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Tradeoffs in the Inductive Inference of Nearly Minimal Size Programs

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Inductive inference machines are algorithmic devices which attempt to synthesize (in the limit) programs for a function while they examine more and more of the graph of the function. There are many possible criteria of success. We study the inference of nearly minimal size programs. Our principal results imply that nearly minimal size programs can be inferred (in the limit) without loss of inferring power *provided* we are willing to tolerate a finite, but not uniformly, bounded, number of anomalies in the synthesized programs. On the other hand, there is a severe reduction of inferring power in inferring nearly minimal size programs if the maximum number of anomalies allowed is any uniform constant. We obtain a general characterization for the classes of recursive functions which can be synthesized by inferring nearly minimal size programs with anomalies. We also obtain similar results for Popperian inductive inference machines. The exact tradeoffs between mind change bounds on inductive inference machines and anomalies in synthesized programs are obtained. The techniques of recursive function theory including the recursion theorem are employed.

1. INTRODUCTION

While doing more and more experiments, a scientist may be able to infer a correct theory or explanation for a phenomenon he is investigating. What are the theoretical capabilities and limitations of this inference process if the scientist is a robot or machine? In order to answer this question, *inductive inference machines* have been defined formally (Blum and Blum, 1975; Gold, 1967) as follows.

N denotes the set of natural numbers.

DEFINITION 1.1. An *inductive inference machine* (abbr: IIM) is an algorithmic device, with no *a priori* bounds on how much time or memory resource it shall use, which takes as its input the graph of a function from N to N an ordered pair at a time, and which from time to time, as it is receiving its input, outputs computer programs.

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We assume without loss of generality all IIMs are order independent (Blum and Blum, 1975; Case and Smith, 1978).

There are many ways to define what it means for an IIM to succeed at eventually synthesizing a correct program. For example, we say that, for $a \in N \cup \{*\}$, an IIM M *EX^a-identifies* f iff M fed f outputs a non-empty finite sequence of computer programs the last of which computes f except on at most a anomalous inputs (if $a = *$, except for finitely many inputs). What we call EX^0-identification was the first criterion of success proposed and is essentially due to Gold (1967). The $a = *$ case was first proposed by Blum and Blum (1975) and they refer to it as *a.e. identification*. The other cases are due to Case (Case and Smith, 1978, 1979). Each IIM may EX^a-identify some functions but fail to EX^a-identify others. EX^a is defined to be the class of all sets $\mathscr{S}$ of recursive functions such that some IIM EX^a-identifies each function in $\mathscr{S}$. In (Case and Smith, 1978, 1979), the introducing of anomalies is motivated by the fact that physicists sometimes employ explanations with anomalies and it is shown that $EX^0 \subset EX^1 \subset \cdots \subset EX^*$. Hence, allowing anomalies in explanatory programs enables individual IIMs to infer explanations for a broader class of phenomena. In this paper we focus on the succinctness of the explanatory programs themselves. Since the quality of a scientific explanation is determined in part by its succinctness (Occam's Razor), we consider the effect on inferring power of restricting the final output programs to be of small, nearly minimal, size (Blum, 1967).

2. Preliminaries

Let $\tilde{N} = (N \cup \{*\})$, where $(\forall n \in N)\ [n < *]$. $\in$, $\subseteq$, and $\subset$ denote, respectively, membership, containment, and proper containment for sets. We let e, $i, j, k, l, m, n, p, s, x$, and y range over N and a, b, c, d range over $\tilde{N}$. We let f, g, h, v, w, z, and G (with or without subscripts) range over total (number theoretic) functions. ψ (with or without subscripts) ranges over partial (number theoretic) functions. φ (with or without 's) ranges over *acceptable numberings* (Machtey and Young, 1978; Rogers, 1958, 1967) of the partial recursive functions. φ_n is the partial recursive function computed by program n in the acceptable numbering φ; we then speak of n as being a *φ program* or *φ index*. In what follows φ should denote a fixed acceptable numbering and when no explicit reference to a particular acceptable numbering is made, φ will be the numbering referred to. For example, then, $\min(\psi)$ denotes $\min_\varphi(\psi)$, the minimal φ program of ψ. Φ denotes an arbitrary Blum complexity measure (Blum, 1967) associated with φ. $\mathscr{R}$ denotes the class of recursive functions. $\mathscr{A}$, $\mathscr{B}$, $\mathscr{S}$ (with or without subscripts) range over classes of recursive functions. We define $\psi[x = \{(y, \psi(y)) \mid y < x\}$. $Card(A)$ denotes the cardinality of the set A. $\psi_1 =^n \psi_2$ means that $Card\{x \mid \psi_1(x) \neq$

$\psi_2(x)\} \leqslant n$, and $\psi_1 =^* \psi_2$ means that $Card\{x \mid \psi_1(x) \neq \psi_2(x)\}$ is finite. $\psi_1 \subseteq^n \psi_2$ means that $Card\{x \mid \psi_1(x) \neq \psi_2(x),\ x \in \text{domain}(\psi_1)\} \leqslant n$. $\lambda x[\psi(x)]$ denotes ψ (Rogers, 1967). i_φ denotes a fixed program for $\lambda x[0]$ in φ. $\lambda x[\uparrow]$ denotes the everywhere undefined function. We define $\varphi_i^s = \{(y, \varphi_i(y)) \mid y < s$ and $\Phi_i(y) \leqslant s\}$. M (with or without subscripts or 's) ranges over inductive inference machines. τ ranges over finite functions; whereas, σ ranges over finite functions with domain an initial segment of N. $M(\sigma)$ denotes M's last output (if any) just after it has been fed all of σ. For Sections 3 through 5 we adopt (without loss of generality) the convention that any given IIM M outputs i_φ before being fed any input so that for all σ, $M(\sigma)$ is defined. $M(\tau)$ denotes $M(\sigma)$, where $\sigma \subseteq \tau$ and domain of σ is the largest initial segment of the domain of τ. $M(f)$ denotes M's last output, on input f, if M has a last output; else, $M(f)$ is undefined. $\lambda i, j[\langle i, j\rangle]$ denotes the *pairing function* such that $\langle i, j\rangle = [(i+j+1)(i+j)/2] + i$ (Rogers, 1969). ψ is *limiting partial recursive* iff there exists a partial recursive function ψ' such that, for all x, $\text{limit}_y\, \psi'(x, y) = \psi(x)$ (Schubert, 1974). f is *limiting recursive* iff f is limiting partial recursive and total. Any unexplained notation or terminology is from Rogers (1967).

3. Inferring Nearly Minimal Size Programs

Freivald (1975) considered the problem of EX^0-identification of *minimal* size programs but was able to show that this notion of inference is acceptable numbering *dependent*, i.e., what can be done depends on the particular acceptable numbering employed. Freivald also considered EX^0-identification of programs which are of minimal size module a recursive (fudge) factor. This notion, precisely defined below, turns out to be acceptable numbering independent. We generalize it below to the anomaly hierarchy of Case and Smith (1978, 1979).

Definition 3.1. (a) An IIM M *MEX^a-identifies* $\mathscr{S}$ (written: $\mathscr{S} \subseteq MEX^a(M)$) iff there is recursive function h such that for all $f \in \mathscr{S}$, M EX^a-identifies f and $M(f) \leqslant h(\min(f))$. (b) $MEX^a = \{\mathscr{S} \mid (\exists M)[\mathscr{S} \subseteq MEX^a(M)]\}$.

To facilitate showing that MEX^a-identification is acceptable numbering independent, we say that for an acceptable numbering φ' and recursive function h, an IIM M *$MEX^a(h, \varphi')$-identifies* $\mathscr{S}$ (written: $\mathscr{S} \subseteq MEX^a(M, h, \varphi')$) iff h witnesses that M MEX^a-identifies $\mathscr{S}$ in the numbering φ'. $MEX^a(h, \varphi')$ is defined to be $\{\mathscr{S} \mid (\exists M)[\mathscr{S} \subseteq MEX^a(M, h, \varphi')]\}$. It suffices to show that for every φ', $\bigcup_{h \in \mathscr{R}} MEX^a(h, \varphi) = \bigcup_{h \in \mathscr{R}} MEX^a(h, \varphi')$.

