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Fuzzy Proximity Structures and Fuzzy Ultrafilters

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1. INTRODUCTION

A theory of fuzzy subsets was initiated by Zadeh [11] as an alternative to the classical theory of subsets of a set. The concept of fuzzy subsets was exploited by Chang [3] and others to develop fuzzy topological spaces, by Lowen [6] for putting forward the idea of fuzzy uniform spaces, and recently by Katsaras [5] to introduce fuzzy proximity spaces. The present authors [8] defined fuzzy proximity bases and subbases, and investigated some of the properties of fuzzy proximity spaces in terms of fuzzy proximity bases and subbases. In a subsequent article [9] we have defined fuzzy symmetric generalized proximity spaces and have developed part of its theory as an extension of the Lodato proximity theory.

Thron [10] presented a new approach to proximity structures based on the recognition that many of the entities important in the theory are grills, a concept introduced by Choquet [4] in 1947. Taking a start from Thron's work Azad [1] defined fuzzy stacks, fuzzy grills, and fuzzy basic proximities and obtained a characterization of a fuzzy proximity using fuzzy grills.

It is well known that every filter $\mathscr{F}$ on a set X is the intersection of ultrafilters containing it [2, Proposition 7, p. 61]. We observe in Section 2 that this property need not hold in the fuzzy setting. Thron emphasized that one of the important properties of grills is that they are unions of ultrafilters. A counterexample has been given to show that a fuzzy grill may not contain any fuzzy ultrafilter.

Thus in the fuzzy theory the behaviour of ultrafilters in relation to grills and filters differs radically from that in the ordinary subset theory. But fuzzy ultrafilters continue to be crucial in the development of the theory of fuzzy proximity structures. The object of this article is to substantiate this theme with results for fuzzy basic proximity spaces in general and for fuzzy *LO*-proximity spaces in particular. Part of the theory presented by Thron has been investigated in the context of fuzzy subsets.

In Section 3 fuzzy closure spaces are defined and following the definition

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of fuzzy basic proximity as given by Azad [1] we have shown that every fuzzy basic proximity Π on X induces a fuzzy closure space (X, C_π). It has been further shown that if u is a fuzzy ultrafilter, then $\Pi(u)$ is a fuzzy grill. We have also investigated various properties of $\Pi(u)$. Proceeding with the definitions of a fuzzy Π-clan and a fuzzy Π-cluster we have developed various equivalences for fuzzy Π-clusters. We have also established that every fuzzy Π-cluster is a maximal fuzzy Π-clan.

In Section 4 we have made a further study of fuzzy LO-proximity as defined by us in [9]. In a fuzzy LO-proximity space a condition has been investigated under which a fuzzy grill is the union of fuzzy ultrafilters. If (X, Π) is a fuzzy symmetric generalized proximity space, i.e., a fuzzy LO-proximity space, then it has been proved that $\Pi(\lambda)$ is the union of fuzzy ultrafilters contained in it. We have been able to derive various other properties of $\Pi(\lambda)$ and $\Pi(u)$ in a fuzzy LO-space (X, Π). A definition of a fuzzy Π-bunch is given and it has been shown that every fuzzy Π-cluster is a fuzzy Π-bunch but not the converse. A few equivalent conditions for a fuzzy Π-bunch have been given. We have also obtained a characterization of fuzzy LO-proximity in terms of fuzzy basic proximity. Lastly, if (X, Π) is a fuzzy LO-proximity space and $\lambda \in \Pi(\mu)$, then we have shown that there exists a fuzzy Π-bunch containing both λ and μ.

2. Fuzzy Ultrafilters and Fuzzy Grills

Let X be a nonempty set and I be the closed unit interval of the real line. A fuzzy set in X is an element of the set I^X of all functions from the set X into I. If $\lambda \in I^X$, then we define $\operatorname{supp} \lambda = \{x \in X: \lambda(x) \neq 0\}$. A fuzzy point $\mu_x \in I^X$ is defined as

$$\mu_x(x) = p, \qquad \text{where} \quad 0 < p \leqslant 1,$$
$$\mu_x(y) = 0, \qquad y \neq x.$$

Suppose $\mu \in I^X$ and $\mu(y) \neq 0$. Then the fuzzy point μ_y^* is defined as

$$\mu_y^*(y) = \mu(y),$$
$$\mu_y^*(z) = 0, \qquad z \neq y.$$

2.1[1]. A fuzzy stack S on X is a subfamily of I^X satisfying the following condition:

$$\mu \geqslant \lambda \in S \qquad \text{implies} \quad \mu \in S.$$

2.2[5]. A fuzzy filter $\mathscr{F}$ on X is a nonempty fuzzy stack on X satisfying the conditions

(i) $\lambda, \mu \in \mathscr{F}$ implies $\lambda \wedge \mu \in \mathscr{F}$,

(ii) $0 \notin \mathscr{F}$.

