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On the duality of fusion frames

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Abstract

The fusion frames were considered recently by P.G. Casazza, G. Kutyniok and S. Li in connection with distributed processing and are related to the construction of global frames from local frames. In this paper we give new results on the duality of fusion frames in Hilbert spaces.

Keywords: Hilbert space; Frame; Dual frame; Fusion frame

1. Introduction

First we will recall the definitions and some properties of frames in Hilbert spaces. For basic results on frames, see [3,4,7–9,13–18].

Let $\mathcal{H}$ be a Hilbert space and I be a set which is finite or countable. We denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and by $I_{\mathcal{H}}$ the identity operator on $\mathcal{H}$.

A system $\mathcal{F} = \{f_i\}_{i\in I}$ is called a *frame* for $\mathcal{H}$ if there exist the constants $A, B > 0$ such that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{i\in I} \left|\langle f, f_i\rangle\right|^2 \leqslant B\|f\|^2.$$

The constants A and B are called *frame bounds*. If $A = B$ we call this frame an *A-tight frame* and if $A = B = 1$ it is called a *Parseval frame*. A frame is *exact* if it is ceases to be frame whenever

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any single element is deleted from the system $\{f_i\}_{i\in I}$. It is known that a frame is exact if and only if it is a Riesz basis. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a *Bessel sequence*.

If $\{f_i\}_{i\in I}$ is a Bessel sequence, then the following operators are linear and bounded

$$T_{\mathcal{F}}: l^2(I)\to\mathcal{H},\qquad T_{\mathcal{F}}(c_i)=\sum_{i\in I}c_i f_i\quad\text{(synthesis operator)}$$

$$T_{\mathcal{F}}^*: \mathcal{H}\to l^2(I),\qquad T_{\mathcal{F}}^* f=\{\langle f, f_i\rangle\}_{i\in I}\quad\text{(analysis operator)}$$

$$S_{\mathcal{F}}: \mathcal{H}\to\mathcal{H},\qquad S_{\mathcal{F}} f=T_{\mathcal{F}}T_{\mathcal{F}}^* f=\sum_{i\in I}\langle f, f_i\rangle f_i\quad\text{(frame operator).}$$

The operator $\theta=T_{\mathcal{F}}^*$ is called also the *frame transform* of $\{f_i\}_{i\in I}$. It is the adjoint of $T_{\mathcal{F}}$.

If $\{f_i\}_{i\in I}$ is a frame, then θ is injective; θ is invertible if and only if $\{f_i\}_{i\in I}$ is a Riesz basis. If $\{f_i\}_{i\in I}$ is a frame, then $S_{\mathcal{F}}$ is an invertible operator and the following *reconstruction formula* holds for all $f\in\mathcal{H}$:

$$f=\sum_{i\in I}\langle f, S_{\mathcal{F}}^{-1}f_i\rangle f_i=\sum_{i\in I}\langle f, f_i\rangle S_{\mathcal{F}}^{-1}f_i.$$

Then the family $\{\tilde{f}_i\}_{i\in I}$, where $\tilde{f}_i=S_{\mathcal{F}}^{-1}f_i$, $i\in I$, is also a frame for $\mathcal{H}$, called the *canonical dual* of $\{f_i\}_{i\in I}$.

In general, the Bessel sequence $\{g_i\}_{i\in I}$ is called an *alternate dual* of the frame $\{f_i\}_{i\in I}$ if the following formula holds, for all $f\in\mathcal{H}$:

$$f=\sum_{i\in I}\langle f, g_i\rangle f_i.$$

If denote by θ_1 the frame transform of $\{f_i\}_{i\in I}$ in $\mathcal{H}$ and by θ_2 the frame transform of $\{g_i\}_{i\in I}$, then Han and Larson [13] proved that $\{g_i\}_{i\in I}$ is an alternate dual of $\{f_i\}_{i\in I}$ if and only if $\theta_1^*\theta_2=I_{\mathcal{H}}$. Then $\{g_i\}_{i\in I}$ is also a frame for $\mathcal{H}$ since

$$\|f\|=\|\theta_1^*\theta_2(f)\|\leqslant\|\theta_1^*\|\|\theta_2(f)\|,\qquad f\in\mathcal{H}.$$

It follows also that $\theta_2^*\theta_1=I_{\mathcal{H}}$, hence then $\{f_i\}_{i\in I}$ is the alternate dual of $\{g_i\}_{i\in I}$.