Without loss of generality, we assume from now on that h is monotone nondecreasing: Suppose $\mathscr{S} \subseteq MEX^a(M, g, \varphi)$ and $h(x) = max\{g(y) \mid y \leqslant x\}$; clearly $\mathscr{S} \subseteq MEX^a(M, h, \varphi)$ and h is monotone nondecreasing.

THEOREM 3.1.

$$(\forall h \in \mathscr{R})(\forall \varphi', \varphi'')(\exists h' \in \mathscr{R})[MEX^a(h, \varphi') \subseteq MEX^a(h', \varphi'')].$$

Proof. Since both φ' and φ'' are acceptable numberings, there exist recursive functions w_1 and w_2 such that for every $i \in N$, $\varphi'_i = \varphi''_{w_1(i)}$ and $\varphi''_i = \varphi'_{w_2(i)}$. Suppose $\mathscr{S} \subseteq MEX^a(M, h, \varphi')$, where h is recursive and monotone nondecreasing. We define an IIM M' thus. On input f, M' simulates M, and, whenever M outputs a program i, M' outputs $w_1(i)$. Let $h'(x) = max\{w_1(y) \mid y \leqslant h(w_2(x))\}$. Clearly, h' is recursive. We claim that $\mathscr{S} \subseteq MEX^a(M', h', \varphi'')$. To see this, suppose $f \in \mathscr{S}$. Since h is monotone nondecreasing, $M(f) \leqslant h(\min_{\varphi'}(f)) \leqslant h(w_2(\min_{\varphi''}(f)))$. $M'(f) = w_1(M(f)) \leqslant max\{w_1(y) \mid y \leqslant M(f)\} \leqslant max\{w_1(y) \mid y \leqslant h(\min_{\varphi'}(f))\} \leqslant max\{w_1(y) \mid y \leqslant h(w_2(\min_{\varphi''}(f)))\} = h'(\min_{\varphi''}(f))$. ∎

COROLLARY 3.2.

$$(\forall \varphi', \varphi'')[\bigcup \{MEX^a(h, \varphi') \mid h \in \mathscr{R}\} = \bigcup \{MEX^a(h, \varphi'') \mid h \in \mathscr{R}\}].$$

We write $MEX^a(M, h)$ for $MEX^a(M, h, \varphi)$.

We next extend Freivald's characterization of (what we call) MEX^0 to an interesting and technically useful characterization of MEX^a.

DEFINITION 3.2. $\mathscr{S}$ is *a-limiting standardizable with a recursive estimate* (abbr: $\mathscr{S} \in LSR^a$) iff there exist recursive functions G and v, such that for all $f \in \mathscr{S}$ and $i \in N$, if $\varphi_i = f$, then (i) $\text{limit}_s\, G(i, s)$ exists and computes f with at most a anomalous inputs, (ii) for all j, if $\varphi_j = f$, then $\text{limit}_s\, G(i, s) = \text{limit}_s\, G(j, s)$, and (iii) $Card\{G(i, s) \mid s \in N\} \leqslant v(i)$.

We write $\mathscr{S} \subseteq LSR^a(G, v)$ iff the recursive functions G and v witness that $\mathscr{S} \in LSR^a$.

Clearly LSR^a is numbering independent. Intuitively $\mathscr{S} \in LSR^a$ means that from any program i for an $f \in \mathscr{S}$, we can find *in the limit* an approximate (up to a anomalies) canonical program for f and there is a recursive bound $v(i)$ to limit the number of possibilities for the approximate canonical program. Being able to fix an approximate canonical program in the limit is equivalent to being able to solve in the limit the program equivalence problem approximately, at least for programs that compute functions in $\mathscr{S}$.

THEOREM 3.3. *The following three statements are equivalent.*

(1) $\mathscr{S} \in MEX^a$.

(2) *There exist recursive functions G and v such that* $\mathscr{S} \subseteq LSR^a(G, v)$ *and there exists an* IIM *M such that M* EX^a*-identifies every function in* $\mathscr{S}$ *and* $(\forall i \mid \varphi_i \in \mathscr{S})$ $[M(\varphi_i) = \text{limit}_s\, G(i, s)]$.

(3) *There exist recursive functions G and v and an* IIM *M such that M* EX^a*-identifies* $\mathscr{S}$ *and for every* $f \in \mathscr{S}$, $\text{limit}_s\, G(\min(f), s) = M(f)$ *and* $Card\{G(\min(f), s) \mid s \in N\} \leqslant v(\min(f))$.

Proof. (1) ⇒ (2). Suppose $\mathscr{S} \subseteq MEX^a(M, h)$. Let $v(i) = h(i) + 2$ and define G as follows:

$$G(i, s) = M(\varphi_i^s), \quad \text{if } M(\varphi_i^s) \leqslant h(i);$$
$$= i_\varphi, \quad \text{otherwise.}$$

It is clear that thanks to the conventions of Section 2, both G and v are recursive. If $\varphi_i \in \mathscr{S}$, then (i) $\text{limit}_s\, G(i, s) = M(\varphi_i) =^a \varphi_i$, (ii) for every j, if $\varphi_j = \varphi_i$, then $\text{limit}_s\, G(j, s) = \text{limit}_s\, G(i, s) = M(\varphi_i)$, and (iii) $\{G(i, s) \mid s \in N\} \subseteq \{k \mid k \leqslant h(i)\} \cup \{i_\varphi\}$. Hence, $Card\{G(i, s) \mid s \in N\} \leqslant v(i) = h(i) + 2$. Therefore, $\mathscr{S} \in LSR^a$ and $(\forall i \mid \varphi_i \in \mathscr{S})$ $[M(\varphi_i) = \text{limit}_s\, G(i, s)]$.

(2) ⇒ (3). Immediate, since $\min(f)$ is one of the programs which compute f.

(3) ⇒ (1). Suppose that for every $f \in \mathscr{S}$, (i) $f \in EX^a(M)$, (ii) $\text{limit}_s\, G(\min(f), s) = M(f)$, and (iii) $Card\{G(\min(f), s) \mid s \in N\} \leqslant v(\min(f))$. Without loss of generality, we assume that for every i, $Card\{G(i, s) \mid s \in N\} \leqslant v(i)$. Intuitively we need to construct a machine M' which simulates M on input f to get $M(f)$ in the limit and simutaneously dovetails to search the least program i such that $\text{limit}_s\, G(i, s) = M(f)$. Then from i, it is possible to construct a program which computes $\varphi_{M(f)}$ and is h-minimal. By the $s-m-n$ theorm (Rogers, 1967), there is a recursive function z such that for all i, j, x,

$$\varphi_{z(i,j)}(x) = \varphi_p(x), \quad \text{if there exist a least } m \text{ such that } Card\{G(i, k) \mid k \leqslant m\} = j \text{ and } p = G(i, m);$$
$$= \text{undefined}, \quad \text{otherwise.}$$

We define an IIM M' thus. On input $f \lceil x$, M' computes $M(f \lceil x)$ and M' searches for the least $i \leqslant x$ for which there exists $y \leqslant x$ such that $G(i, y) = M(f \lceil x)$. If such an i exists, M' picks the least y such that $G(i, y) = M(f \lceil x)$. Let $j = Card\{G(i, k) \mid k \leqslant y\}$. If $z(i,j)$ is not the last program output by M' so far, M' then outputs $z(i,j)$.