2.3[5]. A maximal, with respect to set inclusion, fuzzy filter on X is called a fuzzy ultrafilter on X. Let φ be a fuzzy filter on X. Then:

(i) φ is a fuzzy ultrafilter on X iff every $\mu \in I^X$, with $\mu \wedge \rho \neq 0$ for all $\rho \in \varphi$, belongs to φ.

(ii) If φ is a fuzzy ultrafilter on X and $\mu_1 \vee \mu_2 \in \varphi$, then either $\mu_1 \in \varphi$ or $\mu_2 \in \varphi$.

(iii) If φ is a fuzzy ultrafilter on X, then for each $\mu \in I^X$ either $\mu \in \varphi$ or $1 - \mu \in \varphi$.

2.4[1]. A fuzzy grill $\mathscr{G}$ on X is a fuzzy stack on X satisfying

(i) $0 \notin \mathscr{G}$,

(ii) $\lambda \vee \mu \in \mathscr{G}$ implies $\lambda \in \mathscr{G}$ or $\mu \in \mathscr{G}$.

2.5. *Notation.* For a fixed nonempty set X, we shall denote by $\Sigma(X)$, $\Phi(X)$, $\Gamma(X)$, and $\Omega(X)$, respectively, the family of all fuzzy stacks on X, all fuzzy filters on X, all fuzzy grills on X, and all fuzzy ultrafilters on X. Then u and v will generally stand for fuzzy ultrafilters.

2.6[1]. For all $S \in \Sigma(X)$, $c(S)$ and $d(S)$ are defined as

(i) $c(S) = \{\lambda \in I^X : 1 - \lambda \notin S\}$,

(ii) $d(S) = \{\lambda \in I^X : \lambda \wedge \mu \neq 0 \ \forall \mu \in S\}$,

Here, $c: \Sigma(X) \to \Sigma(X)$ and $d: \Sigma(X) \to \Sigma(X)$.

2.7. THEOREM. *The operators c and d satisfy the following*:

(i) $c(S) \subset d(S) \ \forall S \in \Sigma(X)$,

(i) $c(c(S)) = S \ \forall S \in \Sigma(X)$,

(iii) *c is a bijection from* $\Sigma(X)$ *to* $\Sigma(X)$,

(iv) *c is a bijection from* $\Gamma(X)$ *to* $\Phi(X)$,

(v) *c is a bijection from* $\Phi(X)$ *to* $\Gamma(X)$,

(vi) $c(\bigcup_i S_i) = \bigcap_i c(S_i)$, $S_i \in \Sigma(X) \ \forall i$,

(vii) $c(\bigcap_i S_i) = \bigcup_i c(S_i)$, $S_i \in \Sigma(X) \ \forall i$,

(viii) $c(u) \subset u$, $u \in \Omega(X)$,

(ix) *c is order reversing*,

(x) *d is order reversing,*

(xi) $d(d(S)) \supset S \ \forall S \in \Sigma(X)$,

(xii) *If* $u \in \Omega(X)$, *then* $d(u) = u$.

Proof. Conditions (i)–(v) are known [1, Theorem 2.6]. Conditions (vi), (vii), (ix), and (x) are easily verified and (xii) is obtained with the help of 2.3(i). Then $\lambda \in c(u)$ implies $1 - \lambda \notin u$ and hence it follows that $\lambda \in u$ by 2.3(iii). This proves (viii). If $\lambda \notin d(d(S))$, then $\lambda \wedge \mu = 0$ for some $\mu \in d(S)$ and therefore $\lambda \notin S$. Hence $S \subset d(d(S))$, as required in (xi).

2.8. Examples 1 and 2 illustrate that the inclusions in (i) and (viii) may be proper. That there is no inclusion relation between S and $c(S)$ is demonstrated by Examples 3 and 4. Likewise, Examples 5 and 6 show that no definite inclusion can hold between S and $d(S)$. Example 5 further illustrates that the inclusion in (xi) may be proper.