For recent applications of frame theory see references of [12]. Generalizations of frame theory were give in [1,5,6,10,11,16,19].

The theory of fusion frames of Hilbert spaces were developed recently by Casazza et al. [5,6]. See also [2].

Let $\{V_i\}_{i\in I}$ be a family of closed subspaces of Hilbert space $\mathcal{H}$ and $\{v_i\}_{i\in I}$ be a family of weights, i.e. $v_i>0$, $i\in I$. The family $\mathcal{V}=\{(V_i,v_i)\}_{i\in I}$ is a *fusion frame* (*frame of subspaces*), if there exist constants $0<C\leqslant D<\infty$ such that

$$C\|f\|^2\leqslant\sum_{i\in I}v_i^2\|\pi_{V_i}f\|^2\leqslant D\|f\|^2,\qquad\text{for all } f\in\mathcal{H},$$

where for the closed subspace $V\subset\mathcal{H}$, π_V denotes the orthogonal projection of $\mathcal{H}$ on V. The constants C and D are called the *fusion frame bounds*. If we only have the upper bound, we call $\{(V_i,v_i)\}_{i\in I}$ a *Bessel fusion sequence*.

Let $\mathcal{V}=\{(V_i,v_i)\}_{i\in I}$ be a fusion frame. Then the frame operator $S_{\mathcal{V}}$ for $\{(V_i,v_i)\}_{i\in I}$ is defined by

$$S_{\mathcal{V}}(f)=\sum_{i\in I}v_i^2\pi_{V_i}(f),\qquad f\in\mathcal{H}.$$

Casazza and Kutyniok proved that $S_{\mathcal{V}}$ is positive, self-adjoint, invertible operator on $\mathcal{H}$ and the following *reconstruction formula* holds for all $f \in \mathcal{H}$:

$$f = \sum_{i \in I} v_i^2 \; S_{\mathcal{V}}^{-1} \pi_{V_i}(f).$$

This relation proves that the family of operators $\{v_i^2 S_{\mathcal{V}}^{-1} \pi_{V_i}(f)\}_{i \in I}$ is a resolution of identity. We recall that a family of bounded operators $\{T_i\}_{i \in I}$ on $\mathcal{H}$ is called *a resolution of identity* on $\mathcal{H}$ if for all $f \in \mathcal{H}$ we have

$$f = \sum_{i \in I} T_i f$$

(and series converges unconditionally for all $f \in \mathcal{H}$). The family $\{(S_{\mathcal{V}}^{-1} V_i, v_i)\}_{i \in I}$ is called *the dual fusion frame*. To prove that the dual fusion frame is a fusion frame, Casazza and Kutyniok stated the following result:

Proposition 1.1. *Let $\{(V_i, v_i)\}_{i \in I}$ be a fusion frame and let $T : \mathcal{H} \to \mathcal{H}$ be an invertible operator on $\mathcal{H}$. Then $\{(T V_i, v_i)\}_{i \in I}$ is a fusion frame.*

To prove this result, the authors used the formula:

$$\pi_{T V_i} = T \pi_{V_i} T^{-1}, \quad i \in I. \tag{1}$$

Remark 1.2. The problem with Eq. (1) is that the right-hand side is always a projection onto $T V_i$, but it is not an *orthogonal projection* unless $T^* T V_i \subset V_i$ (see Section 2). In particular, this would happen if T is an unitary operator.

In Section 2 of this paper we prove that (1) in general is not true, but we show in another way that Proposition 1.1 is true.

In Section 3 we consider an operator associated to a pair of Bessel sequences of subspaces and we prove some new resolutions of identity.

2. Duals of fusion frames

The formula (1) is equivalent with

$$\pi_{T V} T = T \pi_V.$$

First, we prove the following result on operators.

