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SIMILAR CONSTRUCTIONS FOR YOUNG TABLEAUX AND INVOLUTIONS, AND THEIR APPLICATION TO SHIFTABLE TABLEAUX

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1. Introduction

It is well known (c.f. [2]) that the number of Young tableaux on the set $[n] = \{1, 2, \ldots, n\}$ equals the number of involutions of $[n]$. Therefore the recurrence

$$a_n = a_{n-1} + (n-1)a_{n-2}, \tag{1}$$

which has a straightforward combinatorial proof when a_n is the number of involutions of $[n]$, must also hold when a_n is the number of Young tableaux on $[n]$.

In this paper, we first give (Section 3) a combinatorial proof of (1) for Young tableaux which agrees under the Robinson–Schensted correspondence with the proof for involutions. The proof involves a recursive construction which depends in part on Schensted's insertion algorithm [6].

An immediate consequence of this construction is a straightforward proof of the fact that the number of Young tableaux on $[n]$ equals the number of involutions on $[n]$. The usual proof requires the use of the result that if a permutation corresponds to the pair of tableaux (P, Q), then its inverse corresponds to the pair (Q, P), but the proof here does not. Another consequence is a bijective proof of a theorem of Schützenberger, simpler than in [8], that the number of fixed points of an involution equals the number of columns of odd length in the corresponding tableau. These proofs are discussed in Section 4.

We then restrict the construction to shiftable tableaux (definition below) and obtain (in Section 5) a method for listing all shiftable tableaux. We also give (in Section 6) a characterization, in terms of certain decreasing subsequences, of the involutions which correspond to shiftable tableaux under the Robinson–Schensted correspondence, and a recursive description of these involutions.

2. Definitions

If $\lambda = (u_1 \geqslant u_2 \geqslant \cdots \geqslant u_r > 0)$ is a partition of n, a *generalized Young tableau* of shape λ is an array of n integers into r left justified rows, with u_i elements in row i, such that the rows are nondecreasing and the columns are strictly increasing. A *generalized shifted Young tableau* of shape $(u_1 > u_2 > \cdots > u_r > 0)$ is defined similarly, except that row i has been indented $(i-1)$ spaces. Shifted tableaux have been studied by Thrall [9], Sagan [5] and Worley [10], among others. A tableau is called *standard* if the integers it contains are $1, 2, \ldots, n$ and it is then said to be a tableau on $[n]$. T_n will denote the set of standard tableaux on $[n]$.

A tableau Y satisfies the *northeast condition* if each element $y_{ij} \in Y$ is greater than its neighbor $y_{i-1,j+1}$ diagonally upward and to its right, whenever such a neighbor exists. A Young tableau Y is called *shiftable* if it satisfies the northeast condition and has a strict shape (i.e., $u_1 > u_2 > \cdots > u_r > 0$). Thus a Young tableau is shiftable if and only if a shifted Young tableau results from indenting its ith row by $i-1$ spaces $(i = 1, 2, \ldots, r)$. The Robinson–Schensted correspondence [4, 6] associates with a permutation π of $[n]$ a pair $(P(\pi), Q(\pi))$ of Young tableaux on $[n]$ of the same shape. It is well known that if the permutation happens to be an involution, then $P(\pi) = Q(\pi)$. Thus, when we refer to the "involution that corresponds to the tableau Y", we mean of course the involution that corresponds to the pair of tableaux (Y, Y).

3. Constructions

In this section, we give the constructions for standard Young tableaux and involutions which can be used to prove eq. (1) for each case. The essential steps of Algorithm 3.1 for Young tableaux have been used before (Lascoux [3] and Burge [1]), but the relationship between this algorithm and Algorithm 3.2 for involutions has not previously been studied.

Algorithm 3.1 below gives a mapping from the set $T_{n-1} \cup ([n-1] \times T_{n-2})$ to the set T_n.

Algorithm 3.1. Given an element E of $T_{n-1} \cup ([n-1] \times T_{n-2})$, construct an element of T_n.

(A) If $E = X \in T_{n-1}$, then adjoint n to the end of the first row of X.

(B) If $E = (j, X)$, $j < n$ and $X \in T_{n-2}$, then

(1) Increase all elements $\geqslant j$ of X by one, obtaining a Young tableau on $[n-1] - \{j\}$.

(2) Insert j using Schensted's algorithm. (Put j in the place of the first element y of row 1 that is greater than j, bumping y out of row 1. If there is no such y, then put j at the end of row 1 and stop. Next, insert y into row 2 in a similar manner. Repeat this process until some

element is inserted at the end of a row, then stop.)

(3) Place n at the end of row $q+1$, where q is the row in which Schensted's insertion algorithm stops.

We denote the resulting tableau by $X+(j, n)$, where $j=n$ in Case A, and $j<n$ in Case B. For example, if X is the tableau on [11] shown in Fig. 1, then $X+(5, 13)$ is the tableau on [13] shown in Fig. 2.

It is easy to give an algorithm that recovers the input of Algorithm 3.1 from its output and then establish that the mapping is a bijection.

