JOURNAL OF DIFFERENTIAL EQUATIONS 84, 62–72 (1990)

The Dirichlet Problem for Quasilinear Elliptic Differential Equations in Unbounded Domains

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Communicated by Jack K. Hale

Received July 29, 1988; revised March 24, 1989

This paper is devoted to the second order, quasilinear elliptic Dirichlet problem of nondivergence type. We mainly consider the existence and uniqueness of classical solutions which radially converge at infinity under certain hypotheses. © 1990 Academic Press, Inc.

1. INTRODUCTION

The second order, quasilinear elliptic Dirichlet problem of nondivergence type, i.e., the problem of the form

$$\begin{aligned} Qu &= a^{ij}(x, Du)\, D_{ij}u + b(x, u, Du) = 0 && \text{in } \Omega, \\ u(x) &= \varphi(x) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

will be considered, where Ω is a unbounded subdomain of R^m with $C^{2,\alpha}$ boundary, $m \geqslant 2$, $\alpha \in (0, 1)$.

The elliptic boundary problem in unbounded domains is an important and active area in partial differential equations. A lot of results have been obtained for linear or semilinear second order elliptic boundary problems in exterior domains. N. Meyers and J. Serrin [2] discussed the existence and uniqueness of solutions for linear exterior problems. E. S. Noussair [3], concerning the Dirichlet problem and the third boundary problem of semilinear equations, presented some conditions permitting the existence of nonnegative solutions, positive solutions, maximal solutions, bounded solutions, and solutions which converge to zero uniformly at infinity. Y. Furusho and Y. Ogura [5, 6] considered the existence of bounded positive solutions of semilinear equations. In recent years, the boundary problems for quasilinear equations in exterior domains, especially positive solution problems, have also been paid much attention (see [7]). But for general

0022-0396/90 $3.00

unbounded domains we only know the results on the existence of positive solutions of semilinear elliptic boundary value problems (see [4]).

In this paper, we are interested in the existence and uniqueness of classical solutions of the problem (1.1) which radially converge at infinity (see Definition 2.1).

Similarly, the third boundary problem

$$Qu = 0 \qquad \text{in } \Omega,$$
$$\frac{\partial u}{\partial v} + \beta u = \varphi(x) \qquad \text{on } \partial\Omega,$$

may be discussed, where v is an outer normal direction at $\partial\Omega$.

2. Preliminaries

The unbounded domain Ω in (1.1) is assumed to have the form

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n,$$

where $\{\Omega_n\}$ is a sequence of bounded domains with the following properties:

(1) $\bar{\Omega}_n \subset \bar{\Omega}_{n+1} \subset \bar{\Omega}$;

(2) $\partial\Omega_n \in C^{2,\alpha}$, $n = 1, 2, ...$;

(3) $\{\Omega_n\}$ uniformly satisfies the exterior sphere condition, i.e., there exists a sphere B of constant volume such that $\{x\} = \bar{B} \cap \bar{\Omega}_n$ for every $x \in \partial\Omega_n$, $n = 1, 2, \ldots$.

We assume throughout that for every bounded domain $G \subset \Omega$:

(Q_1) $a^{ij}(x, p) \in C^1(\bar{G} \times R^m)$ and $b(x, z, p) \in C^1(\bar{G} \times R \times R^m)$, $i, j = 1, 2, ..., m$;

(Q_2) Q is uniformly elliptic in G, if $\lambda(x, p)$ and $\Lambda(x, p)$ denote respectively the minimum and maximum eigenvalues of the matrix $[a^{ij}(x, p)]$, this means that

$$\frac{\lambda(x, p)}{\Lambda(x, p)} \leqslant \mathscr{M}_1 \qquad \text{for every} \quad (x, p) \in \bar{G} \times R^m;$$

(Q_3) $D_z b(x, z, p) \leqslant 0$;

(Q_4) $b(x, z, p) \operatorname{sing} z \leqslant \lambda(x, p)\, \mathscr{M}_2(1 + |p|)$;

(Q_5) $|D_{x_k} a^{ij}(x, p)|$, $(1 + |p|)\, |D_{p_k} a^{ij}(x, p)| \leqslant \lambda(x, p)\, \mathscr{M}_3(|z|)$;

(Q_6) $|b(x, z, p)|$, $|D_{x_k} b(x, z, p)|$, $|D_z b(x, z, p)|$, $(1 + |p|)\, |D_{p_k} b(x, z, p)|$

$\leqslant \lambda(x, p)\, \mathscr{M}_4(|z|)(1+|p|)$, where $\mathscr{M}_1$, $\mathscr{M}_2$ are constants, and $\mathscr{M}_3(|z|)$, $\mathscr{M}_4(|z|)$ are positive and increasing functions of $|z|$.

