lambda_2+1))^{-k}}{1-(c+1)/(\lambda_2+1)}\right] \\ &= \frac{\gamma}{c(c+1)^{-k}} \left[\frac{\lambda_1+1}{\lambda_1-c} + \frac{\lambda_2+1}{\lambda_2-c}\right] \\ &\quad - \frac{\gamma}{c}\left[\frac{(\lambda_1+1)^{k+1}}{\lambda_1-c} + \frac{(\lambda_2+1)^{k+1}}{\lambda_2-c}\right]. \end{aligned}$$

Taking into account that $\lambda_1 - c = -\lambda_2 - L = -\lambda_2(\lambda_1+1)$ and similarly $\lambda_2 - c = -\lambda_1(\lambda_2+1)$, we obtain

$$\begin{aligned} \frac{\sum_{j=k+1}^{0} \varphi_j (c+1)^{-j}}{c(c+1)^{-k}} &= -\frac{\gamma}{c(c+1)^{-k}}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \\ &\quad + \frac{\gamma}{c}\left[\frac{(\lambda_1+1)^k}{\lambda_2} + \frac{(\lambda_2+1)^k}{\lambda_1}\right] \\ &= -\frac{\gamma(c-L)}{cL(c+1)^{-k}} + \frac{\gamma}{cL}\left[\lambda_1(\lambda_1+1)^k + \lambda_2(\lambda_2+1)^k\right]. \end{aligned}$$

Since $\alpha = \gamma(c-L)/cL$, we can conclude that

$$\frac{\alpha}{(c+1)^{-k}} + \frac{\sum_{j=k+1}^{0} \varphi_j (c+1)^{-j}}{c(c+1)^{-k}} = \frac{\Delta\varphi_k}{cL}. \tag{34}$$

Let us consider, now

$$\begin{aligned} \frac{\sum_{j=-\infty}^{k} \varphi_j}{c} &= \frac{\sum_{j=-\infty}^{0} \varphi_j}{c} - \frac{\sum_{j=k+1}^{0} \varphi_j}{c} \\ &= \frac{\gamma}{c}\left(1 + \frac{1}{\lambda_1} + 1 + \frac{1}{\lambda_2}\right) - \frac{\gamma}{c}\left(1 + \frac{1}{\lambda_1}\right)\left[1 - (\lambda_1+1)^k\right] \\ &\quad - \frac{\gamma}{c}\left(1 + \frac{1}{\lambda_2}\right)\left[1 - (\lambda_2+1)^k\right] \\ &= \frac{\gamma}{c}\left(1 + \frac{1}{\lambda_1}\right)(\lambda_1+1)^k + \frac{\gamma}{c}\left(1 + \frac{1}{\lambda_2}\right)(\lambda_2+1)^k \\ &= \frac{\varphi_k}{c} + \frac{\gamma}{cL}\left[\lambda_2(\lambda_1+1)^k + \lambda_1(\lambda_2+1)^k\right] \\ &= \frac{\varphi_k}{c} + \frac{\gamma}{cL}\left[(c-L-\lambda_1)(\lambda_1+1)^k + (c-L-\lambda_2)(\lambda_2+1)^k\right]. \end{aligned}$$

Therefore, it holds

$$\frac{\sum_{j=-\infty}^{k}\varphi_j}{c}=\frac{(c+1)\varphi_k}{cL}-\frac{\varphi_{k+1}}{cL}$$

and combining with (34), we obtain

$$\frac{\alpha}{(c+1)^{-k}}+\frac{\sum_{j=k+1}^{0}\varphi_j(c+1)^{-j}}{c(c+1)^{-k}}+\frac{\sum_{j=-\infty}^{k}\varphi_j}{c}=\frac{\varphi_k}{L}.$$

