NOTE

Discontinuity and Fixed Points

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In this paper we deal with the open question [7, p. 242] on the existence of a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point and provide one such contractive definition. We also study this problem for a pair of mappings and establish a situation in which the common fixed point is a point of discontinuity. It may be observed in this context that it is known since the paper of Kannan [3] in 1968 that there exist maps that have a discontinuity in their domain but which have fixed points. However, in all the known cases the maps involved were continuous at the fixed point.

The study of fixed points of contractive type mappings has centered around compatible mappings for more than a decade. However, the study of common fixed points of noncompatible mappings is equally interesting and this author [5] has initiated some work along these lines. Interestingly enough, the best examples of noncompatible mappings are found among pairs of mappings which are discontinuous at their common fixed point.

Two self-maps \( f \) and \( g \) of a metric space \((X, d)\) are called compatible [2] if \( \lim_{n \to \infty} d(fg^n, gf^n) = 0 \) whenever \((x_n)\) is a sequence such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \) in \( X \). This implies that \( f \) and \( g \) will be noncompatible if there exists a sequence \((x_n)\) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \) in \( X \) but \( \lim_{n \to \infty} d(fg^n, gf^n) \) is either nonzero or nonexistent. In 1994, the present author [4] introduced the notion of \( R \) weakly commuting maps and in a work Pathak, Cho, and Kang [6] gave an interesting analogue of \( R \)-weak commutativity by defining \( R \)-weak commutativity of type \((A)\). Two self-maps \( f \) and \( g \) of a metric space \((X, d)\) are \( R \) weakly
 commuting of type $(A_9)$ if there exists some positive real number $R$ such that

$$d(fff, gff) \leq Rd(fff, gff)$$

for all $x$ in $X$. This notion implies commutativity at coincidence points and is useful in studying common fixed points of noncompatible mappings. Using this notion we obtain a common fixed point theorem under such a contractive condition which otherwise does not guarantee a common fixed point.

RESULTS

If $f$ is a self-map of a metric space $(X, d)$, in the first theorem we denote:

$$m(x, y) = \max\{d(x, xf), d(y, yf)\}.$$ 

Also, let $\phi: R_+ \to R_+$ denote a function such that $\phi(t) < t$ for each $t > 0$.

**Theorem 1.** Let $f$ be a self-mapping of a complete metric space $(X, d)$ such that for any $x, y$ in $X$

(i) $d(fff, gff) \leq \phi(m(x, y))$

(ii) given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon < m(x, y) < \varepsilon + \delta \implies d(fff, gff) \leq \varepsilon .$$

Then $f$ has a unique fixed point, say $z$. Moreover, $f$ is continuous at $z$ if and only if $\lim_{x \to z} m(x, z) = 0$.

**Proof.** By virtue of (i), we have

(iii) $d(fff, gff) < m(x, y)$, whenever $m(x, y) > 0$.

Now, the contractive conditions (ii) and (iii) above are particular cases of the corresponding conditions in Lemma 2.2 of Jachymski [1] if we let $A = B = f$ and $S = T = \text{identity map in Jachymski's lemma}$.

Let $x_0$ be any point in $X$. Define a sequence $(x_n)$ in $X$ given by $x_{n+1} = fx_n$, $n \geq 0$. Then using Lemma 2.2 of Jachymski [1], we conclude that $(x_n)$ is a Cauchy sequence. Since $X$ is complete, $\lim_{n \to \infty} x_n = z$ for some $z$ in $X$. Also, $\lim_{n} fx_n = \lim_{n} x_{n+1} = z$. If $z \neq fz$, then using (i), for large values of $n$ we get

$$d(fff, gff) \leq \phi(\max\{d(x_n, x_{n+1}), d(z, z)\})$$

$$= \phi(d(z, fz)).$$
On letting \( n \to \infty \) this yields \( d(z, f_2) \leq \phi(d(z, f_2)) < d(z, f_2) \), a contradiction. Hence \( z = f_2 \) and \( z \) is a fixed point of \( f \). Uniqueness of the fixed point is a consequence of (i).

Now, let \( f \) be continuous at the fixed point \( z \) and \( x_n \to z \). Then \( f x_n \to f z = z \) and \( d(x_n, f x_n) \to 0 \). Hence
\[
\lim_n m(x_n, z) = \lim_n \max\{d(x_n, f x_n), d(z, f z)\} = 0.
\]

On the other hand, if \( \lim_{x \to z} m(x, z) = 0 \), then \( d(x, f x) \to 0 \) as \( x_n \to z \). This implies that \( f x_n \to z = f z \), that is, \( f \) is continuous at \( z \). This establishes the theorem.