The recurrence of eq. (1) has an easy combinatorial proof for involutions, which is analogous to the proof for Young tableaux. Let I_n denote the set of involutions of $[n]$. An involution of $[n]$ is specified by a list of the pairs of elements interchanged by the involution and the singletons fixed by it. Algorithm 3.2 gives a mapping from the set $I_{n-1} \cup ([n-1] \times I_{n-2})$ to the set I_n.

X =

1	2	4	7	8
3	5	9	10	
6				
11				

Fig. 1.

X + (5,13) =

1	2	4	5	9
3	6	8	11	
7	10			
12	13			

Fig. 2.

Algorithm 3.2. Given an element $E \in I_{n-1} \cup ([n-1] \times I_{n-2})$, construct an element of I_n.

(A) If $E = \pi \in I_{n-1}$, then add the singleton n to π.

(B) If $E = (j, \pi)$, $j<n$ and $\pi \in I_{n-2}$, then increase by one all elements $\geqslant j$ of π (obtaining an involution of $[n-1]-\{j\}$) and add the pair (j, n).

We will denote the resulting involution by $\pi+(j, n)$, where $j=n$ in Case A, and $j<n$ in Case B. For example, if π is the involution of [11] whose pairs are indicated in Fig. 3, then $\pi+(5, 13)$ is the involution of [13] with pairs shown in Fig. 4. It is easily seen that the mapping given by the algorithm is a bijection.

The algorithm can also be described in terms of the sequence that represents an involution. If $\pi(1), \pi(2), \ldots, \pi(n-1)$ is in I_{n-1}, then $\pi+(n, n)=\pi(1), \pi(2), \ldots, \pi(n-1), n$. If $\pi(1), \pi(2), \ldots, \pi(n-2)$ is in I_{n-2} and $j<n$, then $\pi+(j, n)$ is the involution obtained by inserting n between $\pi(j-1)$ and $\pi(j)$,

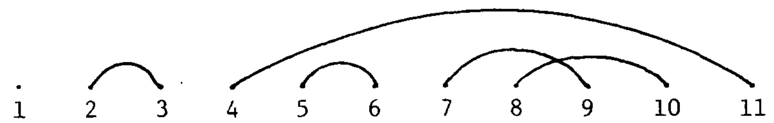

Fig. 3. An involution π.

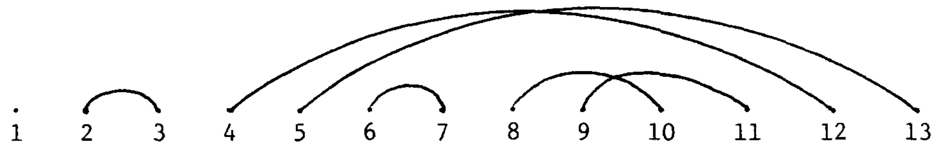

Fig. 4. The involution $\pi + (5, 13)$.

increasing by one of elements of π that are $\geqslant j$ and adjoining the element j at the end of the sequence. For example, from the sequence $\pi = 1, 3, 2, 11, 6, 5, 9, 10, 7, 8, 4$, that represents the involution in Fig. 3, we obtain in this manner the sequence $\pi + (5, 13) = 1, 3, 2, 12, 13, 7, 6, 10, 11, 8, 9, 4, 5$ that represents the involution of Fig. 4.

Note that the involutions of Figs. 3 and 4 correspond under the Robinson–Schensted correspondence to the tableaux of Figs. 1 and 2, respectively. More generally, the Robinson–Schensted correspondence transforms Algorithm 3.2 into Algorithm 3.1, as stated more precisely in the following theorem.

Theorem 3.1. *Let $P(\pi)$ be the tableau that corresponds under the Robinson–Schensted correspondence to the involution π. Then $P(\pi + (j, n)) = P(\pi) + (j, n)$.*

Proof. The theorem is obviously true for $j = n$, so assume $j < n$. Let $P(\pi)^+$ be the tableau which results from increasing the elements $\geqslant j$ in Step B1 in the construction of $P(\pi) + (j, n)$.

For the involution $\tau = \pi + (j, n)$, let τ_k denote the subsequence $\tau(1), \tau(2), \ldots, \tau(k)$. Since τ_{n-1} is obtained from π by inserting n between $\pi(j-1)$ and $\pi(j)$, and increasing by one the elements of π that are $\geqslant j$, it is not difficult to see that $P(\tau_{n-1})$ is the same as $P(\pi)^+$, except that $P(\tau_{n-1})$ contains n at the end of some row.

$P(\tau)$ is constructed by inserting j into $P(\tau_{n-1})$, according to Schensted's algorithm, and thus $P(\tau)$ is the same as the tableau which results after inserting j into $P(\pi)^+$ in Step B2, except that $P(\tau)$ has n at the end of some row. We will show that n is at the end of row $q + 1$ of $P(\tau)$ (where q is as described in B3), and the proof will be complete, since this is its position after Step B3 in the construction of $P(\pi) + (j, n)$.

Let $c(k)$ be the cell occupied by k in the tableau $P(\pi)$. It is a well known property of the Robinson–Schensted correspondence that, since π is an involution, $c(k)$ is the new cell which is added to the shape when $\pi(k)$ is inserted into $P(\pi_{k-1})$. It follows that for $j < k < n$, the cells occupied by $\tau(1), \tau(2), \ldots, \tau(k)$ are the cells $c(1), c(2), \ldots, c(k-1)$, and the cell which contains n. Furthermore, if cell $c(k)$ $(k<n)$ is occupied by n when $\tau(k+1)$ is inserted, then n will be bumped to a new cell at the current end of the next row; otherwise, cell $c(k)$ will be the new cell added to the shape.