Since $g(v_k)\le\varphi_k$ for every $k\in\mathbb{Z}\setminus\mathbb{N}$, according to (11) and previous equality, we get

$$\begin{aligned}T(v_k)&\le\frac{\alpha}{(c+1)^{-k}}+\frac{\sum_{j=k+1}^{0}\varphi_j(c+1)^{-j}}{c(c+1)^{-k}}+\frac{\sum_{j=-\infty}^{k}\varphi_j}{c}\\&=\frac{\varphi_k}{L}=\frac{g\left(g^{-1}(\varphi_k)\right)}{L}\le g^{-1}(\varphi_k).\end{aligned}\tag{35}$$

Hence, $T(Q)\subseteq Q$. Now we show that T is a continuous operator. For this purpose, let us consider $\{z_k\}_k$, $\{\{v_k^n\}_k\}_{n\in\mathbb{N}}\subset Q$ such that $\sup_{k\in\mathbb{Z}\setminus\mathbb{N}}|v_k^n-z_k|\to 0$ when $n\to+\infty$. We get, for every $k\in\mathbb{Z}\setminus\mathbb{N}$,

$$\begin{aligned}\left|T\left(v_k^n\right)-T(z_k)\right|&\le\frac{\sum_{j=-\infty}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|}{c}\\&\quad+\frac{\sum_{j=k+1}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|(c+1)^{-j}}{c(c+1)^{-k}}\\&\quad+\frac{\sum_{j=-\infty}^{k}\left|g\left(v_j^n\right)-g(z_j)\right|}{c}\\&\le\frac{\sum_{j=-\infty}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|}{c}\\&\quad+\frac{\sum_{j=k+1}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|}{c(c+1)}+\frac{\sum_{j=-\infty}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|}{c}\\&\le\sum_{j=-\infty}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|\left(\frac{2}{c}+\frac{1}{c(c+1)}\right).\end{aligned}$$

For every $j\in\mathbb{Z}\setminus\mathbb{N}$, we have $\lim_{n\to+\infty}|v_j^n-z_j|=0$, so $\lim_{n\to+\infty}|g(v_j^n)-g(z_j)|=0$, by (4). Moreover, the definition of Q yields $|g(v_j^n)-g(z_j)|\le 2\varphi_j$. The convergence of $\sum_{j=-\infty}^{0}\varphi_j$ implies the convergence of $\sum_{j=-\infty}^{0}|g(v_j^n)-g(z_j)|$. Hence, applying the discrete Lebesgue's dominated convergence theorem, we get

$$\lim_{n\to+\infty}\sum_{j=-\infty}^{0}\left|g\left(v_j^n\right)-g(z_j)\right|=\sum_{j=-\infty}^{0}\lim_{n\to+\infty}\left|g\left(v_j^n\right)-g(z_j)\right|=0.$$

Therefore,

$$\lim_{n\to+\infty}\sup_{k\in\mathbb{Z}\setminus\mathbb{N}}\left|T\left(v_k^n\right)-T(z_k)\right|=0,$$

and so T is continuous. Finally, we show that $T(Q)$ is relatively compact. Indeed, according to the monotonicity both of g and $\{\varphi_k\}_k$, it follows that $\sup_{k\in\mathbb{Z}\setminus\mathbb{N}} |T(v_k)| \leq g^{-1}(\gamma)$ for every $\{v_k\}_k \in Q$, which implies that $T(Q)$ is a bounded subset of ℓ^∞. Moreover, given $\varepsilon > 0$, since $\lim_{k\to-\infty} \varphi_k = 0$, it is possible to take $m \in \mathbb{Z}\setminus\mathbb{N}$ such that $\varphi_m \leq (L\varepsilon/2)$. Recalling (35), it follows, for every $\{v_k\}_k \in Q$ and every $k, i < m$, that

$$|T(v_k) - T(v_i)| \leq T(v_k) + T(v_i) \leq \frac{\varphi_k + \varphi_i}{L} \leq \frac{2\varphi_m}{L} \leq \varepsilon,$$

because $\{\varphi_k\}_k$ is increasing, so $T(Q)$ is uniformly Cauchy in ℓ^∞. Since $T(Q)$ is also bounded, it is relatively compact (see [11, Theorem 3.3]). Thus, all the hypotheses of the Schauder's fixed-point theorem are satisfied and so T has a fixed point $\{u_k\}_{k\in\mathbb{Z}\setminus\mathbb{N}} \in Q$ which is a solution of