Let $h(i) = max\{z(k, 1) \mid k \leqslant i \text{ and } 1 \leqslant v(k)\}$. For any $f \in \mathscr{S}$, let x_0 be a sufficiently large number such that (i) $(\forall x \geqslant x_0)$ $[M(f \lceil x_0) = M(f \lceil x)]$, and (ii) $x_0 \geqslant max(i_0, y_0)$, where i_0 is the least program for which there exists y

such that $G(i_0, y) = M(f)$ and y_0 is the least such y. Since $\text{limit}_s\, G(\min(f), s) = M(f)$, we observe that $i_0 \leqslant \min(f)$. Therefore, by the definition of M', for all $x \geqslant x_0$, $M'(f \mid x) = z(i_0, j_0)$, where $j_0 = Card\{G(i_0, y) \mid y \leqslant y_0\}$. Furthermore, for every x, $\varphi_{z(i_0,j_0)}(x) = \varphi_{G(i_0,y_0)}(x) = \varphi_{M(f)}(x)$. Hence, $\varphi_{z(i_0,j_0)} =^a f$. Since $i_0 \leqslant \min(f)$ and $j_0 \leqslant v(i_0)$, $z(i_0, j_0) \leqslant h(\min(f))$. ∎

Remark. If in the definition of MEX^a, we had allowed the functions h to be limiting recursive, then Theorem 3.3 would go through if we require the function v (but not G) to be limiting recursive.

COROLLARY 3.4. *Let* $\mathcal{S}^a = \{f \mid \varphi_{f(0)} =^a f\}$. *Then* $\mathcal{S}^a \in MEX^a$.

Proof. It suffices to show (i), there exists recursive G, v such that $\mathcal{S}^a \subseteq LSR^a(G, v)$ and (ii) there exists M such that $\mathcal{S}^a \subseteq EX^a(M)$ and for all i such that $\varphi_i \in \mathcal{S}^a$, $M(\varphi_i) = \text{limit}_s\, G(i, s)$. Let $v = \lambda x[2]$.

$$\begin{aligned} G(i, s) &= \varphi_i(0), && \text{if} \quad \Phi_i(0) \leqslant s; \\ &= 0, && \text{otherwise.} \end{aligned}$$

We define an IIM M thus. M on input f, outputs $f(0)$ only. Clearly $\mathcal{S}^a \subseteq LSR^a(G, v)$, $\mathcal{S}^a \subseteq EX^a(M)$, and for all i such that $\varphi_i \in \mathcal{S}^a$, $M(\varphi_i) = \text{limit}_s\, G(i, s)$. ∎

Case and Smith (1978, 1979) have shown that $\mathcal{S}^{n+1} \in (EX^{n+1} - EX^n)$. Hence we have

COROLLARY 3.5. $(\forall n)[MEX^{n+1} \not\subseteq EX^n]$.

COROLLARY 3.6. $MEX^0 \subset MEX^1 \subset \cdots \subset MEX^*$.

4. COMPARISON OF INFERENCE WITH AND WITHOUT SIZE RESTRICTIONS

In this section we examine the cost in anomalies and inferring power of inferring nearly minimal size programs. Theorem 4.1 below shows that with the cost of finitely many anomalies, nearly minimal size programs can be inferred without reducing the inferring power of IIMs. On the other hand, Theorem 4.3 implies that there is a severe reduction of inferring power in inferring nearly minimal size programs if the maximum number of anomalies allowed is any uniform constant.

THEOREM 4.1. $MEX^* = EX^*$.

Proof. It suffices to show that $EX^* \subseteq MEX^*$.

By Corollary 3.2 it then suffices, for each IIM M, to define an acceptable numbering φ', a recursive function h and an IIM M' such that $EX^*(M) \subseteq MEX^*(M', h, \varphi')$.

We define $\varphi'_{2n}(x) = \varphi_{M(\varphi_n^x)}(x)$ and $\varphi'_{2n+1} = \varphi_n$. It is clear that φ' is an acceptable numbering since φ can be *reduced* (Rogers, 1958) to φ' by the recursive function $\lambda n[2n+1]$. Also, let w be a recursive function such that φ' is reduced to φ by w.

We define an IIM M' thus. On input $f \lceil x$, M' computes $M(f \lceil x)$ and then searches for the least $n \leqslant x$ such that $M(\varphi_n^x) = M(f \lceil x)$ and $\varphi_n^x \subseteq f \lceil x$. If such n exists and the last program output by M', so far, is not $2n$, then M' outputs $2n$.

We claim that M' on input $f \in \mathscr{S}$ will output a last program $2n$ such that $n \leqslant \min(f)$ and $\varphi'_{2n} =^* f$. If our claim is true, we will have that $2n \leqslant h(\min_{\varphi'}(f))$, where $h(i) = 2 \cdot w(i)$, which implies the theorem. It remains to prove our claim. Suppose $f \in \mathscr{S}$. Consider the set of φ programs $A = \{m \mid (\exists x)(\forall y \geqslant x)\ [M(\varphi_m^y) = M(f)$ and $\varphi_m^y \subseteq f \lceil y]\}$. It is clear that for every i such that $\varphi_i = f$, $i \in A$. Let n be the least φ program in A. Hence, $n \leqslant \min_\varphi(f)$. We want to show that for all sufficiently large x, $M'(f \lceil x) = 2n$. Let x_0 be so large that (i) $(\forall y \geqslant x_0)$ $[M(\varphi_n^y) = M(f)$ and $\varphi_n^y \subseteq f \lceil y]$ and (ii) for every $m < n$, if $\varphi_m \not\subseteq f$, then $\varphi_m^{x_0} \not\subseteq f \lceil x_0$. We then proceed to show that for every $y \geqslant x_0$, $M'(f \lceil y) = 2n$. By (i) and the fact that n is the least program in A, we have $M'(f \lceil y) \leqslant 2n$ for all $y \geqslant x_0$. By (ii) and the convention of Section 2, for all m, if $\varphi_m \not\subseteq f$, then $M'(f \lceil y) \neq 2m$ for all $y \geqslant x_0$. We then consider any program $m < n$ such that $\varphi_m \subseteq f$. Since $m < n \leqslant \min_\varphi(f)$ and $\varphi_m \subseteq f$, φ_m is not total. Let x' be the least number which is not in the domain of φ_m. Then $M(\varphi_m \lceil y) = M(\varphi_m \lceil x')$ for all $y \geqslant x'$. By the conventions of section 2, $m < n$, the least program in A, so $M(\varphi_m \lceil y) \neq M(f)$ for all $y \geqslant x'$, a contradiction.

Therefore, $M'(f) = 2n$. We next show that $\varphi'_{2n} =^* f$. Since $(\exists x)$ $(\forall y \geqslant x)$ $[M(\varphi_n^y) = M(f \lceil y)]$ and $\varphi'_{2n}(x) = \varphi_{M(\varphi_n^x)}(x)$, we have $(\forall y \geqslant x)$ $[\varphi'_{2n}(y) = \varphi_{M(\varphi_n^y)}(y) = \varphi_{M(f \lceil y)}(y)]$. Hence $\varphi'_{2n} =^* \varphi_{M(f)} =^* f$. ∎

COROLLARY 4.2. $(\forall n)[EX^n \subset MEX^*]$.

Let $ZERO^* = \{f \mid f =^* \lambda x[0]\}$. Clearly (for example, by Blum and Blum's (1975) enumeration technique) $ZERO^* \in EX^0$. Kinber (Freivald, 1975; Kinber, 1977) has claimed without proof that $ZERO^* \notin MEX^0$. Our next theorem extends this result.

THEOREM 4.3. $(\forall n)[ZERO^* \notin MEX^n]$.

Proof. Suppose by way of contradiction, there exists a recursive, monotone nondecreasing h and an IIM M such that $ZERO^* \subseteq MEX^n(M, h)$. By implicit use of the recursion theorem (Rogers, 1967), we define a selfreferential program i as follows. Let $(\varphi_i)^s$ denote the finite part of φ_i constructed by the beginning of stage s of program i. $x_1^s, x_2^s, x_3^s, \ldots$ denote the least, second least, third least,... elements $\notin$ *domain*$((\varphi_i)^s)$. Let $(\varphi_i)^0 = \varnothing$.

Begin Program i. On input x, successively execute the stages $s \geqslant 0$ below until (if ever) $\varphi_i(x)$ is defined.

Before stage 0 no program is cancelled.

Begin stage s. Let $j = M((\varphi_i)^s)$.