Since q is the row in which the insertion of j into $P(\pi)^+$ stops, there is a sequence $j=k_0<k_1<\cdots<k_{q-1}$, where k_i is the first element of row i of $P(\pi)^+$ which is greater than k_{i-1} $(i=1,\ldots,q-1)$, and k_{q-1} is larger than all elements of row q (this is the sequence of elements bumped to the next row by the insertion of j).

Let $q\geqslant 2$. We claim that just after the insertion of $\tau(k_i)$ into $P(\tau_{k_i-1})$, n is in cell $c(k'_{i+1})$ $(i=0,\ldots,q-2)$, where $k'=k$ if $k<j$ and $k'=k-1$ if $k>j$. Since $n=\tau(j)$, the tableau contains only cells $c(1), c(2),\ldots,c(j-1)$ just prior to the insertion of n. When inserted, n goes to the current end of row 1, which is cell $c(k'_1)$. Thus the claim is true for $i=0$.

Let $1\leqslant i\leqslant q-2$ (and therefore $k_i>j$) and suppose the claim is true for $i-1$. By the inductive hypothesis, n is in cell $c(k'_i)$ of row i just after $\tau(k_{i-1})$ is inserted. It will be bumped from this cell when $\tau(k_i)$ is inserted, and will go to the current end of row $i+1$.

The tableau contains only cells $c(1), c(2),\ldots,c(k'_i)$ just prior to the insertion of $\tau(k_i)$ and since k_{i+1} is the first element of row $i+1$ larger than k_i, cell $c(k'_{i+1})$ is the first cell of this row that is not in $P(\tau_{k_i-1})$. Thus cell $c(k'_{i+1})$ is the cell of row $i+1$ to which n is bumped, as claimed.

Thus, after the insertion of $\tau(k_{q-2})$, n is in cell $c(k'_{q-1})$ and when $\tau(k_{q-1})$ is inserted, n is bumped to the current end of row q $(q\geqslant 2)$. For $q=1$, the insertion of $\tau(k_{q-1})=n$ also results in the placement of n at the current end of row q. For $q\geqslant 1$, since k_{q-1} is larger than all elements in row q, all the cells of row q of $P(\pi)$ have been filled before n is placed in this row and thus the cell into which n is placed does not exist in $P(\pi)$. Since this cell does not exist in $P(\pi)$, n will not be bumped again and thus $P(\tau_{n-1})$ contains n at the end of row q.

When j is inserted into $P(\tau_{n-1})$ to get $P(\tau)$, element k_i is bumped to row $i+1$ $(i=0,1,\ldots,q-1, q\geqslant 1)$ and n is bumped to row $q+1$. This completes the proof. □

Algorithms 3.1 and 3.2 can be applied recursively to give a mapping ϕ between involutions and Young tableaux and this mapping turns out to agree with the Robinson–Schensted correspondence. Specifically, for an involution τ, reverse the steps of Algorithm 3.2 to find π and (j,n) such that $\tau=\pi+(j,n)$. Then apply Algorithm 3.1 to find $\phi(\tau)=\phi(\pi)+(j,n)$. A simple induction argument, using Theorem 3.1, shows that $\phi(\tau)=P(\tau)$.

4. Applications

In this section, we discuss two consequences of the combinatorial proofs of (1), implied by Algorithms 3.1 and 3.2, and their relationship, given by Theorem 3.1.

The first consequence is that the cardinalities of I_n and T_n are equal, since they satisfy the same recurrence. The usual method of proving this fact is to show that the Robinson–Schensted correspondence gives a bijection between I_n and the set of pairs (P, Q) of tableaux for which $P = Q$, and thus, in effect, a bijection between I_n and T_n. To show this, one usually must use Schützenberger's result [7] that if a permutation corresponds to the pair (P, Q), then its inverse corresponds to (Q, P). The advantage of using the algorithms here to describe the mapping is that the proof that it is indeed a bijection between I_n and T_n is immediate and does not rely on other results.

A second consequence, observed by G. Viennot, is a direct proof of the following theorem of Schützenberger [8]:

Theorem 4.1 (Schützenberger). *The number of fixed points of an involution equals the number of columns of odd length in the corresponding tableau.*

Proof. The proof is by induction on n. The theorem is clearly true for $n = 1$, so let $n > 1$ and let τ be an involution of $[n]$. Either τ was built from an involution of $[n-1]$ in Step A of Algorithm 3.2, or from an involution of $[n-2]$ in Step B.

In the first case, $\tau = \pi + (n, n)$ where $\pi \in I_{n-1}$ and τ has one more fixed point (specifically, $\tau(n) = n$) than π. From Theorem 3.1, the corresponding tableau is $P(\tau) = P(\pi) + (n, n)$, which has one more column of odd length (a column of length 1 containing n) then $P(\pi)$. By the inductive hypothesis, π has the same number of fixed points as $P(\pi)$ has columns of odd length, and therefore the same is true for τ and $P(\tau)$.