Proposition 2.1. *Let $T \subset \mathcal{L}(\mathcal{H})$ and $V \in \mathcal{H}$ be a closed subspace. Then the following are equivalent*:

(i) $\pi_{\overline{T V}} T = T \pi_V$;
(ii) $T^* T V \subset V$.

Proof. (i) $\Rightarrow$ (ii). We take $h \in V^{\perp}$. We have

$$\pi_{\overline{T V}} T h = T \pi_V h = 0,$$

hence $Th \in (\overline{TV})^{\perp} = (TV)^{\perp}$. But $\langle Th, Tv\rangle = 0$, for all $v \in V \Leftrightarrow \langle h, T^*Tv\rangle = 0$, for all $v \in V \Leftrightarrow h \in (T^*TV)^{\perp}$.

(ii) $\Rightarrow$ (i). If $v \in V$ then

$$\pi_{\overline{TV}} Tv = Tv \quad \text{and} \quad T\pi_V v = Tv.$$

If $h \in V^{\perp}$ then $T\pi_V h = T0 = 0$ and, from the hypothesis, we have $h \in (T^*TV)^{\perp}$.

As before, we have now $Th \in (\overline{TV})^{\perp}$, hence $\pi_{\overline{TV}} Th = 0$. □

Corollary 2.2. *There exist Hilbert space $\mathcal{H}$, an invertible operator $T \in \mathcal{L}(\mathcal{H})$ and V a closed subspace of $\mathcal{H}$ such that*

$$\pi_{TV} T \neq T\pi_V.$$

Proof. We take $\mathcal{H} = \mathbb{R}^2$, $V = \{(x, 0)/x \in \mathbb{R}\}$ and

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (x + y, y)$$

for all $(x, y) \in \mathbb{R}^2$. Then the adjoint of T is

$$T^*(x, y) = (x, x + y) \quad \text{for all } (x, y) \in \mathbb{R}^2.$$

We have

$$T^*T(x, 0) = T^*(x, 0) = (x, x) \notin V \quad \text{for all } x \neq 0.$$

From Proposition 2.1 it follows that $\pi_{TV} T \neq T\pi_V$. □

To prove the main result of this section, we give the following lemma:

Lemma 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ and $V \in \mathcal{H}$ be a closed subspace. Then we have*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

Proof. If $f \in \mathcal{H}$, then

$$f = \pi_{\overline{TV}} f + g, \quad g \in (\overline{TV})^{\perp} = (TV)^{\perp}.$$

It follows

$$T^* f = T^* \pi_{\overline{TV}} f + T^* g.$$

But, for $v \in V$, we have

$$\langle T^* g, v\rangle = \langle g, Tv\rangle = 0,$$

hence $T^* g \in V^{\perp}$. It follows

$$\pi_V T^* f = \pi_V T^* \pi_{\overline{TV}} f + \pi_V T^* g = \pi_V T^* \pi_{\overline{TV}} f. \qquad □$$

Theorem 2.4. *Let $\{(V_i, v_i)\}_{i\in I}$ be a fusion frame with frame bounds C, D. If $T \in \mathcal{L}(\mathcal{H})$ is an invertible operator, then $\{(TV_i, v_i)\}_{i\in I}$ is a fusion frame with frame bounds*

$$\frac{C}{\|T^*\|^2\|T^{*-1}\|^2} \quad \textit{and} \quad D\|T^*\|^2\|T^{*-1}\|^2.$$

Proof. From Lemma 2.3 we have

$$\|\pi_{V_i} T^* f\| \leqslant \|T^*\| \cdot \|\pi_{TV_i} f\|$$

hence

$$C\|T^* f\|^2 \leqslant \sum_{i\in I} v_i^2 \|\pi_{V_i} T^* f\|^2 \leqslant \|T^*\|^2 \sum_{i\in I} v_i^2 \|\pi_{TV_i} f\|^2$$

and since T^* is invertible, we obtain

$$\sum_{i\in I} v_i^2 \|\pi_{TV_i} f\|^2 \geqslant \frac{C}{\|T^*\|^2 \|T^{*-1}\|^2} \|f\|^2.$$