Denote

$$\Gamma = \{x/|x| : x \in \bar{\Omega} \backslash \{0\}\}.$$

DEFINITION 2.1. Let f be a continuous function on S^{m-1}. A function $u(x)$ belonging to $C^2(\Omega) \cap C^0(\bar{\Omega})$ is a solution of (1.1) which radially converges to f at infinity, if it satisfies (1.1) and the condition

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} u(x) = f(\omega) \qquad \text{for every} \quad \omega \in \Gamma.$$

The conditions $\varphi(x) \in C^{2,\alpha}(\bar{\Omega})$ $(0 < \alpha < 1)$ and

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} \varphi(x) = f(\omega) \qquad \text{for every} \quad \omega \in \Gamma$$

will be assumed throughout this paper. From this, we know that $\varphi(x)$ is bounded. Let $|\varphi(x)| \leqslant M_0$, $M_0 > 0$.

DEFINITION 2.2. A function $v^+(x)$ $(v^-(x))$ $\in C^2(\Omega) \cap C^0(\bar{\Omega})$ is called a global super- (sub-) function relative to Q, $\varphi(x)$, and f, if the conditions

$$\begin{aligned} Qv^+(x) &\leqslant 0 \qquad (Qv^-(x) \geqslant 0) \qquad \text{in } \Omega, \\ v^+(x) &\geqslant \varphi(x) \quad (v^-(x) \leqslant \varphi(x)) \qquad \text{on } \bar{\Omega}, \end{aligned} \tag{2.1}$$

are satisfied.

LEMMA 2.3. *Suppose that there exist a superfunction $v^+(x)$ and a subfunction $v^-(x)$. Then we can find a sequence of functions $u_n(x) \in C^0(\bar{\Omega})$, $n = 1, 2, \ldots$, with the following properties*:

$$\begin{aligned} &\text{(i)} \quad u_n(x) \in C^{2,\alpha}(\bar{\Omega}_n), && n = 1, 2, \ldots; \\ &\text{(ii)} \quad Qu_n(x) = 0 && \textit{in } \Omega_n, \quad n = 1, 2, \ldots; \\ &\qquad u_n(x) = \varphi(x) && \textit{on } \partial\Omega_n, \\ &\text{(iii)} \quad v^-(x) \leqslant u_n(x) \leqslant v^+(x), && n = 1, 2, \ldots. \end{aligned}$$

Proof. By Theorem 15.10 in [1], we know that the problem

$$\begin{aligned} Qu(x) &= 0 && \text{in } \Omega_n, \\ u(x) &= \varphi(x) && \text{on } \partial\Omega_n, \end{aligned}$$

has a solution $U_n(x) \in C^{2,\alpha}(\bar{\Omega}_n)$, $n = 1, 2, \ldots$. Then the functions $u_n(x)$ defined by

$$u_n(x) = \begin{cases} U_n(x), & x \in \Omega_n, \\ \varphi(x), & x \in \Omega \backslash \Omega_n, \end{cases} \quad n = 1, 2, \ldots,$$

are of $C^0(\bar{\Omega})$ and satisfy (i), (ii). Moreover, we get (iii) from the comparison principle (Theorem 10.1 in [1]). The proof is complete.

