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J. Math. Anal. Appl. 321 (2006) 114–123

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Stability of a Jensen type equation in the space of generalized functions

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Received 21 March 2005

Available online 8 September 2005

Submitted by S.R. Grace

Abstract

We reformulate and solve the stability problem of a Jensen type functional equation

$$3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)-2f\left(\frac{x+y}{2}\right)-2f\left(\frac{y+z}{2}\right)-2f\left(\frac{z+x}{2}\right)=0,$$

in the spaces of some generalized functions such as tempered distributions and Fourier hyperfunctions.

Keywords: Gauss transforms; Heat kernel; Jensen type equation; Generalized functions; Popoviciu inequality; Stability

1. Introduction

The stability problem of functional equations was first raised by S.M. Ulam [15] in 1940, who proposed a question:

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.07.053

Let $(G_1, \circ)$ be a group and $(G_2,*)$ a metric group with a metric d. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies

$$d\big(f(x \circ y), f(x) * f(y)\big) \leqslant \delta, \quad x, y \in G_1,$$

then there exists a homomorphism $h : G_1 \to G_2$ with

$$d\big(f(x), h(x)\big) \leqslant \varepsilon, \quad x \in G_1?$$

This problem was first studied by D.H. Hyers [8] in 1941, who solved the stability problem for the Cauchy functional equation as follows:

Theorem 1. *Let $f : E_1 \to E_2$ with E_1, E_2 Banach spaces be an ε-additive, that is, f satisfies*

$$\big\| f(x+y) - f(x) - f(y) \big\| \leqslant \varepsilon, \tag{1}$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $g : E_1 \to E_2$ such that

$$\big\| f(x) - g(x) \big\| \leqslant \varepsilon$$

for all $x \in E_1$. Here, an additive mapping $g : E_1 \to E_2$ means the inequality (1) *satisfies for $\varepsilon = 0$.*

After the work of Hyers, the stability problems of various functional equations, such as Pexider equation [10], D'Alembert functional equation [7], Quadratic equation [3], and so on, have been investigated by a number of authors.

The functional equation

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = 0$$

is called Jensen equation, and in 1989, Z. Kominek [9] solved the stability problem for this equation. There is another interesting functional equation which is very similar to the Jensen equation:

$$\begin{aligned} &3f\left(\frac{x+y+z}{3}\right) + f(x) + f(y) + f(z) - 2f\left(\frac{x+y}{2}\right) \\ &\quad - 2f\left(\frac{y+z}{2}\right) - 2f\left(\frac{z+x}{2}\right) = 0. \end{aligned} \tag{2}$$

This equation has been studied for the first time by T. Popoviciu, who verified that if f is a convex function on a nonempty interval then the left-hand side of (2) is nonnegative. This is known as *the Popoviciu inequality*.

The first result on the stability of this Jensen type equation (2) was obtained by T. Trif [14], as follows:

Theorem 2. *Let X be a real normed linear space and Y a real Banach space. If $f : X \to Y$ is a function satisfying*

$$\left|3f\left(\frac{x+y+z}{3}\right)+f(x)+f(y)+f(z)-2f\left(\frac{x+y}{2}\right)\right.$$
$$\left.-2f\left(\frac{y+z}{2}\right)-2f\left(\frac{z+x}{2}\right)\right|\leqslant\varepsilon, \tag{3}$$

for all $x, y, z \in X$, then there exists a unique additive mapping $A: X \to Y$ such that

$$\left|f(x)-f(0)-A(x)\right|\leqslant\varepsilon,$$

for all $x \in X$.

In this paper, we consider the stability of the Jensen type equation (2) in the space of some generalized functions, such as tempered distributions and Fourier hyperfunctions. Note that the above inequality (3) itself cannot be applied to the space of generalized functions. But following the methods used in the papers [1,2,4,11], we can reformulate the inequality to the space of generalized functions, as follows:

Let A, P_i and Q_i be the functions

$$A(x,y,z)=x+y+z, \qquad P_1(x,y,z)=x,$$
$$P_2(x,y,z)=y, \qquad P_3(x,y,z)=z,$$
$$Q_1(x,y,z)=y+z, \qquad Q_2(x,y,z)=z+x, \quad \text{and} \quad Q_3(x,y,z)=x+y,$$

for all $x, y, z \in \mathbb{R}^n$. Then the inequality (3) can be naturally extended as

$$\left\|3u\circ\frac{A}{3}+u\circ P_1+u\circ P_2+u\circ P_3-2u\circ\frac{Q_3}{2}-2u\circ\frac{Q_2}{2}-2u\circ\frac{Q_1}{2}\right\|\leqslant\varepsilon, \tag{4}$$

where $\circ$ means the pullback of generalized functions, and $\|v\|\leqslant\varepsilon$ means that $|\langle v,\varphi\rangle|\leqslant\varepsilon\|\varphi\|_{L^1}$ for all test functions φ.