On the other hand, from Lemma 2.3, we obtain, with T^{-1} instead of T:

$$\pi_{TV_i} = \pi_{TV_i} T^{*-1} \pi_{V_i} T^*,$$

hence

$$\|\pi_{TV_i} f\| \leqslant \left\|T^{*-1}\right\| \cdot \|\pi_{V_i} T^* f\|.$$

It follows

$$\begin{aligned}\sum_{i\in I} v_i^2 \|\pi_{TV_i} f\|^2 &\leqslant \left\|T^{*-1}\right\|^2 \sum_{i\in I} v_i^2 \|\pi_{V_i} T^* f\|^2 \\ &\leqslant \left\|T^{*-1}\right\|^2 D \|T^* f\|^2 \\ &\leqslant D \cdot \left\|T^{*-1}\right\|^2 \|T^*\|^2 \|f\|^2. \qquad \square\end{aligned}$$

Corollary 2.5. *The dual fusion frame of fusion frame $\mathcal{V} = \{(V_i, v_i)\}_{i\in I}$ with C, D frame bounds is a fusion frame with the frame bounds*

$$\frac{C}{\|S_{\mathcal{V}}\|^2 \|S_{\mathcal{V}}^{-1}\|^2} \quad \textit{and} \quad D\|S_{\mathcal{V}}\|^2 \left\|S_{\mathcal{V}}^{-1}\right\|^2.$$

Proof. We take in Theorem 2.4, $T = S_{\mathcal{V}}^{-1}$. $\square$

Corollary 2.6. *Let $\{(V_i, v_i)\}_{i\in I}$ be a fusion frame with C, D frame bounds and U a unitary operator on $\mathcal{H}$. Then $U\mathcal{V} := \{(UV_i, v_i)\}_{i\in I}$ is a fusion frame with C, D frame bounds and frame operator $US_{\mathcal{V}}U^*$.*

Proof. For the first part we apply Theorem 2.4, with $T = U$. For the second part, we apply the Proposition 2.1:

$$S_{U\mathcal{V}} f = \sum_{i\in I} v_i^2 \pi_{UV_i}(f) = \sum_{i\in I} v_i^2 U \pi_{V_i} U^*(f) = US_{\mathcal{V}}U^* f, \quad f \in \mathcal{H}. \qquad \square$$

We give now a form of the reconstruction formula with the help of the dual fusion frame. By Lemma 2.3, we have

$$\pi_{V_i} S_{\mathcal{V}}^{-1} = \pi_{V_i} S_{\mathcal{V}}^{-1} \pi_{S_{\mathcal{V}}^{-1} V_i},$$

hence

$$S_{\mathcal{V}}^{-1}\pi_{V_i} = \pi_{S_{\mathcal{V}}^{-1}V_i} S_{\mathcal{V}}^{-1}\pi_{V_i}.$$

Then the reconstruction formula has the form

$$f = \sum_{i\in I} v_i^2 \pi_{S_{\mathcal{V}}^{-1}V_i} S_{\mathcal{V}}^{-1}\pi_{V_i}(f), \quad f \in \mathcal{H}. \tag{2}$$

This leads us to introduce the following definition:

Definition 2.7. Let $\mathcal{V} = \{(V_i, v_i)\}_{i\in I}$ be a fusion frame and let $S_{\mathcal{V}}$ be the frame operator. We consider also $\mathcal{W} = \{(W_i, w_i)\}_{i\in I}$ a Bessel fusion sequence. We say that $\mathcal{W}$ is an alternate dual of $\mathcal{V}$ if we have

$$f = \sum_{i\in I} v_i w_i \pi_{W_i} S_{\mathcal{V}}^{-1}\pi_{V_i}(f), \tag{3}$$

for all $f \in \mathcal{H}$.

By the relation (2) we have that *the dual fusion frame of $\mathcal{V}$ is an alternate dual frame*.