LEMMA 2.4. *Suppose that $b(x, 0, 0) = 0$, then the sequence $\{u_n(x)\}$ in Lemma 2.3 is uniformly bounded.*

Proof. We can rewrite

$$Qu_n(x) = a^{ij}(x, Du_n(x)) D_{ij} u_n(x) + b(x, u_n(x), Du_n(x)) = 0$$

as

$$Lu_n(x) = a_n^{ij}(x) D_{ij} u_n(x) + b_n^k(x) D_k u_n(x) + c_n(x) u_n(x) = 0,$$

where

$$a_n^{ij} = a^{ij}(x, Du_n(x)),$$

$$b_n^k(x) = \int_0^1 D_{p_k} b(x, u_n(x), tDu_n(x))\, dt,$$

$$c_n(x) = \int_0^1 D_z b(x, tu_n(x), 0)\, dt.$$

Then the maximum principle (Theorem 3.1 in [1]) yields that

$$|u_n(x)|_{0,\Omega} \leqslant |\varphi(x)|_{0,\partial\Omega_n} + |\varphi(x)|_{0,\Omega\backslash\Omega_n} \leqslant 2M_0 \qquad \text{for all} \quad n = 1, 2, \ldots,$$

which is just the desired result.

Denote

$$\Omega_{N,\varepsilon} = \{x \in \Omega_N : \operatorname{dist}(x, \partial\Omega_N \backslash \partial\Omega > \varepsilon)\},$$

where $\varepsilon > 0$, N is a positive integer.

LEMMA 2.5. *Let $\{u_n(x)\}$ in Lemma 2.3 be uniformly bounded in $\bar{\Omega}$, i.e., $|u_n(x)| \leqslant M$ for all $x \in \bar{\Omega}$, where the constant M is independent of n. Then for any integer $N > 0$ there are two constants C and $\beta \in (0, 1)$ such that*

$$|u_n(x)|_{2,\beta,\bar{\Omega}_{N,\rho}} \leqslant C_N \qquad \textit{for every} \quad n \geqslant N, \tag{2.2}$$

where ρ is the radius of the exterior sphere B assumed in (3), *and* C_N, β *depend on* m, M_0, M, $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3(M)$, $\mathcal{M}_4(M)$, ρ, $\operatorname{diam} \Omega_N$, β *yet depends on* α, *both independent of* n.

Proof. According to the hypotheses, we see that the structure condition (14.9) and the conditions of Theorems 15.2 and 13.6 in [1] are satisfied. Hence, we get

$$|u_n(x)|_{1, \alpha_0, \Omega_{N, \rho/2}} \leqslant C \qquad \text{for all} \quad n \geqslant N,$$

where the constant C is independent of n. Further, we conclude (2.2) by applying Schauder's interior estimate (Corollary 6.3 in [1]) and local boundary estimate (Lemma 6.5 in [1]).

In Lemma 2.5, we gave a uniform $C^{2,\beta}$ estimate of $u_n(x)$ in $\bar{\Omega}_{N,\rho}$, but not in $\bar{\Omega}_N$. This is because a uniform $C^{1,\alpha}$ estimate and $C^{2,\beta}$ estimate for $\{u_n(x)\}$ on $\partial\Omega_N \backslash \partial\Omega$ cannot be directly acquired under the assumptions.

THEOREM 2.6. *Under the hypotheses of Lemmas* 2.3 *and* 2.5, *the problem* (1.1) *has a solution* $u(x) \in C^{2,\alpha}_{\mathrm{loc}}(\bar{\Omega})$ *satisfying*

$$v^-(x) \leqslant u(x) \leqslant v^+(x).$$

Proof. Let $\{u_n(x)\}$ be the sequence in Lemma 2.3. For each positive integer N, by lemma 2.5, there exists a positive constant C_N independent of n such that

$$|u_n(x)|_{2, \beta, \bar{\Omega}_{N,\rho}} \leqslant C_N$$

for all $n \geqslant N$ and some $\beta \in (0, 1)$. Then, by applying inductively the Arzelà–Ascoli theorem to $\{u_n(x)\}$ in Ω_N ($N = 1, 2, ...$) and the diagonal method, we may obtain a solution $u(x)$ of the problem (1.1) in $C^2(\bar{\Omega})$, and hence in $C^{2,\alpha}(\bar{\Omega})$ by a standard regularity argument based on Schauder's estimate of linear equations. Since $v^+(x) \leqslant u_n(x) \leqslant v^-(x)$ in $\bar{\Omega}_N$ for each $n = 1, 2, ...,$ we know that the function $u(x)$ also satisfies $v^+(x) \leqslant u(x) \leqslant v^-(x)$ in $\bar{\Omega}$.