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k + g(u_{k+1}) &= 0, \\ u_0 &= \alpha, \\ \lim_{k\to-\infty} u_k &= 0. \end{aligned}$$

Finally, according to (33), for all $k \leq -1$ it holds

$$\Delta u_k \geq \frac{c\alpha - \sum_{j=-\infty}^{k} g(u_j)}{(c+1)^{-k}}.$$

Since $\{u_k\}_k \in Q$, then $\sum_{j=-\infty}^{0} g(u_j)$ converges. Hence, $\lim_{k\to-\infty} \sum_{j=-\infty}^{k} g(u_j) = 0$ and this implies the existence of $\bar{k} \in \mathbb{Z}\setminus\mathbb{N}$ such that $\Delta u_k > 0$ for all $k \leq \bar{k}$. Therefore, since (29) is autonomous, the sequence $\{w_k\}_k = \{u_{\bar{k}+k}\}_k$ is a strictly monotone solution of (29) satisfying $\lim_{k\to-\infty} w_k = 0$ and the proof is complete. ∎

REMARK 6. Consider the equation $\Delta^2 u_k - c\Delta u_k + g(u_{k+1}) = 0$, with $k \in \mathbb{Z}$, $c > L + 2\sqrt{L}$, and g satisfying conditions (11) and (12). Notice that, with no loss of generality, the bounded increasing solution $\{w_k\}_{k\in\mathbb{Z}\setminus\mathbb{N}}$ of

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k + g(u_{k+1}) &= 0, \\ \lim_{k\to-\infty} u_k &= 0, \end{aligned}$$

obtained in Theorem 7 can be taken such that $w_0 \in (0, 1)$. Moreover, since equation (29) is a special case of (1) with $r_k = (c+1)^{1-k}$ and $q_k = (c+1)^{-k}$, hence, satisfying $\mathcal{S}_r = \mathcal{S}_{rq} = \infty$, according to (12) we can apply Theorem 4 and Proposition 3 in order to state the existence of a bounded increasing solution $\{v_k\}_{k\in\mathbb{N}}$ of the boundary value problem

$$\begin{aligned} \Delta^2 u_k - c\Delta u_k + g(u_{k+1}) &= 0, \\ u_0 &= w_0, \\ \lim_{k\to+\infty} u_k &= 1. \end{aligned}$$

Consider now the sequence $\{u_k\}_{k\in\mathbb{Z}}$ defined by

$$u_k = \begin{cases} w_k, & \text{for } k \in \mathbb{Z}\setminus\mathbb{N}, \\ v_k, & \text{for } k \in \mathbb{N}. \end{cases}$$

Of course it satisfies $\lim_{k\to-\infty} u_k = 0$ and $\lim_{k\to+\infty} u_k = 1$. Moreover, it is a solution of (29) both for $k \in \mathbb{Z}\setminus\mathbb{N}$ and for $k \in \mathbb{N}$. In conclusion, though $\{u_k\}_k$ is not exactly a solution of the boundary value problem (10) when k varies on all $\mathbb{Z}$, nevertheless it can be seen as a wavefront profile for the reaction-diffusion equation (9) which monotonically connects its two stationary solutions $u \equiv 0$ and $u \equiv 1$.

7. CONCLUDING REMARK

(i) It turns out that some of the above approaches might be successfully extended to the investigation of difference systems with similar types of nonlinearities as the function g is. For example, we could consider a nonlinear system of the form

$$\Delta(R_k\Delta u_k) + Q_k F(u_{k+1}, v_{k+1}) = 0,$$
$$\Delta(P_k\Delta v_k) + S_k G(u_{k+1}, v_{k+1}) = 0,$$

with F, G nonnegative and nontrivial. Such system appears in studying discrete mathematical models, where the existence of bounded solutions is related to the appearance of travelling wave solutions. In our opinion, its investigation is interesting also from a purely mathematical viewpoint. Hence, it is an object of our examination in future research.

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