In the second case, $\tau = \pi + (j, n)$, where $\pi \in I_{n-2}$ and $j < n$, and τ has the same number of fixed points as π. From Theorem 3.1, $P(\tau) = P(\pi) + (j, n)$, which is obtained from $P(\pi)$ by adding two cells, one each at the end of rows q and $q + 1$. If these cells are in the same column, the number of cells in that column does not change in parity. If they are in different columns, the two columns will change in length from $q - 1$ to q and from q to $q + 1$, leaving the number of columns of odd length unchanged. Thus $P(\tau)$ has the same number of columns of odd length as $P(\pi)$. Now apply the inductive hypothesis and the proof is complete. □

Burge [1] gave a bijection which provides a proof of Schützenberger's theorem in the special case of involutions with no fixed points. In this case, his map is essentially the same as our ϕ. Schützenberger [8, p. 94] mentioned Burge's bijection, but remarked that he had been unable to find the setting in which Burge's construction is natural. It seems that the recursive constructions of Young tableaux and involutions given by Algorithms 3.1 and 3.2 provide this setting.

5. Shiftable tableaux

In this section, we determine when the construction given by Algorithm 3.1 produces shiftable tableaux, and we give a method for constructing all standard shiftable n-tableaux.

If X is the tableau on [9] shown in Fig. 5, then $X+(4, 11)$ and $X+(5, 11)$ are the tableaux on [11] shown in Figs. 6 and 7. Note that X and $X+(5, 11)$ are shiftable. That X is shiftable is a necessary condition for $X+(j, n)$ to be, as we will show in Lemma 5.1, but it is not a sufficient one, as is illustrated by the nonshiftable tableau $X+(4, 11)$.

1	2	4	6	7
3	5	9		
8				

Fig. 5. The tableau X.

1	2	4	7	8
3	5	10		
6				
9				
11				

Fig. 6. The tableau $X+(4, 11)$.

1	2	4	5	8
3	6	7		
9	10			
11				

Fig. 7. The tableau $X+(5, 11)$.

Lemma 5.1. *If $Y=X+(j, n)$ is a shiftable Young tableau, then X is shiftable.*

Proof. *Case* A. $X \in T_{n-1}$ and $j=n$.

Then Y is formed by adjoining n to the first row of X. Let $u_1 > u_2 > \cdots > u_r > 0$ be the shape of Y. Since Y is shiftable and $y_{2u_2} < n = y_{1u_1}$, n could not be the northeast neighbor of y_{2u_2} and therefore $u_1 - 1 > u_2$. Thus the shape $(u_1 - 1, u_2, \ldots, u_r)$ of X is strict, and since the northeast condition obviously holds as well, X is a shiftable tableau.

Case B. $X \in T_{n-2}$ and $j < n$.

X can be obtained from Y by deleting n, reversing Schensted's insertion procedure (beginning with the element at the end of row q, where $q + 1$ is the row which contained n) to delete an element j, and decreasing by one all elements of the resulting tableau that are greater than j. Let Z be the tableau that results in this process after deleting n and j, but before decreasing the elements. Clearly, X is shiftable if Z is, so we will show that Z is a shiftable tableau.

We first check that the northeast condition holds for Z. Let z_{ik} (resp., y_{ik}) be the element in cell (i, k) of Z (resp., Y).

Either $z_{ik} = y_{ik}$ or $z_{ik} = y > y_{ik}$, where y is in row $i + 1$ of Y. Thus $y_{ik} \leqslant z_{ik}$.

Now consider the northeast neighbor $z_{i-1,k+1}$ of z_{ik}. Either $z_{i-1,k+1} = y_{i-1,k+1} < y_{ik}$ (the inequality because Y is shiftable) or $z_{i-1,k+1} = x$, where x is a number from row i of Y that is no larger than y_{ik} (if $x > y_{ik}$, then $x = y_{i,k+m}$ $(m \geqslant 1)$ and in the deletion procedure it would have replaced $y_{i-1,h}$ for some $h \geqslant k + 2$, since $y_{i,k+m} \geqslant y_{i,k+1} > y_{i-1,k+2}$, instead of replacing $y_{i-1,k+1}$). Thus $z_{i-1,k+1} \leqslant y_{ik}$.

Therefore $z_{i-1,k+1} \leqslant y_{ik} \leqslant z_{ik}$. Obviously, these inequalities cannot both be equalities, thus $z_{i-1,k+1} < z_{ik}$ and the northeast condition holds in Z.

We also must check that the shape of Z is strict. The deletion of n and j from the tableau changes the shape by removing the cells at the end of rows q and $q + 1$ of Y. If n is in the last row of Y, the shape of Z is obviously strict. If n is not in the last row, then since Y is shiftable, the row containing n is at least 2 cells longer than the row below it, by an argument similar to that in Case A. Therefore, if $(v_1, \ldots, v_r)$ is the shape of Z, then $v_{q+1} > v_{q+2}$ and since the lengths of the other rows obviously still satisfy $v_i > v_{i+1}$, Z has a strict shape.