To solve the problem, we apply the heat kernel method [12]. The basic idea of the proof is to reduce the problem to the case of the continuous version by convolving the heat kernel on each side of (4).

2. Preliminary

To write derivatives of higher order concisely, we use the multi-index notations. A multi-index (or, to be precise, an n-multi-index) is an n-tuple $\alpha=(\alpha_1,\dots,\alpha_n)$ of nonnegative integers. Here, we define some notations: $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$, $\alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!$, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $\partial^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, for $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, where $\partial_j=\partial/\partial x_j$. The set of all n-multi-indices is denoted by $\mathbb{N}_0^n$.

We briefly give the definition of tempered distributions and Fourier hyperfunctions. See [5,6,13] for more details.

Definition 3. The space of all infinitely differentiable functions φ on $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{\alpha,\beta}=\sup_x\left|x^\alpha\partial^\beta\varphi(x)\right|<\infty \tag{5}$$

for all α, $\beta \in \mathbb{N}_0^n$, is called the Schwartz space and is denoted by $\mathcal{S}(\mathbb{R}^n)$. The semi-norms $\|\cdot\|_{\alpha,\beta}$ (5) generate a topology on the space $\mathcal{S}(\mathbb{R}^n)$, and the strong dual of $\mathcal{S}(\mathbb{R}^n)$, equipped with this topology is denoted by $\mathcal{S}'(\mathbb{R}^n)$ and its elements are called tempered distributions.

Definition 4. The space of all infinitely differentiable functions φ on $\mathbb{R}^n$ satisfying

$$\|\varphi\|_{h,k} = \sup_{x\in\mathbb{R}^n,\ \alpha\in\mathbb{N}_0^n} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty, \tag{6}$$

for some $h, k > 0$, is called the Sato space and is denoted by $\mathcal{F}(\mathbb{R}^n)$. The topology on this space is given in the sense that a sequence $(\varphi_j) \in \mathcal{F}(\mathbb{R}^n)$ is said to converge to zero if $\|\varphi_j\|_{h,k} \to 0$ as $j \to \infty$ for some $h, k > 0$. The strong dual of $\mathcal{F}(\mathbb{R}^n)$ is denoted by $\mathcal{F}'(\mathbb{R}^n)$ and its elements are called Fourier hyperfunctions.

It is easy to see the following inclusions are continuous:

$$\mathcal{F}\big(\mathbb{R}^n\big) \hookrightarrow \mathcal{S}\big(\mathbb{R}^n\big), \qquad \mathcal{S}'\big(\mathbb{R}^n\big) \hookrightarrow \mathcal{F}'\big(\mathbb{R}^n\big),$$

so that it is natural to say that the space of Fourier hyperfunctions is an extension of the space of tempered distributions.

Now, we introduce the heat kernel method. The n-dimensional heat kernel is the fundamental solution $E_t(x)$ of the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}_x^n \times \mathbb{R}_t^+$ given by

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

Using the Fourier transformation, it is easy to see the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x), \tag{7}$$

which is very useful later. Note that for each $t > 0$, E_t belongs to Sato space $\mathcal{F}(\mathbb{R}^n)$ and Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and so for each $u \in \mathcal{F}'(\mathbb{R}^n)$ (or $\mathcal{S}'(\mathbb{R}^n)$) the convolution

$$Gu(x,t) = (u * E_t)(x) = u_y\big(E_t(x-y)\big), \quad x \in \mathbb{R}^n,\ t > 0,$$

is well defined. We call $Gu(x,t)$ the Gauss transform of u. As a matter of fact it is shown in [12] that the Gauss transform $Gu(x,t)$ of u is a C^∞ solution of the heat equation and converges to u as $t \to 0^+$ in the following sense of generalized functions:

$$\big\langle Gu(\cdot,t), \varphi\big\rangle = \int Gu(x,t)\varphi(x)\,dx \to \langle u, \varphi\rangle \quad \text{as } t \to 0^+,$$

for all test function φ.