Condition 1. Either $j \leqslant h(i)$ and j is already cancelled or $j > h(i)$. Then let $(\varphi_i)^{s+1} = (\varphi_i)^s \cup \{(x_1^s, 0)\}$.

Condition 2. $j \leqslant h(i)$ and j is not yet cancelled.

Dovetail execution of the following two steps until either terminates.

Step 1. Search for $\sigma \supset (\varphi_i)^s$, where *range*$(\sigma - (\varphi_i)^s) = \{0\}$ such that $M(\sigma) \neq M((\varphi_i)^s)$. Terminate step 1 when (if ever) σ is found.

Step 2. Dovetail computing $\varphi_j(x_1^s)$, $\varphi_j(x_2^s), \ldots$ until $n+1$ of them converge, then terminate step 2.

If step 1 terminates before step 2, set $(\varphi_i)^{s+1} = \sigma$. If step 2 terminates before step 1, let $y_1^s, y_2^s, \ldots, y_{n+1}^s$ be $n+1$ points at which φ_j converges. Set $(\varphi_i)^{s+1} = (\varphi_i)^s \cup \{(y_1^s, 1 \dot{-} \varphi_j(y_1^s)), \ldots, (y_{n+1}^s, 1 \dot{-} \varphi_j(y_{n+1}^s))\}$ and cancel j.

End stage s.

End program i.

If φ_i is a finite function, then there must be a stage s such that condition 2 is true and both steps 1 and 2 do not halt. Let $f = (\varphi_i)^s \cup \{(x, 0) \mid x \notin$ *domain*$((\varphi_i)^s)\}$. It is clear that $f \in ZERO^*$. Since step 1 does not halt, there is a j such that $M(f) = j$. Since step 2 does not halt, φ_j is a finite function; hence $\varphi_j \neq^n f$.

Suppose now that φ_i is not a finite function. Then every time condition 2 is true at a stage s, stage s must terminate. Since there are but finitely many $j \leqslant h(i)$ and for every stage at which step 2 terminates before step 1, a different $j \leqslant h(i)$ is cancelled, for all sufficiently large stages s, if condition 2 holds at stage s, step 1 terminates before step 2. Therefore, φ_i is total. Hence, in this case, let $f = \varphi_i$. $\varphi_i \in ZERO^*$ and $\varphi_{M(\varphi_i)} \neq^n f$. ∎

Remarks. (i) The proof of Theorem 4.3 actually shows that $\{f \mid f$ is a characteristic function of a finite set$\} \notin MEX^n$. Also (ii) the proof can be easily modified to show that $ZERO^* \not\subseteq MEX^n(M, h)$, where h is *limiting* recursive: Suppose $h = \text{limit } h'$, then in program i change all occurrences of $h(i)$ in stage s to $h'(i, s)$; lastly observe that for all sufficiently large stages s, $h'(i, s) = h(i)$.

Our next corollary is a slightly strengthened version of a result of Meyer (1972). Meyer's result is the $n = 0$ case.

COROLLARY 4.4. *Suppose h is recursive in the halting problem and n is any given natural number. Then there is a program e which computes the characteristic function of a finite set such that for any loop program p (Meyer and Ritchie,* 1967) *for which* $\varphi_p =^n \varphi_e$, $p > h(e)$.

Proof. Let $p_0, p_1, p_2, \ldots$ be a recursive enumeration of all loop programs. Let $\mathscr{B} = \{f \mid f$ is a characteristic function of a finite set$\}$. It is clear that for every $f \in \mathscr{B}$, there is a loop program which computes f. Let M be an IIM which given any input $f \lceil x$, searches for the smallest program p in $\{p_0, p_1, \ldots, p_x\}$ such that $\varphi_p \lceil x \subseteq^n f \lceil x$. If such program p exists and differs from the last program output by M so far, M then outputs p. Clearly, for every $f \in \mathscr{B}$, $M(f)$ is the smallest loop program such that $\varphi_{M(f)} =^n f$. By the remarks, after the proof of Theorem 4.3, $\mathscr{B} \not\subseteq MEX^n(M, h)$. Hence, there is a $f \in \mathscr{B}$ such that $h(\min(f)) < M(f)$, i.e., for any loop program p such that $\varphi_p =^n f$, $p > h(\min(f))$. ∎

COROLLARY 4.5. $(\forall n)[EX^0 \not\subseteq MEX^n]$.

COROLLARY 4.6. $(\forall n)[MEX^n \subset EX^n \subset MEX^*]$.

Schubert (1974) conjectured that $\lambda x[\mu y[\varphi_y \supseteq \varphi_x]]$ is not limiting recursive. Our next corollary is a strengthening of Schubert's conjecture. Royer (private communication) independently proved Schubert's conjecture itself by using the techniques in Section 5 of Meyer (1972).

COROLLARY 4.7. *Suppose* $n \in N$ *and h is limiting recursive. Then there is no limiting partial recursive function* ψ *such that* $(\forall x)[\varphi_x \in ZERO^* \Rightarrow [\psi(x)$ *converges and* $\varphi_{\psi(x)} =^n \varphi_x$ *and* $\psi(x) \leqslant h(\min_\varphi(\varphi_x))]]$.

Proof. Suppose by way of contradiction otherwise. As noted in Case (in press), there is a recursive function g such that $\psi \subseteq$ limit g. Let M be an IIM such that $ZERO^* \subseteq EX^0(M)$. We define IIM M' thus. M', on input f, simulates M. If M on input f just output a new program p, M' outputs, surpressing co-final repetitions, $g(p, 0)$, $g(p, 1)$, $g(p, 2), \ldots$ until (if ever) M changes its output. Clearly $ZERO^* \subseteq MEX^n(M', h)$, contradicting Remark (ii) following the proof of Theorem 4.3. ∎

COROLLARY 4.8 (Schubert's conjecture). $\lambda x[\mu y[\varphi_y \supseteq \varphi_x]]$ *is not limiting recursive.*

5. POPPERIAN MACHINES

We next extend our results to Popperian machines (Case and Ngo Manguelle, in press), where a *Popperian* IIM, by definition, outputs only programs for total functions.

DEFINITION 5.1 (Case and Ngo Manguelle, in press; Case and Smith, 1979). Suppose I is any previously defined criterion such as EX^a, MEX^a, $MEX^a(h, \varphi)$. (a) An IIM M *PI-identifies* $\mathcal{S}$ (written: $\mathcal{S} \subseteq PI(M)$) iff M is Popperian and M I-identifies $\mathcal{S}$. (b) $PI = \{\mathcal{S} \mid (\exists M)[\mathcal{S} \subseteq PI(M)]\}$.

PEX^0 is a mathematically natural class with many characterizations and closure properties (Case and Ngo Manguelle, in press). For example: (1) $PEX^0 = PEX^1 = \cdots = PEX^*$ (Case and Ngo Manguelle, in press); hence, we write PEX for PEX^0. (2) $PEX = \{\mathcal{S} \mid \mathcal{S}$ is contained in some recursively enumerable class of recursive functions$\}$ (Barzdin and Freivald, 1972; Case and Ngo Manguelle, in press; Case and Smith, 1979). (3) $PEX = \{\mathcal{S} \mid (\exists \text{ recursive } t)(\forall f \in \mathcal{S})(\exists i)[\varphi_i = f$ and $(\exists y)(\forall x \geqslant y)[\Phi_j(x) \leqslant t(x)]]\}$ (Barzdin and Freivald, 1972; Blum and Blum, 1975; Case and Ngo Manguelle, in press). (4) PEX is closed under finite union, i.e., if $\mathcal{S}_1$, $\mathcal{S}_2 \in PEX$ then $\mathcal{S}_1 \cup \mathcal{S}_2 \in PEX$. In contrast, EX^a is not closed under union (Blum and Blum, 1975; Case and Ngo Manguelle, in press).

We define $PLSR^a$, an analogue of LSR^a, as follows.

DEFINITION 5.2. $\mathcal{S} \in PLSR^a$ iff there exist recursive functions G and v such that $\mathcal{S} \subseteq LSR^a(G, v)$ and the range of G contains only programs for total functions.