COROLLARY 2.7. *Suppose that* $\varphi(x) \geqslant 0$, $b(x, 0, 0) \geqslant 0$ *for all* $x \in \bar{\Omega}$, *and there is a nonnegative global super-function* $v_0(x)$, *then the problem* (1.1) *has a nonnegative solution* $u(x) \in C^{2,\alpha}_{\mathrm{loc}}(\bar{\Omega})$ *satisfying* $0 \leqslant u(x) \leqslant v_0(x)$.

Corollary 2.7 follows by taking $v^-(x) = 0$ *and* $v^+(x) = v_0(x)$ in Theorem 2.6.

3. The Existence and Uniqueness of Solutions

Theorem 3.1. *Let* $u_1(x)$, $u_2(x) \in C^2(\Omega) \cap C^0(\bar{\Omega})$ *satisfy* $Qu_1(x) = Qu_2(x) = 0$ *in* Ω, $u_1(x) = u_2(x)$ *on* $\partial\Omega$, *and converge radially to* $f(\omega)$ *for some* $f \in C^0(S^{m-1})$. *Then* $u_1(x) = u_2(x)$ *in* $\bar{\Omega}$.

Proof. By the assumptions, it is easy to get

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} (u_1(x) - u_2(x)) = 0$$

for any $\omega \in \Gamma$. Thus, for any $n \in \mathbb{Z}$ and any $\omega \in \Gamma$, there exist constants $\delta_n(\omega) > 0$ and $R_n(\omega) > 0$ such that

$$|u_1(x) - u_2(x)| < 1/n,$$

provided that $x \in \bar{\Omega} \setminus \{0\}$, $|x/|x| - \omega| < \delta_n(\omega)$, and $|x| > R_n(\omega)$.

Letting

$$S_n(\omega) = \{x/|x| : x \in \bar{\Omega} \setminus \{0\},\ |x/|x| - \omega| < \delta_n(\omega)\},$$

$n = 1, 2, ...,$ then we have

$$\bigcup_{\omega \in \Gamma} S_n(\omega) \supset \bar{\Gamma}.$$

Hence, according to Borel's covering theorem, for each $n \in \mathbb{Z}$ we can choose an integer $k > 0$ and corresponding constants $\delta_n(\omega_1), ..., \delta_n(\omega_k)$, such that

$$\bigcup_{i=1}^{k} S_n(\omega_i) \supset \bar{\Gamma},$$

namely, for every $x \in \bar{\Omega} \setminus \{0\}$ there exists an integer i_0 $(1 \leqslant i_0 \leqslant k)$ such that

$$|x/|x| - \omega_{i_0}| < \delta_n(\omega_{i_0}).$$

Writing

$$R_n = \max_{1 \leqslant i \leqslant k} \{R_n(\omega_i)\},$$

then we have

$$|u_1(x) - u_2(x)| < 1/n$$

provided that $x \in \bar{\Omega}$ and $|x| > R_n$. Moreover, since

$$\lim_{n \to \infty} \Omega_n = \Omega,$$

for each $n \in \mathbb{Z}$ we can find an integer $i > 0$ such that

$$\partial\Omega_{\text{in}} \backslash \partial\Omega \subset \{x \in R^m : |x| > R_n\}.$$

So, from the maximum principle it follows that

$$\sup_{\Omega_{\text{in}}} |u_1(x) - u_2(x)| \leqslant \sup_{\partial\Omega_{\text{in}}} |u_1(x) - u_2(x)| < 1/n.$$

Letting $n \to \infty$, we thus conclude that $u_1(x) = u_2(x)$ in $\bar{\Omega}$. This completes the proof.

A direct treatment for the existence of the problem (1.1) is invalid since the usual estimates for bounded domains cannot be extend to unbounded domains. We plan to approximate a solution of the problem (1.1) by the solutions $u_n(x)$ of the analogous problems in bounded subdomains Ω_n of Ω, $n = 1, 2, \ldots$. So, the key is to obtain a uniform bound for $\{u_n(x)\}$.

DEFINITION 3.2. A function $v^+(x)$ $(v^-(x))$ is called a global upper (lower) barrier in Ω relative to Q, $\varphi(x)$, and f if it satisfies that (i) $v^+(x)$ $(v^-(x))$ is a super- (sub-) function relative to Q, φ, and f, and (ii)

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} v^+(x)\ (v^-(x)) = f(\omega) \qquad \text{for all} \quad \omega \in \Gamma.$$

It is obvious that the barrier is bounded. Hence, we conclude the following existence theorem by virtue of Theorem 2.6.