Remark 1. The last part of Theorem 1 can alternatively be stated as: \( f \) is discontinuous at the fixed point \( z \) if and only if \( \lim_{x \to z} m(x, z) \neq 0 \).

We now give an example to illustrate Theorem 1.

Example 1. Let \( X = [0, 2] \) and \( d \) be the usual metric on \( X \). Define \( f : X \to X \) as
\[
fx = 1 \text{ if } x \leq 1, \quad fx = 0 \text{ if } x > 1.
\]

Then \( f \) satisfies the conditions of Theorem 1 and has a unique fixed point \( x = 1 \). It can be verified in this example that
\[
d(fx, fy) = 0 \text{ and } 0 < m(x, y) \leq 1 \quad \text{when } x, y \leq 1,
\]
\[
d(fx, fy) = 0 \text{ and } 1 < m(x, y) \leq 2 \quad \text{when } x, y > 1,
\]
and
\[
d(fx, fy) = 1 \text{ and } 1 < m(x, y) \leq 2 \quad \text{when } x \leq 1, y > 1.
\]

Hence \( f \) satisfies the contractive condition (i) with \( \phi(t) = 1 \) for \( t > 1 \) and \( \phi(t) = \frac{1}{2} \) for \( t \leq 1 \). Also, \( f \) satisfies the contractive condition (ii) with \( \delta(\varepsilon) = 1 \) for \( \varepsilon \geq 1 \) and \( \delta(\varepsilon) = 1 - \varepsilon \) for \( \varepsilon < 1 \). However, neither \( \phi(t) \) is upper semicontinuous at \( t = 1 \) nor \( \delta(\varepsilon) \) is lower semicontinuous at \( \varepsilon = 1 \). It can also be easily seen that \( \lim_{x \to 1} m(x, 1) \neq 0 \) and that \( f \) is discontinuous at the fixed point \( x = 1 \).

In the next theorem \( f \mathcal{X} \) denotes the closure of the range of the mapping \( f \).

Theorem 2. Let \( f \) and \( g \) be noncompatible self-mappings of a metric space \( (X, d) \) such that \( f \mathcal{X} \subset g \mathcal{X} \) and

(iv) \( d(fx, fy) < \max\{d(gx, gy), [d(fx, gx) + d(fy, gy)]/2, [d(fx, gy) + d(fy, gx)]/2\} \),
whenever the right-hand side is positive. If $f$ and $g$ be $R$ weakly commuting of type $(A_\chi)$ then $f$ and $g$ have a unique common fixed point and the fixed point is a point of discontinuity.

Proof. Noncompatibility of $f$ and $g$ implies that there exists a sequence $(x_n)$ in $X$ such that

$$\lim_{n} f x_n = \lim_{n} g x_n = t$$

for some $t$ in $X$ but $\lim_{n} d( f g x_n, g f x_n)$ is either nonzero or nonexistent. Since $t \in fX$ and $fX \subset gX$, there exists $u$ in $X$ such that $t = gu$. We assert that $fu = gu$. If not, using (iv) we get

$$d( fu, fx_n) < \max\{d( gu, gx_n), [d( fu, gu) + d( f x_n, gx_n)]/2, \}$$

now letting $n \to \infty$ this yields $d( fu, gu) \leq d( fu, gu)/2$, a contradiction unless $fu = gu$. Since $f$ and $g$ are $R$ weakly commuting mappings of type $(A_\chi)$, we get

$$d( f fu, gfu) \leq Rd( fu, gu) = 0,$$

that is, $ffu = gfu$. If $fu \neq ffu$, using (iv) we get

$$d( fu, ffu) < d( gu, gfu) = d( fu, ffu),$$

a contradiction. Hence $fu = ffu = gfu$ and $fu$ is a common fixed point of $f$ and $g$. Uniqueness of the common fixed point follows from (iv).