It follows that Z, and therefore X, is shiftable and the proof of the lemma is complete. □

Worley [10] and Sagan [5] have developed independently a Schensted-like algorithm for inserting an element into a shifted tableau. The algorithm agrees with Schensted's row insertion in the case where insertion does not result in bumping a diagonal element (an element of the first column in our case of shiftable rather than shifted tableaux). The proof that reversing the steps in this case of their insertion algorithm produces a shifted tableau is equivalent to the key step in Lemma 5.1, that of showing that Z is a shiftable tableau.

The condition that Y has at most $r + 1$ rows in Lemma 5.2 below is equivalent to the condition that the Schensted row insertion algorithm does not end in the first column. Thus the key steps in the proof are a special case in the proof that the insertion algorithm of Sagan and Worley produces a shifted tableau.

Lemma 5.2. *If $Y = X + (j, n)$, where X is a shiftable tableau which has r rows, then Y is shiftable if and only if it has at most $r + 1$ rows.*

Proof. If $X \in T_{n-1}$, then Y is formed as in Case A of Algorithm 3.1 and thus has r rows. It is easy to see that Y is always shiftable, since X is. We consider Case B, the case in which $X \in T_{n-2}$.

Let Y be shiftable and suppose it has more than $r + 1$ rows. Since Y has 2 more cells than X, it must have exactly $r + 2$ rows and these must each be of length one. But then the shape of Y is not strict and thus Y is not shiftable. Therefore Y has at most $r + 1$ rows.

We will show that, conversely, if Y has at most $r + 1$ rows, then it is shiftable. First we show that the shape of Y is strict. Let u_i (resp., v_i) be the length of row i of X (*resp.*, Y). Let q and $q + 1$ be the rows of X in which cells are added in the construction of Y. Since Y has at most $r + 1$ rows, $1 \leqslant q \leqslant r$.

All rows remain the same length in the construction of Y except q and $q + 1$, which each get one new cell. Since X is strict and $q \leqslant r$, it follows that $v_q > v_{q+1}$.

We only need to check that $v_{q-1} > v_q$. If the inequality is not satisfied, then it must be that $u_{q-1} = u_q + 1$. Then, since X is shiftable, the last element of row q of X is larger than all elements of row $q - 1$. But then the element bumped from row $q - 1$ during the insertion of j could not land in a new cell at the end of row q, as required by the definition of q. Therefore the inequality holds and it follows that Y has a strict shape.

Next we will show that Y satisfies the northeast condition. Let X^+ be the tableau that results from renumbering the elements $\geqslant j$ in Step B1 of the construction of Y, and denote its elements by x'_{ik}. This renumbering does not affect shiftability and since X satisfies the northeast condition, X^+ does also. Consider an element $y_{ik} \in Y$ which has a northeast neighbor. Then $i > 1$. There are 3 cases.

Case 1. $y_{ik} = x'_{ik}$ (x'_{ik} is not bumped when j is inserted into X^+).

Then $y_{ik} = x'_{ik} > x'_{i-1,k+1} \geqslant y_{i-1,k+1}$, where the first inequality is because X^+ is shiftable. The second inequality holds as equality if $x_{i-1,k+1}$ is not bumped out of row $i - 1$.

Case 2. $y_{ik} < x'_{ik}$.

Then $y_{ik} = x'_{i-1,h}$ for some h ($x'_{i-1,h}$ is bumped to row i when j is inserted). The elements satisfy $x'_{i-1,h} > x'_{i,k-1} > x'_{i-1,k}$, where the first inequality is because $x'_{i-1,h} = y_{ik}$ moved passed $x'_{i,k-1}$ when it was inserted into row i and the second is because X^+ is shiftable. Thus $x'_{i-1,h} > x'_{i-1,k}$ and since elements in the rows of X^+ increase from left to right, it follows that $h \geqslant k + 1$.

If $h > k + 1$, then $y_{ik} = x'_{i-1,h} > x'_{i-1,k+1} = y_{i-1,k+1}$ (the second equality holds since $x'_{i-1,k+1}$ could not be bumped in this case, because $x'_{i-1,h}$ is).

If $h = k + 1$, then $x'_{i-1,k+1}$ is bumped out of row $i - 1$ by some smaller element z of row $i - 2$ and now $z = y_{i-1,k+1}$. Thus $y_{ik} = x'_{i-1,k+1} > z = y_{i-1,k+1}$.

Case 3. $y_{ik} = n$.

Then obviously $y_{ik} > y_{i-1,k+1}$.

In all cases, $y_{ik} > y_{i-1,k+1}$ and thus the northeast condition holds. It follows that Y is shiftable and the proof is complete. □

Lemma 5.3. *Let $Y = X + (j, n)$, where X is a shiftable tableau with r rows. Then Y is shiftable if and only if there is no sequence $j \leqslant x_1 < x_2 < \cdots < x_r$, where $x_i \in$ row i of X.*

Proof. If $j = n$, there is never such a sequence. Furthermore, Y is formed as in Case A of the algorithm and is easily seen to be shiftable. Thus we consider the case $j < n$, in which Y is formed as in Case B.

If there is such a sequence, then the insertion of j in Step B2 will stop in row $q = r + 1$ and n will be placed into row $r + 2$ of Y. It follows from Lemma 5.2 that Y is not shiftable.