THEOREM 3.3. *Let $f(\omega) \in C^0(S^{m-1})$ and suppose that there exist upper and lower barriers $v^+(x)$, $v^-(x)$ relative to Q, φ, and f. Then the problem* (1.1) *has a* (*unique*) *solution $u(x) \in C^{2,\alpha}_{\text{loc}}(\bar{\Omega})$ which converges radially to f at infinity.*

Proof. Let $|v^\pm(x)|_{0,\Omega} \leqslant C_0$ $(C_0 > 0)$ and $\{u_n(x)\}$ be the sequence in Lemma 2.3, then we have

$$\begin{aligned} |u_n(x)|_{0,\Omega} &\leqslant |u_n(x)|_{0,\Omega_n} + |\varphi(x)|_{0,\Omega\backslash\Omega_n} \\ &\leqslant \max\{|v^+(x)|_{0,\Omega}, |v^-(x)|_{0,\Omega}\} + M_0 \\ &\leqslant C_0 + M_0, \qquad n = 1, 2, \ldots. \end{aligned}$$

Consequently, $\{u_n(x)\}$ is uniformly bounded in $\bar{\Omega}$. Thus, by Theorem 2.6, the problem (1.1) has a solution $u(x) \in C^{2,\alpha}_{\text{loc}}(\bar{\Omega})$ satisfying

$$v^-(x) \leqslant u(x) \leqslant v^+(x).$$

Further,

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} v^{\pm}(x) = f(\omega) \qquad \text{for all} \quad \omega \in \Gamma$$

and it follows that $u(x)$ converges radially to f at infinity. The theorem is proved.

In the case that the sequence $\{u_n(x)\}$ in Lemma 2.3 is uniformly bounded, we shall motivate the definition of a local barrier.

DEFINITION 3.4. Let M^+ (M^-) be a uniform upper (lower) bound of $\{u_n(x)\}$ in $\bar{\Omega}$. A function $v^+(x)$ ($v^-(x)$) in $\bar{\Omega}\backslash\Omega_{n_0}$ is a local upper (lower) barrier relative to Q, φ, f, and M^+ (M^-) at infinity for some integer $n_0 \geqslant 0$, if $v^+(x)$ ($v^-(x)$) $\in C^2(\Omega\backslash\bar{\Omega}_{n_0}) \cap C^0(\bar{\Omega}\backslash\Omega_{n_0})$ and satisfies

(a) $Qv^+(x) \leqslant 0$ ($Qv^-(x) \geqslant 0$) in $\Omega\backslash\bar{\Omega}_{n_0}$,

(b) $v^+(x) \geqslant \varphi(x)$ ($v^-(x) \leqslant \varphi(x)$) in $\bar{\Omega}\backslash\Omega_{n_0}$,

(c) $v^+(x) \geqslant M^+$ ($v^-(x) \leqslant M^-$) on $\partial\Omega_{n_0}\backslash\partial\Omega$,

(d) $\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} v^+(x)$ ($v^-(x)$) $= f(\omega)$ for all $\omega \in \Gamma$.

Remark 1. If $\Omega_{n_0} = \Omega_0 = \varnothing$, the condition (c) may be removed. So, Definition 3.4 coincides with Definition 3.2.

When $b(x, 0, 0) = 0$, we see from Lemma 2.4 that the sequence $\{u_n(x)\}$ is uniformly bounded. In this case, we have the following existence theorem.