3. Main theorem

In this section we show the stability of a Jensen type equation (2) in the sense of generalized functions as follows:

$$3u \circ \frac{A}{3} + u \circ P_1 + u \circ P_2 + u \circ P_3 - 2u \circ \frac{Q_3}{2} - 2u \circ \frac{Q_2}{2} - 2u \circ \frac{Q_1}{2} = 0.$$

It can be easily seen that by convolving $E_r(x)E_s(y)E_t(z)$ in each side of the above equation, we have the following functional equation for smooth functions:

$$3Gu\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + Gu(x,r) + Gu(y,s) + Gu(z,t)$$
$$- 2Gu\left(\frac{x+y}{2}, \frac{r+s}{4}\right) - 2Gu\left(\frac{y+z}{2}, \frac{s+t}{4}\right) - 2Gu\left(\frac{z+x}{2}, \frac{t+r}{4}\right) = 0, \quad (8)$$

for all $x, y, z \in \mathbb{R}^n$ and $r, s, t > 0$. Here, Gu is the Gauss transform of u.

The following two lemmas deal with the solutions of Eq. (8) with some additional conditions which play an essential role to prove the main theorem of our paper.

Lemma 5. *Let $Z : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ be a solution of Eq.* (8) *with*

$$Z(-2x, 2r) = 2Z(x, r), \quad (9)$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Then Z is identically zero.

Proof. Putting $x = y = z = 0$ and $r = s = t$ in (8), we have

$$3Z\left(0, \frac{r}{3}\right) + 3Z(0, r) - 6Z\left(0, \frac{r}{2}\right) = 0. \quad (10)$$

It follows from (9) and (10) that

$$Z(0, r) = 0,$$

for all $r > 0$. On the other hand, putting $y = z = 0$ in (8) and using Eq. (9), we obtain

$$3Z\left(\frac{x}{3}, \frac{r+s+t}{9}\right) + Z(x, r) - Z\left(-x, \frac{r+s}{2}\right) - Z\left(-x, \frac{t+r}{2}\right) = 0. \quad (11)$$

Interchanging r and s in (11), we have

$$3Z\left(\frac{x}{3}, \frac{r+s+t}{9}\right) + Z(x, s) - Z\left(-x, \frac{r+s}{2}\right) - Z\left(-x, \frac{t+s}{2}\right) = 0. \quad (12)$$

It follows from (11) and (12) that

$$Z(x, r) - Z(x, s) - Z\left(-x, \frac{t+r}{2}\right) + Z\left(-x, \frac{t+s}{2}\right) = 0, \quad (13)$$

for all $x \in \mathbb{R}^n$ and $r, s, t > 0$. Putting $t = s$ in (13), we have

$$Z(x, r) - Z(x, s) - Z\left(-x, \frac{s+r}{2}\right) + Z(-x, s) = 0. \quad (14)$$

On the other hand, putting $t = r$ in (13), we obtain

$$Z(x, r) - Z(x, s) - Z(-x, r) + Z\left(-x, \frac{r+s}{2}\right) = 0. \quad (15)$$

It follows form (14) and (15) that

$$2Z(x, r) - 2Z(x, s) + Z(-x, s) - Z(-x, r) = 0, \quad (16)$$

for all $x \in \mathbb{R}^n$ and $r, s > 0$. Replacing x in (16) by $-x$, we have

$$2Z(-x,r) - 2Z(-x,s) + Z(x,s) - Z(x,r) = 0. \tag{17}$$

It follows from (16) and (17) that

$$Z(x,r) - Z(x,s) = 0,$$

for all $x \in \mathbb{R}^n$ and $r, s > 0$. Hence $Z(x,r)$ is independent of $r > 0$. Applying this property to (12), we obtain

$$3Z\left(\frac{x}{3}, r\right) + Z(x,r) - 2Z(-x,r) = 0. \tag{18}$$

Similarly, we also obtain the following equality:

$$3Z\left(\frac{x}{3}, r\right) + 2Z(x,r) + Z(-x,r) - 2Z(x,r) = 0, \tag{19}$$

for all $x \in \mathbb{R}^n$ and $r > 0$ if we put $y = x$, $z = -x$ in (8). Then it follows from (18) and (19) that

$$Z(x,r) - 3Z(-x,r) = 0. \tag{20}$$

Replacing x in (20) by $-x$, we have

$$Z(-x,r) - 3Z(x,r) = 0. \tag{21}$$

It follows from (20) and (21) that

$$Z(-x,r) = -Z(x,r),$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Thus, from Eq. (20) we obtain the result. $\square$