We now show that $f$ and $g$ are discontinuous at the common fixed point $t = fu = gu$. If possible, suppose $f$ is continuous. Then considering the sequence $(x_n)$ of (1) we get $\lim_{n} f x_n = ft = t$ and $\lim_{n} g x_n = gt = t$. $R$-weak commutativity of type $(A_\chi)$ now implies that $d( ff x_n, gg x_n) \leq Rd( f x_n, g x_n)$. On letting $n \to \infty$ this yields $\lim_{n} f g x_n = ft = t$. This, in turn, yields $\lim_{n} d( f g x_n, g f x_n) = d( ft, gt) = 0$. This contradicts the fact that $\lim_{n} d( f g x_n, g f x_n)$ is either nonzero or nonexistent for the sequence $(x_n)$ of (1). Hence $f$ is discontinuous at the fixed point. Next, suppose that $g$ is continuous. Then, for the sequence $(x_n)$ of (1), we get $\lim_{n} g f x_n = gt = t$ and $\lim_{n} gg x_n = gt = t$. In view of these limits, the inequality

$$d( ft, fg x_n) < \max\{d( gt, gg x_n), [d( ft, gt) + d( f g x_n, gg x_n)]/2, \}$$

yields a contradiction unless $\lim_{n} f g x_n = ft = gt$. But $\lim_{n} f g x_n = gt$ and $\lim_{n} g f x_n = gt$ contradicts the fact that $\lim_{n} d( f g x_n, g f x_n)$ is either nonzero or nonexistent. Thus, both $f$ and $g$ are discontinuous at their common fixed point. This establishes the theorem.
We now give an example to illustrate Theorem 2.

**Example 2.** Let \( X = [2, 20] \) and \( d \) be the usual metric on \( X \). Define \( f, g : X \to X \) by

\[
\begin{align*}
f(x) &= 2 \text{ if } x = 2 \text{ or } x > 5, \quad f(x) = 6 \text{ if } 2 < x \leq 5, \\
g(x) &= 2 \text{ if } x = 2, \quad g(x) = 12 \text{ if } 2 < x \leq 5, \quad g(x) = (x + 1)/3 \text{ if } x > 5.
\end{align*}
\]

Then \( f \) and \( g \) satisfy all the conditions of Theorem 2 and have a unique common fixed point at \( x = 2 \). It can be verified in this example that \( fX = \{2\} \cup \{6\}, gX = \{2\} \cup \{12\}, \) and \( fX \subset gX \). It can also be verified that \( f \) and \( g \) are noncompatible but \( R \) weakly commuting mappings of type \( (A_g) \). To see that \( f \) and \( g \) are noncompatible, let us consider the sequence \( \{x_n\} = \{5 + 1/n : n \geq 1\} \) in \( X \). Then \( f(x_n) = 2, \lim_n g(x_n) = 2, \lim_n f(x_n) = 6, \) and \( \lim_n g(x_n) = 2 \). Hence \( f \) and \( g \) are noncompatible. \( f \) and \( g \) are \( R \)-weak commuting of type \( (A_g) \) since \( d(fx_n, gx_n) \leq d(fx, gx) \) for all \( x \) in \( X \). Moreover, both \( f \) and \( g \) are discontinuous at the common fixed point \( x = 2 \).

**Remark 2.** In Example 2 the mappings \( f \) and \( g \) are discontinuous but satisfy the following condition:

\((v)\) \( \lim_n f(x_n) = f(t) \) and \( \lim_n g(x_n) = g(t) \) whenever \( \{x_n\} \) is a sequence such that \( \lim_n f(x_n) = \lim_n g(x_n) = t \) for some \( t \) in \( X \).

As an application of condition \((v)\), we now obtain a common fixed point theorem under a contractive condition for which fixed point theorems have not yet been reported.

**Theorem 3.** Let \( f \) and \( g \) be noncompatible self-mappings of a metric space \((X, d)\) satisfying \((v)\) and

\[(v)\quad d(fx, fy) < \max(d(gx, gy), d(fx, gx), d(fy, gy), [d(fx, gy) + d(fy, gx)]/2),\]

whenever the right-hand side is positive. If \( f \) and \( g \) be \( R \) weakly commuting of type \( (A_g) \) then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Noncompatibility of \( f \) and \( g \) implies that there exists a sequence \( \{x_n\} \) in \( X \) such that \( \lim_n f(x_n) = \lim_n g(x_n) = t \) for some \( t \) in \( X \) but \( \lim_n d(fg(x_n), g(x_n)) \) is either nonzero or nonexistent. Condition \((v)\) then implies that \( \lim_n f(x_n) = f(t) \) and \( \lim_n g(x_n) = g(t) \). Further, \( R \)-weak commutativity of type \( (A_g) \) yields

\[d(f(x_n), g(x_n)) \leq Rd(fx_n, gx_n).\]
On making $n \to \infty$ this yields $ft = gt$. The proof now follows on similar lines as in Theorem 2 by using $R$-weak commutativity and condition (vi). An illustration of Theorem 3 is provided in Example 2.

REFERENCES