THEOREM 5.1. $(\forall a)[PMEX^a = PLSR^a]$.

Proof. Consider the proof of Theorem 3.3. In that proof, if we restrict M to be a Popperian machine and the range of G contains programs for total functions only, then we have the following result: $\mathcal{S} \in PMEX^a$ iff there exist recursive functions G and v witnessing that $\mathcal{S} \in PLSR^a$ and there exists a Popperian IIM M such that M PEX^a-identifies every function in $\mathcal{S}$ and $(\forall i \mid \varphi_i \in \mathcal{S})$ $[M(\varphi_i) = \text{limit}_s\, G(i, s)]$. Hence, it remains to show that if $\mathcal{S} \in PLSR^a$, then there exists an IIM M such that M PEX^a-identifies every function in $\mathcal{S}$ and $(\forall i \mid \varphi_i \in \mathcal{S})$ $[M(\varphi_i) = \text{limit}_s\, G(i, s)]$. Let $F_0, F_1, F_2, \ldots$ be a *canonical indexing* (Machtey and Young, 1978) of all finite functions: $N \to N$. By the $s-m-n$ theorem (Rogers, 1967) there is a recursive function z such that, for all i, j, k and x, if $x \in \text{domain}(F_k)$, then $\varphi_{z(i,j,k)}(x) = F_k(x)$; else, $\varphi_{z(i,j,k)}(x) = \varphi_{G(i,j)}(x)$. Clearly $\mathcal{S}' = \{\varphi_{z(i,j,k)} \mid i, j, k \in N\}$ is an r.e. class of recursive functions and $\mathcal{S}'$ contains $\mathcal{S}$. By property (2) of PEX mentioned immediately after the Definition 5.1, there is a single Popperian IIM M' with PEX-identifies every function in $\mathcal{S}'$. Hence, for every $f \in \mathcal{S}$, $\varphi_{M'(f)} = f$. We then define a Popperian IIM M thus. On input f, M simulates M' on f. If p is the current last program output by M', M then outputs, surpressing co-final repetitions, $G(p, 0)$, $G(p, 1)$,... until M' outputs a new program.

It is clear that the range of M is a subset of the range of G. Hence, M is a

Popperian IIM. For every $f \in \mathscr{S}$, since $\varphi_{M'(f)} = f$, $M(f) = \text{limit}_s\, G(M'(f), s)$ and $\varphi_{M(f)} =^a f$. ∎

THEOREM 5.2. $(\forall n)[PLSR^0 = PLSR^n]$.

Proof. It suffices to show that $PLSR^n \subseteq PLSR^0$. Suppose that there exist recursive functions G and v such that $\mathscr{S} \in PLSR^n(G, v)$. Let $v'(i) = (n+1) \cdot v(i)$. Let $G'(i, s) = p$, where p is the patched version of program $G(i, s)$ defined below. Suppose $x_1, x_2, ..., x_k$ are the distinct points $\leqslant s$ such that $\varphi_i^s(x_j)$ is convergent $\neq \varphi_{G(i,s)}(x_j)$ for each j such that $1 \leqslant j \leqslant k$. If $k \leqslant n$, then let $\varphi_p(x) = \varphi_{G(i,s)}(x)$ for $x \notin \{x_1, x_2, ..., x_k\}$ and $\varphi_p(x) = \varphi_i^s(x)$ for $x \in \{x_1, x_2, ..., x_k\}$; else, let $p = G(i, s)$. The number of different elements in the range of $(\lambda s)[G(i, s)]$ is $\leqslant v(i)$. Each different $G(i, s)$ can contribute at most $n+1$ different programs in the range of $(\lambda s)[G'(i, s)]$. Therefore, the number of elements in the range of $(\lambda s)[G'(i, s)]$ is bounded by $(n+1) \cdot v(i) = v'(i)$. For every $\varphi_i \in \mathscr{S}$, if $\varphi_j = \varphi_i$, then $\text{limit}_s\, G(i, s) = \text{limit}_s\, G(j, s)$. Hence, $\text{limit}_s\, G'(i, s) = \text{limit}_s\, G'(j, s)$. Let $\{x_1, x_2, ..., x_k\}$ be the anomalies of $\text{limit}_s\, G(i, s)$ in computing φ_i. Since $\varphi_i \in \mathscr{S}$, $k \leqslant n$. For every sufficiently large s such that $\varphi_i^s(x_j)$ converges for all $x_j \in \{x_1, x_2, ..., x_k\}$, $\varphi_{G'(i,s)} = \varphi_i$. Therefore, G' and v' witness that $\mathscr{S} \in PLSR^0$. ∎

Corollary 5.3 below shows that allowing a finite uniformly bounded number of anomalies in explanatory programs does not increase the inferring power of Popperian machines in inferring nearly minimal size programs. In contrast, $MEX^n \subset MEX^{n+1}$.

COROLLARY 5.3. $(\forall n)[PMEX^0 = PMEX^n]$.

Let $PMEX$ denote $PMEX^0$. By Theorem 4.3 and $PMEX^n \subseteq MEX^n$, we have the following.

COROLLARY 5.4. $ZERO^* \notin PMEX$.

Corollary 5.4 and Theorem 5.5 below show that with the cost of a finite but not uniformly bounded number of anomalies, nearly minimal size programs can be inferred without reducing the inferring power of Popperian machines.

THEOREM 5.5. $PMEX^* = PEX$.

Proof. Consider the proof of Theorem 4.1. It is straightforward to verify that M' is a Popperian machine if M is Popperian. Hence, the proof shows that $PMEX^* = PEX^* = PEX$. ∎

Since $ZERO^* \in PEX$, by Corollary 5.4 and Theorem 5.5, we have

COROLLARY 5.6. $PMEX \subset PMEX^*$.

Also by Theorem 4.3, we have.

COROLLARY 5.7. $(\forall n)[PMEX^* \nsubseteq MEX^n]$.

By Corollary 3.4 and the fact that $\mathcal{S}^a \notin PEX$ (Case and Ngo Manguelle, in press; Case and Smith, 1979), we have

COROLLARY 5.8. $PMEX \subset MEX^0$.

The proof of next theorem is constructive.

THEOREM 5.9. *If* $\mathcal{S}_1, \mathcal{S}_2 \in PMEX^a$, *then* $\mathcal{S}_1 \cup \mathcal{S}_2 \in PMEX^a$.

Proof. Suppose that there exist Popperian machines M_1 and M_2 such that $\mathcal{S}_1 \subseteq PMEX^a(M_1)$ and $\mathcal{S}_2 \subseteq PMEX^a(M_2)$. We define M thus. M on input $f \lceil x$, computes $M_1(f \lceil x)$ and $M_2(f \lceil x)$. M then compares the number of anomalies n_1 in $\varphi_{M_1(f \lceil x)} \lceil x$ and the number of anomalies n_2 in $\varphi_{M_2(f \lceil x)} \lceil x$.

Case 1. If $a = *$; M then outputs, suppressing co-final repetitions, the least program in $\{M_1(f \lceil x), M_2(f \lceil x)\}$ with fewer anomalies.

Case 2. $a \neq *$. If both n_1 and $n_2 \leqslant a$, then M outputs, surpressing co-final repetitions, the smaller of $M_1(f \lceil x)$ and $M_2(f \lceil x)$. If only one of n_1 and $n_2 \leqslant a$, say $n_1 \leqslant a$ and $n_2 > a$, then M outputs, surpressing co-final repetitions, $M_1(f \lceil x)$.

It can easily be verified that M $PMEX^a$-identifies $\mathcal{S}_1 \cup \mathcal{S}_2$. ∎

COROLLARY 5.10. *Given* M_1 *and* M_2 *witnessing that* $\mathcal{S}_1 \in PMEX$, $\mathcal{S}_2 \in PEX$, *respectively, we effectively find a Popperian machine* M *such that* M *PEX-identifies* $\mathcal{S}_1 \cup \mathcal{S}_2$ *and PMEX-identifies* $\mathcal{S}_1$; *furthermore, if* h *is such that* $\mathcal{S}_1 \subseteq PMEX(M_1, h)$, $\mathcal{S}_1 \subseteq PMEX(M, h)$.