Lemma 6. *Let $L : \mathbb{R}^n \times (0,\infty) \to \mathbb{C}$ be a function. Then L is a solution of Eq.* (8) *with the additional condition*

$$L(-2x,r) = -2L(x,r),$$

for all $x \in \mathbb{R}^n$ and $r > 0$ if and only if L satisfies $L(x,r) = L(x,s)$ and

$$L(x+y,r) = L(x,r) + L(y,r),$$

for all $x, y \in \mathbb{R}^n$ and $r, s > 0$. Moreover, if L is continuous with respect to x, then there exists a constant $a \in \mathbb{C}^n$ such that

$$L(x,r) = a \cdot x.$$

Proof. Since the sufficiency can be easily seen, we now consider the necessity. By the assumption, $L(0,r) = -2L(0,r)$. Hence we obtain

$$L(0,r) = 0,$$

for all $r > 0$. Putting $y = -x$, $z = 0$ in (8), we have

$$\begin{aligned} &L(x,r)+L(-x,s)-2L\left(-\frac{x}{2},\frac{s+t}{4}\right)-2L\left(\frac{x}{2},\frac{t+r}{4}\right) \\ &\quad = L(x,r)+L(-x,s)+L\left(x,\frac{s+t}{4}\right)+L\left(-x,\frac{t+r}{4}\right)=0, \end{aligned} \tag{22}$$

for all $x \in \mathbb{R}^n$ and $r, s, t > 0$. Putting $s = r$ and $t = 3r$ in (22), we obtain

$$L(-x,r) = -L(x,r). \tag{23}$$

Putting $t = 3s$ in (22) it follows from Eq. (23) that

$$L(x,r) - L\left(x, \frac{r+3s}{4}\right) = 0,$$

for all $x \in \mathbb{R}^n$ and $r, s > 0$. Hence $L(x,r)$ is independent of $r > 0$. Applying this property and putting $z = -x - y$ and $r = s = t$ in (8), we have

$$\begin{aligned} &L(x,r)+L(y,r)+L(-x-y,r)-2L\left(\frac{x+y}{2},\frac{r}{2}\right) \\ &\qquad -2L\left(\frac{-x}{2},\frac{r}{2}\right)-2L\left(\frac{-y}{2},\frac{r}{2}\right) \\ &\quad = 2L(x,r)+2L(y,r)-4L\left(\frac{x+y}{2},r\right)=0, \end{aligned}$$

for all $x, y \in \mathbb{R}^n$ and $r > 0$. Hence, for every $r > 0$, $L(\cdot, r)$ satisfies Jensen's equation and $L(0,r) = 0$ so that we have

$$L(x+y,r) = L(x,r) + L(y,r),$$

which gives the result. □

Theorem 7. *Let u be a tempered distribution or a Fourier hyperfunction satisfying*

$$\left\| 3u \circ \frac{A}{3} + u \circ P_1 + u \circ P_2 + u \circ P_3 - 2u \circ \frac{Q_3}{2} - 2u \circ \frac{Q_2}{2} - 2u \circ \frac{Q_1}{2} \right\| \leqslant \varepsilon. \tag{24}$$

Then there exist some constants $a \in \mathbb{C}^n$ and $c \in \mathbb{C}$ such that

$$\|u - a \cdot x - c\| \leqslant 2\varepsilon.$$

Here, the constant c can be given by $c = \lim_{r \to 0^+} u * E_r(x)$.

Proof. Convolving in the each side of (24) the tensor product $E_r(x)E_s(y)E_t(z)$ of n-dimensional heat kernels, we have

$$\begin{aligned} &\left| 3Gu\left(\frac{x+y+z}{3}, \frac{r+s+t}{9}\right) + Gu(x,r) + Gu(y,s) + Gu(z,t) \right. \\ &\quad \left. -2Gu\left(\frac{x+y}{2},\frac{r+s}{4}\right)-2Gu\left(\frac{y+z}{2},\frac{s+t}{4}\right)-2Gu\left(\frac{z+x}{2},\frac{t+r}{4}\right)\right| \leqslant \varepsilon, \end{aligned} \tag{25}$$

for all $x, y, z \in \mathbb{R}^n$ and $r, s, t > 0$. Putting $y = x$, $z = -2x$ and $r = s = t$ in (25), we have

$$\left|3Gu\left(0,\frac{r}{3}\right)+2Gu(x,r)+Gu(-2x,r)-2Gu\left(x,\frac{r}{2}\right)\right.$$
$$\left.-4Gu\left(-\frac{x}{2},\frac{r}{2}\right)\right|\leqslant\varepsilon. \tag{26}$$