THEOREM 3.5. *Let $b(x, 0, 0) = 0$, and suppose that there exist local upper and lower barriers $v^+(x)$, $v^-(x)$. Then the problem* (1.1) *has a* (*unique*) *solution $u(x) \in C^{2,\alpha}_{\text{loc}}(\bar{\Omega})$ which converges radially to f at infinity.*

Proof. Let $v^{\pm}(x) \in C^2(\Omega\backslash\Omega_{n_0}) \cap C^0(\bar{\Omega}\backslash\Omega_{n_0})$, then, according to Tietze's extending theorem (Theorem 2 of Section 2.2 in [8]), we can continuously extend $v^{\pm}(x)$ to $v_1^{\pm}(x)$ ($\in C^0(\bar{\Omega})$). Let

$$\varphi_0^+(x) = \max\{v_1^+(x), \varphi(x)\},$$

$$\varphi_0^-(x) = \min\{v_1^-(x), \varphi(x)\}.$$

It is clear that $\varphi_0^{\pm}(x) \in C^0(\bar{\Omega})$. Therefore the existence theorem for continuous boundary values (Theorem 15.18 in [1]) implies that the problems

$$Qu(x) = 0 \qquad \text{in } \Omega_{n_0}$$

$$u(x) = \varphi_0^{\pm}(x) \qquad \text{on } \partial\Omega_{n_0}$$

have solutions $u_0^\pm(x) \in C^0(\bar{\Omega}_{n0}) \cap C^2(\Omega_{n0})$. Putting

$$v_2^\pm(x) = \begin{cases} v^+(x), & x \in \bar{\Omega}\backslash\bar{\Omega}_{n0}, \\ u_0^\pm(x), & x \in \bar{\Omega}_{n0}, \end{cases} \tag{3.1}$$

we see that $v_2^\pm(x) \in C^0(\bar{\Omega})$ and satisfy

$$v_2^-(x) \leqslant \varphi(x) \leqslant v_2^+(x) \qquad \text{for all} \quad x \in \partial\Omega \cup (\Omega\backslash\Omega_{n0}).$$

Let $\{u_n(x)\}$ be the sequence in Lemma 2.3, then we have

$$v_2^-(x) \leqslant \varphi(x) = u_n(x) \leqslant v_2^+(x) \qquad \text{on } \partial\Omega_n$$

for all $n \geqslant n_0$. Moreover, from the assumptions and above argument we get

$$\begin{aligned} &v_2^\pm(x) = u_0^\pm(x) && \text{for all} \quad x \in \bar{\Omega}_{n0}, \\ &u_0^-(x) \leqslant \varphi(x) = u_n(x) \leqslant u_0^+(x) && \text{for all} \quad x \in \partial\Omega\backslash\partial\Omega_{n0}, \\ &u_0^-(x) = v^-(x) \leqslant u_n(x) \leqslant v^+(x) = u_0^+(x) && \text{for all} \quad x \in \partial\Omega_{n0}\backslash\partial\Omega, \end{aligned}$$

and

$$Qu_0^\pm(x) = Qu_n(x) = 0 \qquad \text{in } \Omega_{n0}.$$

Thus the comparison principle for quasilinear elliptic equations yields that

$$v_2^-(x) \leqslant u_n(x) \leqslant v_2^+(x) \tag{3.2}$$

for all $n \geqslant n_0$ and $x \in \bar{\Omega}_{n0}$. Similarly we also get (3.2) in $\bar{\Omega}_n\backslash\Omega_{n0}$, and hence in $\bar{\Omega}_n$ for all $n \geqslant n_0$. Further, we see from (3.1) that $v_2^\pm(x)$ are bounded in $\bar{\Omega}$ and converge radially to f at infinity. So, (3.2) implies that $\{u_n(x)\}$ is uniformly bounded. We thus obtain the conclusion from Theorem 2.6.

Remark 2. If we replace the condition (Q_2) by the condition

(Q_2') The operator Q is elliptic in $G \subset \Omega$, that is, for any $(x, p) \in G \times R^m$, it holds that

$$\Lambda(x, p) = o(\lambda(x, p)\,|p|^2) \qquad \text{as} \quad |p| \to +\infty,$$

where $\Lambda(x, p)$ and $\lambda(x, p)$ are as above,

the conclusion of Theorem 3.5 is still true.

Remark 3. If Ω is an exterior domain in R_m, i.e., $\Omega = R^m\backslash\bar{G}$, where G is a bounded domain in R^m, the condition $v^-(x) \leqslant \varphi(x) \leqslant v^+(x)$ for $x \in \bar{\Omega}$ in Definition 2.2 as well as in Definition 3.2 can be replaced by the condition $v^-(x) \leqslant \varphi(x) \leqslant v^+(x)$ for $x \in \partial\Omega$.