We have also the following result:

Proposition 2.8. *The alternate dual of fusion frame is a fusion frame.*

Proof. By (3) we obtain

$$\begin{aligned}
\|f\|^2 &= \sum_{i\in I} v_i w_i \langle S_{\mathcal{V}}^{-1}\pi_{V_i}(f), \pi_{W_i}(f)\rangle \\
&\leqslant \sum_{i\in I} v_i w_i \|S_{\mathcal{V}}^{-1}\pi_{V_i}(f)\| \|\pi_{W_i}(f)\| \\
&\leqslant \left(\sum_{i\in I} v_i^2 \|S_{\mathcal{V}}^{-1}\pi_{V_i}(f)\|^2\right)^{1/2} \left(\sum_{i\in I} w_i^2 \|\pi_{W_i}(f)\|^2\right)^{1/2} \\
&\leqslant \|S_{\mathcal{V}}^{-1}\| \sqrt{D} \|f\| \left(\sum_{i\in I} w_i^2 \|\pi_{W_i}(f)\|^2\right)^{1/2},
\end{aligned}$$

where D is the upper bound of the frame $\mathcal{V}$. □

3. Frame operator for a pair of Bessel fusion sequences

In the following, we consider two Bessel fusion sequences: $\mathcal{V} = \{(V_i, v_i)\}_{i\in I}$ with Bessel bound D_1 and $\mathcal{W} = \{(W_i, w_i)\}_{i\in I}$ with Bessel bound D_2. We introduce the operator

$$S_{VW} f := \sum_{i\in I} v_i w_i \pi_{V_i}\pi_{W_i} f, \quad f \in \mathcal{H}.$$

By [5, Lemma 3.9], it follows that series converges unconditionally.

We have also

$$\langle S_{VW} f, g\rangle = \sum_{i\in I} v_i w_i \langle \pi_{W_i} f, \pi_{V_i} g\rangle, \tag{4}$$

for all $f, g \in \mathcal{H}$.

By Cauchy–Schwartz inequality, we have

$$\left|\langle S_{VW} f, g\rangle\right| \leqslant \left(\sum_{i\in I} v_i^2 \|\pi_{V_i} g\|^2\right)^{1/2} \cdot \left(\sum_{i\in I} w_i^2 \|\pi_{W_i} f\|^2\right)^{1/2}. \tag{5}$$

From (5), it follows

$$\left|\langle S_{VW} f, g\rangle\right| \leqslant \sqrt{D_1}\sqrt{D_2}\|g\|\|f\|,$$

hence S_{VW} is a bounded operator and

$$\|S_{VW}\| \leqslant \sqrt{D_1}\sqrt{D_2}.$$

From (5) we have also

$$\|S_{VW} f\| \leqslant \sqrt{D_1}\left(\sum_{i\in I} w_i^2 \|\pi_{W_i} f\|^2\right)^{1/2} \tag{6}$$

and

$$\|S_{VW}^* g\| \leqslant \sqrt{D_2}\left(\sum_{i\in I} v_i^2 \|\pi_{V_i} g\|^2\right)^{1/2}. \tag{7}$$

Moreover, from (4) we have

$$\langle S_{VW} f, g\rangle = \sum_{i\in I} v_i w_i \langle f, \pi_{W_i}\pi_{V_i} g\rangle = \langle f, S_{WV} g\rangle,$$

hence

$$S_{VW}^* = S_{WV}.$$

Theorem 3.1. *The following are equivalent*:

(i) S_{VW} *is bounded below*;
(ii) $(\exists)K \in \mathcal{L}(\mathcal{H})$ *such that* $\{T_i\}_{i\in I}$ *is a resolution of identity, where*

$$T_i = v_i w_i K \pi_{V_i}\pi_{W_i}, \quad i \in I.$$

If one of conditions holds, then $\mathcal{W}$ *is a fusion frame.*

Proof. (i) ⇒ (ii). If S_{VW} is bounded below, then there exists $K \in \mathcal{L}(\mathcal{H})$ such that $K S_{VW} = I_{\mathcal{H}}$. It follows

$$f = \sum_{i\in I} v_i w_i K \pi_{V_i}\pi_{W_i} f.$$

(ii) ⇒ (i). If (ii) holds, then for $f \in \mathcal{H}$ we have

$$f = \sum_{i\in I} v_i w_i K \pi_{V_i}\pi_{W_i}.$$

It follows

$$f = K\left(\sum_{i\in I} v_i w_i \pi_{V_i}\pi_{W_i} f\right),$$

hence $I_{\mathcal{H}} = K S_{VW}$. It follows that S_{VW} is bounded below.