Proof. The corollary follows from the $a \neq *$ case of the proof of Theorem 5.9. ∎

In (Chen, 1981), we show that there exist $\mathcal{S}_1, \mathcal{S}_2 \in MEX^0$ such that $\mathcal{S}_1 \cup \mathcal{S}_2 \notin EX^*$.

6. Bounded Mind Changes

We say that an IIM changes its mind when it outputs a new program. The bound on the number of changes of output is a first approximation to a bound on the complexity of IIMs. For example, $\mathcal{S}^0 = \{f \mid \varphi_{f(0)} = f\}$ can be EX-identified with no mind changes (Case and Smith, 1978, 1979). On the other hand, it can be easily shown that there is no *uniform* upper bound in mind changes for any IIM to EX-identify $ZERO^*$.

DEFINITION 6.1. Suppose $Q \in \{P, PM, M, \Lambda\}$, where Λ denotes the empty string.

(a) An IIM M QEX^a_b-identifies $\mathcal{S}$ (written: $\mathcal{S} \subseteq QEX^a_b(M)$) iff for all $f \in \mathcal{S}$, M QEX^a-identifies f and M fed f makes no more than b (if $b = *$, finitely many) mind changes.

(b) $QEX^a_b = \{\mathcal{S} \mid (\exists M)[\mathcal{S} \subseteq QEX^a_b(M)]\}$.

Obviously, QEX^a_* is QEX^a. Since the construction in the proof of Theorem 3.1 does not affect the number of mind changes, we have that for each $Q \in \{P, PM, M, \Lambda\}$, QEX^a_b is acceptable numbering independent.

In (Case and Smith, 1979), it is shown that $[EX^a_b \subseteq EX^c_d] \Leftrightarrow [a \leqslant c$ and $b \leqslant d]$. Hence, all of the tradeoffs between bounds on number of anomalies and bounds on number of mind changes are partial. On the other hand, it is shown in Case and Ngo Manguelle (in press) that $PEX^a_b \subseteq PEX^c_d \Leftrightarrow [[a \leqslant c$ and $b \leqslant d]$ or $d = *$ or $[a, b, c, d \in N$ and $G(a, b, c) \leqslant (1 + d)]]$, where $G(a, b, c) = [1 + \text{floor}(a/(c+1))] \cdot (1 + b)$. Hence, for Popperian machines, it is possible to completely tradeoff anomaly for mind change bounds. In this section we determine, for each $Q \in \{PM, M\}$, the exact containment relations between the classes QEX^a_b. We shall see that, as in the case of EX^a_b, the tradeoffs between anomaly and mind change bounds are partial for both $PMEX^a_b$ and MEX^a_b.

We suppose without loss of generality that if an IIM M QEX^a_m-identifies a class of recursive functions, then for any input f, M changes its mind at most m times.

THEOREM 6.1. *Suppose $Q \in \{P, \Lambda\}$. Then for all m, $QEX^a_m \subseteq QMEX^a$.*

Proof. Suppose that $\mathcal{S} \in QEX^a_m$, then there exists IIM M which QEX^a_m-identifies $\mathcal{S}$. Let $v(i) = m + 2$. Let

$$G(i, s) = M(\varphi^s_i), \quad \text{if } M(\varphi^s_i) \text{ is defined;}$$
$$= i_\varphi, \quad \text{otherwise.}$$

Clearly G and v witness that $\mathcal{S} \in LSR^a$ and for every $\varphi_i \in \mathcal{S}$, $M(\varphi_i) = \text{limit}_s\, G(i, s)$. By Theorem 3.3, we have that $\mathcal{S} \in MEX^a$. If $\mathcal{S} \in PEX^a_m$, then M is Popperian. Hence, the range of G contains only programs for total functions; therefore, $\mathcal{S} \subseteq PLSR^a$. ∎

COROLLARY 6.2. *Suppose $Q \in \{P, \Lambda\}$. Then for all m, $QEX^a_m \subset QMEX^a$.*

Proof. By the previous theorem we have $QEX_m^a \subseteq QMEX^a$. In Case and Smith (1979) it is shown that $EX_m^a \subset EX_{m+1}^a$, and in Case and Ngo Manguelle (in press), it is shown that $PEX_m^a \subset PEX_{m+1}^a$; hence, we have $QEX_m^a \neq QMEX^a$. ∎

We next proceed to obtain a useful property of MEX_m^n.

DEFINITION 6.2. Suppose $\mathscr{A}$ and $\mathscr{B}$ are classes of recursive functions. $\mathscr{B}$ is an *a-cover* of $\mathscr{A}$ iff for each $f \in \mathscr{A}$, there exists $g \in \mathscr{B}$ such that $g =^a f$.

DEFINITION 6.3. $\mathscr{S}$ is *a-immune* iff (i) $\mathscr{S}$ is infinite and (ii) every recursively enumerable subclass (Rogers, 1967) of $\mathscr{S}$ has a finite a-cover.

DEFINITION 6.4. $\mathscr{S}$ is *a-isolated* iff $\mathscr{S}$ is finite or $\mathscr{S}$ is a-immune.

LEMMA 6.3. *Suppose that $\mathscr{A}$ is a recursively enumerable class and that M is an IIM which MEX_m^a-identifies $\mathscr{A}$. Then the set $A = \{M(f) \mid f \in \mathscr{A}$ and M on input f changes its mind* exactly *m times$\}$ is finite.*

Proof. Suppose by way of contradiction otherwise. Then A is an infinite set. Let $p_0, p_1, p_2, ...$ be a recursive enumeration of programs for the functions in $\mathscr{A}$. Suppose that h is a recursive, monotone nondecreasing function such that M $MEX_m^a(h)$-identifies $\mathscr{A}$. By implicit use of the recursion theorem, we describe a program e thus.

Begin program e.

On input x, e enumerates $p_0, p_1, p_2, ...,$ and searchs for a program p_i such that M on input φ_{p_i} changes its mind exactly m times and $M(\varphi_{p_i}) > h(e)$. (Such a p_i must exist since A is infinite.) When the first such p_i is found, e just emulates p_i on x.

End program e.

Since $M(\varphi_e) = M(\varphi_{p_i}) > h(e)$, M does not $MEX_m^a(h)$-identify $\varphi_e \in \mathscr{A}$, a contradiction. ∎

THEOREM 6.4. *Suppose $\mathscr{S} \in MEX_m^n$. Then $\mathscr{S}$ is an n-isolated class.*

Proof. The proof is by induction on m. If $m = 0$, the theorem follows immediately from Lemma 6.3. Suppose that the theorem is true for $m = k$. Suppose that M MEX_{k+1}^n-identifies $\mathscr{S}$. If $\mathscr{S}$ contains a recursively enumerable subclass $\mathscr{A}$ which is infinite. It suffices to show the $\mathscr{A}$ is n-immune. Since $\mathscr{A}$ is a recursively enumerable class in MEX_{k+1}^n, by Lemma 6.3 the set $A = \{M(f) \mid f \in \mathscr{A}$ and M on input f changes its mind exactly $k+1$ times$\}$ is finite. Let $\mathscr{B} = \{f \mid (\exists i \in A)[f(x) = \varphi_i(x)$ if $x \in \text{domain}(\varphi_i)$; else $f(x) = 0]\}$. $\mathscr{B}$ is finite since A is. Clearly $\mathscr{B}$ is an n-cover of the class $\mathscr{A}' = \{f \in \mathscr{A} \mid (\exists g \in \mathscr{B})[f =^n g]\}$ and $\{f \in \mathscr{A} \mid M$ on input f changes its mind exactly $k+1$ times$\} \subseteq \mathscr{A}'$. Let $\mathscr{A}'' = \mathscr{A} - \mathscr{A}'$. Hence, M MEX^n_k-identifies $\mathscr{A}''$. $\mathscr{A}''$ is n-isolated. Since $\mathscr{B}$ is finite, $\mathscr{A}''$ is also an r.e. subclass of the r.e. class $\mathscr{A}$. Therefore $\mathscr{A}''$ is finite or n-immune. Hence, $\mathscr{A} = \mathscr{A}' \cup \mathscr{A}''$ is n-immune. ∎

The proof of above theorem does not work for MEX^*_m, but we conjecture that the theorem is true for this case.