As an application for the results of this paper, we consider the Dirichlet problem of the minimal surface equation

$$(\mathrm{p}) \quad \begin{cases} \mathscr{M}u = (1+|Du|^2)\,\Delta u - D_i u\, D_j u\, D_{ij} u = 0 & \text{in } \Omega, \\ u(x) = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

where Ω is an exterior domain in R^m. It is assumed that the mean curvature of the boundary $\partial\Omega$ is everywhere nonnegative.

Let

$$\Gamma_1 = \{x: |x| = 1\}.$$

PROPOSITION 3.6. *Let* $m \geqslant 3$, $\varphi(x) \in C^{2,\alpha}(\bar{\Omega})$, *and let*

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} \varphi(x) = h \qquad (h \text{ is a constant}),\ \omega \in \Gamma_1,$$

then

(1) *the problem* (p) *has a (unique) solution* $u(x) \in C^{2,\alpha}_{\mathrm{loc}}(\bar{\Omega})$ $(0<\alpha<1)$ *satisfying*

$$\lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} u(x) = h, \qquad \omega \in \Gamma_1;$$

(2) *if* $\varphi(x) \geqslant 0$ *on* $\partial\Omega$, *the problem* (p) *has a nonnegative solution* $u(x) \in C^{2,\alpha}(\bar{\Omega})$ *which converges to zero uniformly as* $|x| \to +\infty$;

(3) *if* $\varphi(x) \geqslant 0$ *on* $\partial\Omega$ *with strict inequality holding for at least one point* $x \in \partial\Omega$, *the problem* (p) *has a positive solution which converges to zero uniformly as* $|x| \to +\infty$.

Proof. (1) Taking

$$v^{\pm}(x) = \pm C\,|x|^{2-m} + h,$$

where $C = \max\{1, \max_{\partial\Omega}\{(h + |\varphi(x)|)\,|x|^{m-2}\}\}$, then we have

$$\begin{aligned} &(\mathrm{a}) \quad \mathscr{M}v^{+}(x) = -C^3(m-2)^3\,(m-1)\,|x|^{-3m+2} < 0 \\ &\qquad\ \mathscr{M}v^{-}(x) = C^3(m-2)^3\,(m-1)\,|x|^{-3m+2} > 0 \end{aligned} \qquad \text{in } \Omega,$$

$$(\mathrm{b}) \quad v^{-}(x) \leqslant \varphi(x) \leqslant v^{+}(x) \qquad \text{on } \partial\Omega,$$

$$(\mathrm{c}) \quad \lim_{\substack{x/|x| \to \omega \\ |x| \to +\infty}} v^{\pm}(x) = h \qquad \text{as} \quad m \geqslant 3.$$

This means that $v^{+}(x)$ $(v^{-}(x))$ is a local upper (lower) barrier relative to φ, h. Hence the conclusion (1) follows by Theorem 3.3 and Remarks 2 and 3.

(2) Similar to (1), we know that

$$v_0(x) = C\,|x|^{2-m} \qquad (C \text{ as } (1))$$

is a nonnegative bounded super-function. So, by Corollary 2.7, we see that the problem (p) has a nonnegative solution $u(x) \in C^{2,\alpha}_{\mathrm{loc}}(\bar{\Omega})$ satisfying

$$0 \leqslant u(x) \leqslant C\,|x|^{2-m},$$

and consequently we have $\lim_{|x| \to +\infty} u(x) = 0$ uniformly as $m \geqslant 3$.

(3) According to (2), the problem (p) has a nonnegative solution $u(x)$. Now it suffices to show that $u(x) > 0$ in Ω.

We can rewrite

$$\mathcal{M}u(x) = (1 + |Du(x)|^2)\,\Delta u(x) - D_i u(x)\,D_j u(x)\,D_{ij}(x) = 0$$

as

$$Lu(x) = a^{ij}(x)\,D_{ij}u(x) = 0,$$

where

$$\begin{aligned} a^{ii}(x) &= 1 + |Du(x)|^2 - |D_i u(x)|^2, \\ a^{ij}(x) &= -D_i u(x)\,D_j u(x), \end{aligned} \qquad i \neq j,\ i, j = 1, 2, ..., m.$$

Then the conclusion (3) follows immediately by E. Hopf's strong maximum principle (Theorem 3.5 in [1]).

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