Conversely, if Y is not shiftable, then from Lemma 5.2, n must be in row $r + 2$ of Y and thus the insertion of j in Step B2 must end in row $r + 1$. Then there is a sequence $j < y_1 < y_2 < \cdots < y_r$ of elements, where y_i is the element bumped from row i to row $i + 1$ upon the insertion of j in Step B2. Let $x_i = y_i - 1$, and obtain a sequence of elements of X satisfying the properties described in the lemma and the proof is complete. □

Let J_X be the largest element of row 1 for which there is a sequence $J_X < x_2 < \cdots < x_r$, where $x_i \in$ row i of X. For the tableau of Fig. 5, $J_X = 4$.

Theorem 5.1. *A tableau $Y = X + (j, n)$ is shiftable if and only if X is shiftable and $j > J_X$.*

Proof. The proof follows easily from Lemmas 5.1 and 5.3. □

From Theorem 5.1, we have the following

Method for constructing all shiftable n-tableaux.

(1) Add n to the end of the first row of each shiftable $(n-1)$-tableau.

(2) For each shiftable $(n-2)$-tableau X,

a) Compute J_X: If X has r rows, let $x_r =$ the last element in row r. For $i = r-1, \ldots, 1$, let $x_i =$ the largest element of row i which is smaller than x_{i+1}. Then $J_X = x_1$.

b) Form the tableaux $Y = X + (j, n)$, for $j = J_X + 1, \ldots, n-1$.

Although the construction applies to *shiftable* Young tableaux, *shifted* tableaux can, of course, be obtained by indenting.

6. Involutions and shiftable tableaux

The Robinson–Schensted correspondence is a bijection between permutations π of $[n]$ and pairs $(P(\pi), Q(\pi))$ of Young tableaux on $[n]$ of the same shape. Since $P(\pi) = Q(\pi)$ when a permutation is an involution, the Robinson–Schensted correspondence provides a bijection between involutions and tableaux. An involution will be called *shiftable* if the tableau to which it corresponds is shiftable. In this section, we characterize shiftable involutions in terms of certain decreasing subsequences (Theorem 6.1) and we give a recursive description of shiftable involutions (Theorem 6.2).

We return to Algorithm 3.2 for constructing involutions. If π is the involution of [9] whose pairs are indicated in Fig. 8, then $\pi + (4, 11)$ and $\pi + (5, 11)$ are the involutions of [11] shown in Figs. 9 and 10, respectively. Note that π and $\pi + (5, 11)$ are shiftable involutions but that $\pi + (4, 11)$ is not.

Lemma 6.1. *If $\tau = \pi + (j, n)$ is a shiftable involution, then π is shiftable.*

Proof. The proof follows from Theorem 3.1 and Lemma 5.1. □

Lemma 6.2. *If $\tau = \pi + (j, n)$, where π is a shiftable involution with longest decreasing subsequence(s) of length r, then τ is shiftable if and only if its longest decreasing subsequence(s) is (are) of length at most $r + 1$.*

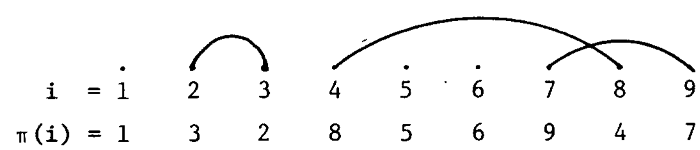

Fig. 8. An involution.

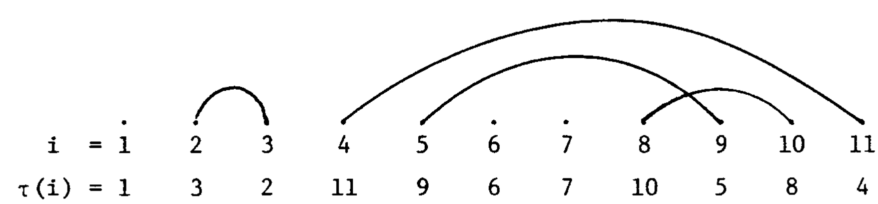

Fig. 9. The involution $\tau = \pi + (4, 11)$.

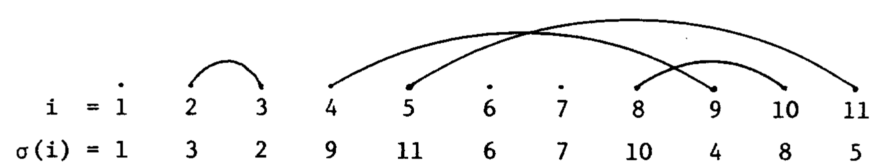

Fig. 10. The involution $\sigma = \pi + (5, 11)$.

Proof. It is well known that the length of a longest decreasing subsequence of a permutation π equals the number of rows in $P(\pi)$. The proof follows from this fact and from Lemma 5.2 and Theorem 3.1. □

For each involution π, there is a set $\emptyset = \pi_0, \pi_1, \pi_2, \ldots, \pi_p = \pi$ of involutions and pairs (j_k, n_k) such that $\pi_{k+1} = \pi_k + (j_k, n_k)$ $(k = 0, 1, \ldots, p-1)$. These involutions (and pairs) can be obtained recursively by reversing the steps of Algorithm 3.2. Lemmas 6.1 and 6.2 together imply that π is shiftable if and only if the length of a longest decreasing subsequence of π_{k+1} exceeds that of π_k by no more than 1 $(k = 0, 1, \ldots, p-1)$. This suggests a method for determining whether an involution is shiftable.