THEOREM 6.5. $(\forall a)(\forall m)[PEX^0_0 \not\subseteq MEX^a_m]$.

Proof. Let $CONST = \{f \mid (\exists i)[f = \lambda x[i]]\}$.

It is clear that $CONST \in PEX^0_0$ and $CONST$ is a recursively enumerable class without any finite a-cover. By Theorem 6.4, $CONST \notin MEX^a_m$ for all $a \in N$. The proof of Theorem 6.4 is easily modified to show that for $\mathscr{A} = CONST$, $\mathscr{A} \notin MEX^*_m$. ∎

COROLLARY 6.6. $(\forall m)(\forall n)[PMEX^{n+1}_0 \not\subseteq MEX^n_m]$.

Proof. Consider the class of recursive functions $ZERO^{n+1} = \{f \mid f =^{n+1} \lambda x[0]\}$. It is clear that $ZERO^{n+1} \in PMEX^{n+1}_0$ and $ZERO^{n+1}$ is a recursively enumerable class without any finite n-cover. By Theorem 6.4, $ZERO^{n+1} \notin MEX^n_m$. ∎

COROLLARY 6.7. *Suppose* $Q \in \{P, A\}$. $(\forall m)[QMEX^a_m \subseteq QMEX^b_m \Leftrightarrow a \leqslant b]$.

The question of whether or not $MEX^{n+1}_0 \not\subseteq EX^n$ remains open; however, we have

THEOREM 6.8. $(\forall m)(\forall n)[MEX^{n+1}_0 \not\subseteq EX^n_m]$.

Proof. Let $\mathscr{S}^{n+1} = \{f \mid \varphi_{f(0)} =^{n+1} f$ and $f(0) < \min(f)\}$. Clearly, the IIM M_0 which on input f, outputs $f(0)$ only, MEX^{n+1}_0-identifies $\mathscr{S}^{n+1}$. We then show that for all IIMs M, $\mathscr{S}^{n+1} \not\subseteq EX^n_m(M)$. We present the $n=0$ case only; the other cases are similar. Suppose M is any IIM. We implicitly define below a program e which computes a partial function which diverges on exactly one input such that some total extension of φ_e is in $\mathscr{S}^1$ and cannot be EX^0_m-identified by M.

φ_e is defined by stages. Let $(\varphi_e)^s$ denote the finite part of φ_e defined before stage s. Let a^s_1, a^s_2 denote respectively the least and second least numbers which are not in the domain of $(\varphi_e)^s$. By the recursion theorem (Rogers, 1967), we may set $(\varphi_e)^0 = \{(0, e)\}$.

Begin stage s. Write $s = \langle i, j\rangle$. Let $\sigma = (\varphi_e)^s \cup \{(a_1^s, i)\}$. If $M(\sigma) \neq M((\varphi_e)^s)$, then set $(\varphi_e)^{s+1} = \sigma$; otherwise, set $(\varphi_e)^{s+1} = (\varphi_e)^s \cup \{(a_2^s, 0)\}$.
End stage s.

Since by convention M makes no more than m mind changes, for some a, $\text{limit}_s\, a_1^s = a < \infty$. Then $\text{domain}(\varphi_e) = (N - \{a\})$ and for any (total) recursive functions f and g such that $\varphi_e \subset f$ and $\varphi_e \subset g$, $M(f) = M(g)$. Since there are infinitely many f such that $\varphi_e \subset f$, there must exist f and g such that $\varphi_e \subset f$, $\varphi_e \subset g$, $f \neq g$, $\min(f) > e$ and $\min(g) > e$. Pick such a pair f, g. Then $f, g \in \mathcal{S}^1$ and $M(f) = M(g)$ but $f \neq g$. Hence, at least one of them cannot be EX_m^0-identified by M. ∎

It is shown in Case and Ngo Manguelle (in press) that $PEX_{m+1} \not\subseteq EX_m^*$. Our next theorem strengthens this result.

LEMMA 6.9. *Suppose* $Q \in \{P, A\}$.

$$\bigcup_{\varphi'} QMEX_b^a(\lambda x[x], \varphi') \subseteq \bigcup_h QMEX_b^a(h, \varphi).$$

Proof. Suppose $\mathcal{S} \in \bigcup_{\varphi'} QMEX_b^a(\lambda x[x], \varphi')$. Then there exist M and φ' such that $\mathcal{S} \subseteq QMEX_b^a(M, \lambda x[x], \varphi')$. Since both φ and φ' are acceptable numberings, there exist recursive functions w_1 and w_2 such that φ' is reduced to φ by w_1 and φ is reduced to φ' by w_2. Let $h(x) = max\{w_1(y) \mid y \leqslant w_2(x)\}$. We define an IIM M' thus. On input $f\lceil x$, M' simulates M on input $f\lceil x$. If M outputs a program p, M' then outputs $w_1(p)$. Hence, if M EX_b^a-identifies f in φ, then M' EX_b^a-identifies f in φ'. If M is a Popperian IIM, then M' is also a Popperian IIM. Suppose $f \in \mathcal{S}$, it remains to show that $M'(f) \leqslant h(\min_\varphi(f))$. Since $f \in \mathcal{S}$, $M(f) = \min_{\varphi'}(f)$. $M'(f) = w_1(M(f)) = w_1(\min_{\varphi'}(f))$. $h(\min_\varphi(f)) = max\{w_1(y) \mid y \leqslant w_2(\min_\varphi(f))\} \geqslant max\{w_1(y) \mid y \leqslant \min_{\varphi'}(f)\} \geqslant w_1(\min_{\varphi'}(f)) = M'(f)$. ∎

LEMMA 6.10. *Suppose φ' and g are such that* (i) *for all n, φ'_{2n} is total,* (ii) *for all m, n, if $m \neq n$, then $\varphi'_{2m} \neq \varphi'_{2n}$,* (iii) *$g$ is a strictly monotone increasing function for which the range of g contains only even numbers, and* (iv) *there exists a k such that for all n, there are at most $n + k$ different functions in $\{\varphi'_1, \varphi'_3, \varphi'_5, \ldots, \varphi'_{g(n)+1}\}$. Then there are infinitely many φ' minimal indices in the range of g.*

Proof. Suppose by way of contradiction otherwise. Then there are finitely many φ' minimal indices in the range of g. Let n_0 be such that for all $n \geqslant n_0$, $g(n) > \min_{\varphi'}(\varphi'_{g(n)})$ and $\min_{\varphi'}(\varphi'_{g(n)})$ is odd. Hence, for all $n \geqslant n_0$, $\{\varphi'_{g(n_0)}, \varphi'_{g(n_0+1)}, \ldots, \varphi'_{g(n)}\} \subseteq \{\varphi'_1, \varphi'_3, \ldots, \varphi'_{g(n)+1}\}$ since g is a strictly monotone increasing function. By the hypothesis, there is a k such that for all n,

$n+k \geqslant$ the number of different partial functions in $\{\varphi'_1, \varphi'_2,..., \varphi'_{g(n)+1}\} \geqslant$ the number of different partial functions in $\{\varphi'_{g(n_0)}, \varphi'_{g(n_0+1)},..., \varphi'_{g(n)}\}$. Since if $m \neq n$, then $\varphi'_{g(m)} \neq \varphi'_{g(n)}$, the number of different partial functions in $\{\varphi'_{g(n_0)}, \varphi'_{g(n_0+1)},..., \varphi'_{g(n)}\}$ is $n-n_0+1$. Therefore, for every $n \geqslant n_0$ the number of different partial functions in $(\{\varphi'_1, \varphi'_3,..., \varphi'_{g(n)+1}\} - \{\varphi'_{g(n_0)}, \varphi'_{g(n_0+1)},..., \varphi'_{g(n)}\})$ is less than or equal to n_0+k-1. Hence the number of different partial functions computed by odd indices not computed by even indices is finite. Therefore, since the even φ' programs compute only total functions, φ' computes but a finite number of non-total functions, a contradiction. ∎

THEOREM 6.11. $(\forall m)[PMEX^0_{m+1} \nsubseteq EX^*_m]$.