If S_{VW} is bounded below, from (6) it follows that $\mathcal{W}$ is a fusion frame. □

Corollary 3.2. *The following are equivalent*:

(i) S_{VW} *is invertible*;
(ii) $(\exists)K \in \mathcal{L}(\mathcal{H})$ *invertible such that*

$$T_i = v_i w_i K \pi_{V_i} \pi_{W_i}$$

is a resolution of identity.

If one of conditions holds, then $\mathcal{V}$, $\mathcal{W}$ are fusion frames.

Corollary 3.3. *Let $\mathcal{W}$ be a Bessel fusion sequence. Then $\mathcal{W}$ is a fusion frame if and only if there exists $\mathcal{V} = \{(V_i, v_i)\}_{i \in I}$ a Bessel fusion sequence such that S_{VW} is bounded below.*

Proof. If $\mathcal{W}$ is a fusion frame we take $V_i = W_i$, $w_i = v_i$, $i \in I$.
For conversely, we use Theorem 3.1. □

Theorem 3.4. *We assume there exist $\lambda_1 < 1$, $\lambda_2 > -1$ such that*

$$\left\| f - \sum_{i \in I} v_i w_i \pi_{V_i} \pi_{W_i}(f) \right\| \leqslant \lambda_1 \|f\| + \lambda_2 \left\| \sum_{i \in I} v_i w_i \pi_{V_i} \pi_{W_i}(f) \right\|,$$

for any $f \in \mathcal{H}$. Then $\mathcal{W}$ is a fusion frame and

$$\left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2 \frac{1}{D_1} \|f\|^2 \leqslant \sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2, \quad f \in \mathcal{H}.$$

Proof. As before, we denote

$$S_{VW} f = \sum_{i \in I} v_i w_i \pi_{V_i} \pi_{W_i}(f).$$

We have

$$\|f - S_{VW} f\| \leqslant \lambda_1 \|f\| + \lambda_2 \|S_{VW} f\|.$$

Since

$$\|f - S_{VW} f\| \geqslant \big| \|f\| - \|S_{VW} f\| \big|,$$

it follows

$$\lambda_1 \|f\| + \lambda_2 \|S_{VW} f\| \geqslant \|f\| - \big\| S_{VW}(f) \big\|,$$

hence

$$\|S_{VW} f\| \geqslant \frac{1-\lambda_1}{1+\lambda_2} \|f\|.$$

From (6) we obtain

$$\sum_{i \in I} w_i^2 \|\pi_{W_i} f\|^2 \geqslant \frac{1}{D_1} \left(\frac{1-\lambda_1}{1+\lambda_2}\right)^2 \|f\|^2. \qquad \square$$

In the particular case $\lambda_2 = 0$ we have a stronger result.

Corollary 3.5. *We assume that exists* $\lambda \in [0,1)$, *such that*

$$\left\| f - \sum_{i\in I} v_i w_i \pi_{V_i}\pi_{W_i}(f) \right\| \leqslant \lambda \|f\|, \quad f \in \mathcal{H}.$$

Then $\mathcal{W}$ *and* $\mathcal{V}$ *are fusion frames and the following estimates hold*

$$\sum_{i\in I} w_i^2 \|\pi_{W_i} f\|^2 \geqslant \frac{(1-\lambda)^2}{D_1}\|f\|^2,$$

$$\sum_{i\in I} v_i^2 \|\pi_{V_i} f\|^2 \geqslant \frac{(1-\lambda)^2}{D_2}\|f\|^2,$$

for any $f \in \mathcal{H}$.

Proof. We have for $f \in \mathcal{H}$

$$\begin{aligned}\|f - S_{WV}(f)\| &= \|(I_{\mathcal{H}} - S_{VW})^*(f)\| \leqslant \|(I_{\mathcal{H}} - S_{VW})^*\|\|f\| \\ &= \|I_{\mathcal{H}} - S_{VW}\|\|f\| \leqslant \lambda\|f\|\end{aligned}$$

and apply Theorem 3.4. □

Remark 3.6. If in Corollary 3.5 we take $\lambda = 0$, we obtain Proposition 2.7 in [2].

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