We can characterize (Theorem 6.1) the involutions which are shiftable, without referring to the construction of Algorithm 3.2, after observing that the π_k's can also be obtained as follows:

Order the pairs $(i, \pi(i))$ (where $i < \pi(i)$) of the involution lexicographically by right (i.e., largest) endpoint (by "endpoint" we mean element of the pair). Let π_k^* be the involution that consists of the first k arcs (pairs) in this ordering. Then π_k is the involution obtained by renumbering the elements of π_k^* with consecutive integers. For example, if π is the involution of Fig. 11, then π_4^* and π_4 are shown in Figs. 12 and 13, respectively.

Theorem 6.1. *Let π be an involution of $[n]$, S_j be the subsequence of $\pi(1), \pi(2), \ldots, \pi(j)$ which consists of the elements $\leqslant j$, and let $r(j)$ be the length of the longest decreasing subsequence(s) of S_j. Then π is shiftable if and only if $r(j+1) \leqslant r(j) + 1$ for $j = 1, 2, \ldots, n-1$.*

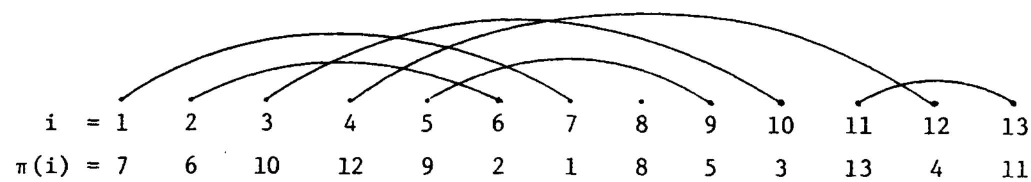

Fig. 11. Involution π.

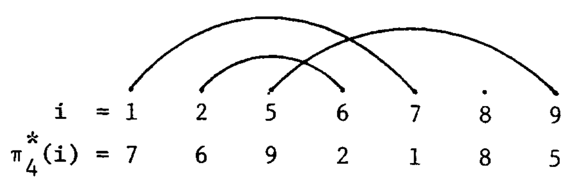

Fig. 12. The involution π_4^*.

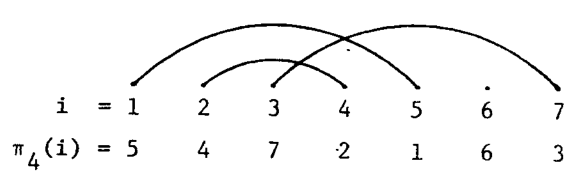

Fig. 13. The involution π_4.

Proof. S_j is the subsequence of π which corresponds to the set of arcs whose largest endpoints are $\leqslant j$. Thus if j_k is the right endpoint of the kth arc, then $S_{j_k} = \pi_k^*$.

Note that $S_j = S_{j_k}$ for all j such that $j_k \leqslant j < j_{k+1}$. Thus $r(j+1) \leqslant r(j) + 1$ for $j = 1, 2, \ldots, n-1$ if and only if $r(j_{k+1}) \leqslant r(j_k) + 1$ for $k = 0, 1, \ldots, p-1$, where $r(j_0) = 0$. This is true if and only if the length of the longest decreasing subsequence(s) of π_{k+1}^* exceeds that of π_k^* by no more than one ($k = 0, 1, \ldots, p-1$). This last statement holds for the π_k's as well as the π_k^*'s. Thus $r(j+1) \leqslant r(j) + 1$ for $j = 1, 2, \ldots, n-1$ if and only if the length of the longest decreasing subsequence(s) of π_{k+1} exceeds that of π_k by no more than ($k = 0, 1, \ldots, p-1$). From Lemmas 6.1 and 6.2 as discussed above, this occurs if and only if π is shiftable, and the proof is complete. □

For example, consider the involution $\pi = 1, 5, 8, 6, 2, 4, 11, 3, 9, 12, 7, 10$. Here $S_7 = 1, 5, 6, 2, 4$ and $S_8 = 1, 5, 8, 6, 2, 4, 3$, and thus $r(7) = 2$ and $r(8) = 4$. Therefore π is not shiftable.

As was the case for tableaux, we can determine (Theorem 6.2) whether an involution $\tau = \pi + (j, n)$ is shiftable from the shiftability of π and the size of j. To do this for an involution π, we consider the set of longest decreasing subsequences of π and order this set lexicographically. We will be interested in the last subsequence in this ordering, called the *last longest decreasing subsequence* (LLDS) of π.

In Lemma 6.3, we show that when an involution is represented as a set of arcs $(i, \pi(i))$, as in Fig. 11, then its LLDS corresponds to a set of nested arcs (where a fixed point is considered to be a degenerate arc). The involution in Fig. 11 has LLDS = 12, 9, 8, 5, 4. The subsequence 12, 9, 8, 5, 3 is a longest decreasing subsequence which does not correspond to a set of nested arcs, but it is not lexicographically last.