Proof. Suppose $e \geqslant 0$. A finite sequence $(x_1, x_2,..., x_e)$ is *strictly monotone* iff $[e=0$ or $x_1 < x_2 < \cdots < x_e]$. ( ) denotes the empty sequence. i.e. the case $e=0$. Let $\Sigma_m =$ the set of strictly monotone finite sequences $(x_1, x_2,..., x_e)$ such that $0 \leqslant e \leqslant m$. A function f is a *step up function* at step up points $(x_1, x_2,..., x_e)$ iff *case* 1. $e=0$; then $f=\lambda x[0]$, and *case* 2. $e>0$; then $f(x)=0$ if $x<x_1$; $f(x)=i$ if $x_i \leqslant x < x_{i+1}$; $f(x)=e$ if $x \geqslant x_e$. For $\alpha = (x_1, x_2,..., x_e)$, f_α denotes the step up function at step up points $(x_1, x_2,..., x_e)$. Let $\mathscr{S}_{(x_1, x_2, ..., x_e)} = \{f_\alpha \mid \alpha = (x_1, x_2,..., x_e, x_{e+1})$, where $x_{e+1} > x_e\}$. $\mathscr{S}_{()} = \{f_\alpha \mid \alpha = (x_1)$, where $x_1 \in N\}$. We also let $\mathscr{S}_0 = \{\lambda x[0]\}$. Note that if α and β are distinct elements of $\Sigma_m \cup \{0\}$, then $\mathscr{S}_\alpha \cap \mathscr{S}_\beta = \varnothing$. Let $\mathscr{S} = \bigcup\{\mathscr{S}_\alpha \mid \alpha \in \Sigma_m \cup \{0\}\}$. Fix a canonical indexing (Machtey and Young, 1978; Rogers, 1967) of the elements of Σ_m. Let α_i be the element of Σ_m with canonical index i. Clearly there is a recursive function w such that $(\forall i)[\{\varphi_{w(i,j)} \mid j \in N\} = \mathscr{S}\alpha_i]$ and $(\forall i,j,k)$ [if $j \neq k$ then $\varphi_{w(i,j)} \neq \varphi_{w(i,k)}$].

We will now construct an acceptable numbering φ' such that a set of φ' programs for the functions in $\mathscr{S}$ are effectively enumerated by some recursive function z such that for every i,j, if $i \neq j$, then $\varphi_{z(i)} \neq \varphi_{z(j)}$ and for each $\alpha \in \Sigma_m$, the range of z contains infinitely many φ' minimal indices for the functions in $\mathscr{S}_\alpha$. We define φ' thus.

$$\begin{array}{lllll} \varphi'_0 = \lambda x[0] & \varphi'_2 = \varphi_{w(0,0)} & \varphi'_4 = \varphi_{w(0,1)} & \varphi'_6 = \varphi_{w(1,0)} & \cdots \\ \varphi'_1 = \lambda x[\uparrow] & \varphi'_3 = \varphi_0 & \varphi'_5 = \lambda x[\uparrow] & \varphi'_7 = \varphi_1 & \cdots \end{array}$$

Generally

$$\varphi'_0 = \lambda x[0], \varphi'_{2(\langle i,j\rangle+1)} = \varphi_{w(i,j)},$$

and

$$\begin{aligned} \varphi'_{2\langle i,j\rangle+1} &= \lambda x[\uparrow], \qquad \text{if} \quad i \neq 0; \\ &= \varphi_j, \qquad \text{otherwise}. \end{aligned}$$

It is clear that φ' is effective and φ can be reduced to φ' by $\lambda x[2 \cdot \langle x, 0\rangle + 1]$. Hence, φ' is an acceptable numbering. Let $z(x) = 2x$. Then z enumerates a set of φ' programs for the functions in $\mathscr{S}$ such that for every i, j, if $i \neq j$, then $\varphi_{z(i)} \neq \varphi_{z(j)}$. We claim that

$$\text{for each } \alpha \in \Sigma_m, \text{ the range of } z \text{ contains infinitely many } \varphi' \text{ minimal indices for the functions in } \mathscr{S}_\alpha. \tag{6.2}$$

To see this, let $g(x) = 2(\langle j, x\rangle + 1)$. Clearly, g is 1–1 monotone increasing and $\mathscr{S}_{\alpha_j} = \{\varphi'_{g(x)} \mid x \in N\}$. We want to show that there are infinitely many φ' minimal indices in the range of g. We see that the number of different functions which are computed by odd indices $\leqslant 2\langle j, x\rangle + 1$, is $\leqslant j + x + 2$. By Lemma 6.10, with $k = j + 2$, there are infinitely many minimal indices in the range of g. This establishes (6.2).

Now there is an IIM M which on input $f \lceil x$, finds the step up points $(x_1, x_2, \ldots, x_e)$ (if any); if $e = 0$, M then outputs 0, otherwise M outputs the even φ' index which computes $f_{(x_1, x_2, \ldots, x_e)}$. M is a Popperian machine since M outputs even indices only. Let $\mathscr{S}' = \{f \mid f \in \mathscr{S}$ and $\min_{\varphi'}(f)$ is even$\}$. By (6.2) we know that for every $\alpha \in \Sigma_m$, $\mathscr{S}' \cap \mathscr{S}_\alpha$ contains infinitely many elements. For every $f \in \mathscr{S}'$, f has at most $m + 1$ step up points, so M on input f, will change its mind at most $m + 1$ times and $\varphi'_{M(f)} = f$. Hence by Lemma 6.9, $\mathscr{S}' \subseteq PMEX_{m+1}(M)$.

Note. $\lambda x[0]$ is in $\mathscr{S}'$ since $\varphi'_0 = \lambda x[0]$.

It remains to show that no IIM EX^*_m-identifies $\mathscr{S}'$. Let us consider the case $m = 0$. Suppose by way of contradiction otherwise. Then there is an IIM M' such that M' EX^*_0-identifies $\mathscr{S}'$. Since $\lambda x[0] \in \mathscr{S}'$, there is a sufficiently large x_1 such that for every $y \geqslant x_1$, $M'(\lambda x[0] \lceil y) = M'(\lambda x[0] \lceil x_1) = p_1$ and $\varphi'_{p_1} =^* \lambda x[0]$. Since $\mathscr{S}_{()} \cap \mathscr{S}'$ contains infinitely many elements, there is an x'_1 such that $x'_1 \geqslant x_1$ and $f_{(x'_1)} \in \mathscr{S}_{()} \cap \mathscr{S}'$. We then consider M' on input $f_{(x'_1)}$. Since $f_{(x'_1)} \in \mathscr{S}'$, there is a sufficiently large x_2 such that for all $y \geqslant x_2$, $M'(f_{(x'_1)} \lceil y) = M'(f_{(x'_1)} \lceil x_2) = p_2$ and $\varphi'_{p_2} =^* f_{(x'_1)}$. It is clear that $p_1 \neq p_2$. We therefore force M' to change its mind, a contradiction. For $m > 0$, similarly there is a step up function $f_{(x'_1, x'_2, \ldots, x'_{m+1})} \in \mathscr{S}'$ such that M' on input $f_{(x'_1, x'_2, \ldots, x'_{m+1})}$ will be forced to change its mind at least $m + 1$ times, a contradiction. ∎

COROLLARY 6.12. *Suppose* $Q \in \{P, A\}$. *Then* $QMEX^a_m \subseteq QMEX^b_n \Leftrightarrow a \leqslant b$ *and* $m \leqslant n$.

Proof. It immediately follows from Theorem 6.6 and 6.11. ∎

COROLLARY 6.13. $(\forall m)[PMEX \nsubseteq MEX^*_m]$.

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