Putting $x=0$ in (26), we obtain

$$\left|3Gu\left(0,\frac{r}{3}\right)+3Gu(0,r)-6Gu\left(0,\frac{r}{2}\right)\right|\leqslant\varepsilon. \tag{27}$$

Applying the triangle inequality in the inequalities (26) and (27) it follows that

$$\left|2Gu(x,r)+Gu(-2x,r)-2Gu\left(x,\frac{r}{2}\right)\right.$$
$$\left.-4Gu\left(-\frac{x}{2},\frac{r}{2}\right)-3Gu(0,r)+6Gu\left(0,\frac{r}{2}\right)\right|\leqslant 2\varepsilon, \tag{28}$$

for all $x\in\mathbb{R}^n$ and $r>0$. Replacing x, r in (28) by $-2x$, $2r$ respectively and dividing the result by 2, we have

$$\left|Gu(-2x,2r)+\frac{1}{2}Gu(4x,2r)-\frac{3}{2}Gu(0,2r)\right.$$
$$\left.-2Gu(x,r)-Gu(-2x,r)+3Gu(0,r)\right|\leqslant\varepsilon.$$

Define $U(x,r)=2Gu(x,r)+Gu(-2x,r)-3Gu(0,r)$. Then, by the above inequality, we obtain

$$\left|U(x,r)-\frac{1}{2}U(-2x,2r)\right|\leqslant\varepsilon,$$

for all $x\in\mathbb{R}^n$ and $r>0$. By the induction argument, we have

$$\left|U(x,r)-\frac{1}{2^n}U\big((-2)^n x,2^n r\big)\right|\leqslant 2\varepsilon. \tag{29}$$

Replacing x, r in (29) by $(-2)^m x$, $2^m r$ respectively and dividing the result by 2^m, we obtain

$$\left|\frac{1}{2^{m+n}}U\big((-2)^{m+n}x,2^{m+n}r\big)-\frac{1}{2^m}U\big((-2)^m x,2^m r\big)\right|\leqslant\frac{\varepsilon}{2^{m-1}},$$

for all $m,n\in\mathbb{N}$, $x\in\mathbb{R}^n$ and $r>0$. Hence $\frac{1}{2^n}U((-2)^n x,2^n r)$ is a Cauchy sequence for all $x\in\mathbb{R}^n$ and $r>0$. Define $A(x,r)=\lim_{n\to\infty}\frac{1}{2^n}U((-2)^n x,2^n r)$. Then we have

$$A(-2x,2r)=2\lim_{n\to\infty}\frac{1}{2^{n+1}}U\big((-2)^{n+1}x,2^{n+1}r\big)=2A(x,r),$$

for all $x\in\mathbb{R}^n$ and $r>0$. On the other hand, U satisfies the inequality (25) so that A satisfies Eq. (8). Hence, by Lemma 5, A is identically zero.

It follows from (29) that

$$\left|Gu(x,r)-Gu(0,r)-\left(-\frac{1}{2}\right)\big[Gu(-2x,r)-Gu(0,r)\big]\right|\leqslant\varepsilon, \tag{30}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Define $F(x,r) = Gu(x,r) - Gu(0,r)$. Then it follows from the above inequality that

$$\left| F(x,r) - \left(-\frac{1}{2}\right) F(-2x,r) \right| \leqslant \varepsilon.$$

By the induction argument, we have

$$\left| F(x,r) - \left(-\frac{1}{2}\right)^n F\bigl((-2)^n x,r\bigr) \right| \leqslant 2\varepsilon,$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Using the similar method we have done, it can be seen that

$$L(x,r) := \lim_{n\to\infty} \left(-\frac{1}{2}\right)^n F\bigl((-2)^n x,r\bigr)$$

satisfies the following two properties:

$$\bigl| F(x,r) - L(x,r) \bigr| \leqslant 2\varepsilon \tag{31}$$

and

$$L(-2x,r) = -2L(x,r),$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Moreover, it is easy to see that L satisfies Eq. (8). Hence, by Lemma 6, there exists a constant $a \in \mathbb{C}^n$ such that

$$L(x,r) = a \cdot x.$$

In view of (31) we have

$$\bigl| Gu(x,r) - Gu(0,r) - a \cdot x \bigr| \leqslant 2\varepsilon, \tag{32}$$

for all $x \in \mathbb{R}^n$ and $r > 0$. Letting $r \to 0^+$ in (32) we have the result. □

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