Lemma 6.3. *The LLDS of an involution τ of n elements is of the form $\tau(i_1) > \tau(i_2) > \cdots > \tau(i_r)$, where $\tau(i_k) = i_{r-k+1}$, $k = 1, 2, \ldots, r$. In other words, the LLDS corresponds to a set of nested arcs.*

Proof. The proof is by induction on n. The statement obviously holds for $n \leqslant 2$, so let $n > 2$ and assume it is true for involutions of m elements, where $m < n$.

Let $j = \tau(n)$. Let $Y = y_1, y_2, \ldots, y_r$ be the LLDS of τ and let $X = x_1, x_2, \ldots, x_s$ be the LLDS of the involution $\tau - (n, j)$ obtained by deleting the pair (n, j) from τ. By induction, X is of the desired nested form. If $X = Y$, then Y is of the desired form and we are done.

Suppose $X \neq Y$ and let I be the subsequence of τ that consists of the elements common to both X and Y. If $I = \emptyset$, then the LLDS of τ is the same as the LLDS of $\tau - X$. Since X is nested, it consists of a complete set of pairs of the involution

τ, and therefore $\tau - X$ is an involution. By induction, the LLDS of $\tau - X$, and thus of τ, is nested as desired.

If $I \neq \emptyset$, then we claim that Y begins with n and ends with j (since $X \neq Y$ and $I \neq \emptyset$, it follows that $j \neq n$). Let z_1 be the first element of I. Then X begins with elements $x_1 > x_2 > \cdots > x_p > z_1$ and Y begins with elements $y_1 > y_2 > \cdots > y_q > z_1$. Clearly, $p \leq q$, for otherwise the elements $y_1, \ldots, y_q$ could be replaced by $x_1, \ldots, x_p$ to obtain a longer decreasing subsequence of τ then Y.

If $p = q$, the sequence $y_1, \ldots, y_q$, must be lexicographically larger than $x_1, \ldots, x_p$, for otherwise it could be replaced by $x_1, \ldots, x_p$ to obtain a decreasing subsequence of τ that has the same length but is lexicographically larger than Y. Therefore, the sequence $y_1, \ldots, y_q$ must contain an element that is not in the involution $\tau - (n, j)$, for otherwise it could replace $x_1, \ldots, x_p$ to obtain a decreasing subsequence of $\tau - (n, j)$ that has the same length but is lexicographically larger than X. Since $j = \tau(n)$, z_1 precedes j in τ and thus this element could not be j and therefore must be n. Since $y_1, \ldots, y_q$ is decreasing, it follows that $y_1 = n$.

If $p < q$, then again, $y_1, \ldots, y_q$ must contain an element not in $\tau - (n, j)$ for otherwise, $y_1, \ldots, y_q$ could replace $x_1, \ldots, x_p$ to obtain a longer subsequence of $\tau - (n, j)$ than X. It follows that this element is $y_1 = n$.

Thus Y begins with n. A very similar argument (involving the elements of X and Y that follow the last element of I) can be used to show that Y ends with j, and complete the proof of the claim.

Consider the subinvolution τ^* of τ which consists of the pairs in which both elements of the pair are larger than j. Since Y ends with j, all its elements are $\geq j$ and thus $Y' = y_2, \ldots, y_{r-1}$ is a subsequence of τ^*. Clearly, Y' is the LLDS of τ^* and thus, by induction, Y' satisfies the desired nested property. Since $Y = n, y_2, \ldots, y_{r-1}, j$, Y satisfies this property as well, and the proof of the lemma is complete. □

Let L_π be the index of the first element in the LLDS of π. Thus the LLDS begins with $\pi(L_\pi)$.

Theorem 6.2. *An involution $\tau = \pi + (j, n)$ is shiftable if and only if π is shiftable and $j > L_\pi$.*

Proof. If $j = n$, then the theorem is clearly true. Assume $j < n$, so that τ is formed as in Case B of Algorithm 3.2.

If τ is shiftable, then from Lemma 6.1, π is shiftable. We claim that $j > L_\pi$. Let $\pi(L_\pi) = \pi(i_1) > \cdots > \pi(i_r)$ be the LLDS of π.

From Lemma 6.3, $\pi(i_r) = i_1 = L_\pi$. If $L_\pi \geq j$, then $\pi(i_r) \geq j$ and τ contains the decreasing subsequence $n > \pi(L_\pi) + 1 > \pi(i_2) + 1 > \cdots > \pi(i_r) + 1 > j$. This subsequence has length $r + 2$ and hence from Lemma 6.2, τ is not shiftable. This is a contradiction and therefore $j > L_\pi$.

Conversely, suppose π is shiftable and $j > L_\pi$. It is easy to check that no decreasing subsequence of length greater than $r+1$ could be created by the addition of the pair (n, j) in the construction of τ. Thus, from Lemma 6.2, $\tau = \pi + (j, n)$ is shiftable and the proof is complete. □

Theorem 6.2 implies a method for constructing all shiftable involutions, similar to the method described in Section 5 for shiftable tableaux.

We close with the following corollary which relates the numbers L_π to the numbers J_X of Section 5.

Corollary 6.1. *Let π be a shiftable involution and let $X = P(\pi)$ be the tableau which corresponds to π under the Robinson–Schensted correspondence. Then $L_\pi = J_X$.*

Proof. The proof follows from Theorems 5.1, 3.1, and 6.2. □

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