EXAMPLE 1. Let $a \in X$. Consider the fuzzy point λ_a given by

$$\lambda_a(a) = \tfrac{1}{4},$$

and the fuzzy stack S generated by λ_a, i.e.,

$$S = \{\mu \in I^X : \mu \geqslant \lambda_a\}.$$

Define $\mu \in I^X$ as follows:

$$\mu(x) = 1, \qquad x \neq a, \qquad \mu(a) = \tfrac{3}{4}.$$

Then $\mu \in d(S)$ but $\mu \notin c(S)$.

EXAMPLE 2. Consider a fuzzy point λ_x given by $\lambda_x(x) = \frac{2}{3}$. Let $u = \{\mu \in I^X : \lambda_x \wedge \mu \neq 0\}$. We have $u \in \Omega(X)$ and $c(u) = \{\nu \in I^X : \nu(x) = 1\} \subset u$. Take $\gamma \in I^X$ such that $\gamma(x) = \frac{1}{4}$ and $\gamma(y) = 1$ for $y \neq x$. Obviously, $\gamma \in u$ but $\gamma \notin c(u)$. It can be verified that $c(u)$ is a fuzzy filter and, for every $\lambda \in c(u)$, $\gamma \wedge \lambda \neq 0$. Thus, by 2.3(i), we infer that $c(u) \notin \Omega(x)$.

EXAMPLE 3. Consider the fuzzy point λ_x of Example 2. Let $S_1 = \{\lambda \in I^X : \lambda \geqslant \lambda_x\}$. Then $c(S_1) = \{\lambda \in I^X : \lambda(x) > \frac{1}{3}\} \supsetneqq S_1$.

EXAMPLE 4. Let μ_x be a fuzzy point such that $\mu_x(x) = \frac{1}{3}$. Let $S_2 = \{\lambda \in I^X : \lambda \geqslant \mu_x\}$. Then $c(S_2) = \{\lambda \in I^X : \lambda(x) > \frac{2}{3}\} \subsetneqq S_2$.

EXAMPLE 5. *Let* $S_1 = \{1 \in I^X : 1(x) = 1 \ \forall x \in X\} = \{1\} \in \Sigma(X)$. Then $d(S_1) = I^X - \{0\} \supsetneqq S_1$, $d(d(S_1)) = \{\lambda \in I^X : \operatorname{supp} \lambda = X\} \supsetneqq S_1$.

EXAMPLE 6. Let $S_2 = I^X - \{0\} \in \Sigma(X)$. Here $d(S_2) = \{\lambda \in I^X\colon \operatorname{supp}\lambda = X\} \subsetneqq S_2$ and $d(d(S_2)) \equiv I^X - \{0\} = S_2$.

2.9. THEOREM. *For fuzzy grills, fuzzy filters, and fuzzy ultrafilters the following hold*:

(i) *If* $G_i \in \Gamma(X)$ *for all* $i \in \Lambda$, *then* $\bigcup_{i\in\Lambda} G_i \in \Gamma(X)$.

(ii) *Every fuzzy ultrafilter is a fuzzy grill and arbitrary unions of fuzzy ultrafilters are fuzzy grills.*

(iii) $\Omega(X) \subsetneqq \Gamma(X) \cap \Phi(X)$.

(iv) *If* $u \subset G_1 \cup G_2$, *then* $u \subset G_1$ *or* $u \subset G_2$, *where* $u \in \Omega(X)$, $G_1, G_2 \in \Gamma(X)$.

(v) *A fuzzy filter may be properly contained in the intersection of all fuzzy ultrafilters containing it.*

(vi) *A fuzzy grill may not contain any fuzzy ultrafilter.*

Proof. (i) $G_i \in \Gamma(X)$ $\forall i \in \Lambda$ implies $\bigcup_{i\in\Lambda} G_i \in \Sigma(x)$. By virtue of Theorem 2.7 (ii) and (vi), we obtain

$$\bigcup_{i\in\Lambda} G_i = c\left[c\left(\bigcup_{i\in\Lambda} G_i\right)\right] = c\left[\bigcap_{i\in\Lambda} c(G_i)\right].$$

Since $c(G_i)$ are fuzzy filters, by Theorem 2.7(iv), $\bigcap_{i\in\Lambda} c(G_i)$ is a fuzzy filter. Since $\bigcup_{i\in\Lambda} G_i$ is the image of a fuzzy filter under c, it is a fuzzy grill by Theorem 2.7(v).

(ii) If $u \in \Omega(X)$ and $\lambda \vee \mu \in u$, then $\lambda \in u$ or $\mu \in u$ by virtue of 2.3(ii). Thus, $u \in \Gamma(X)$. The second part follows from (i).

(iii) If $u \in \Omega(X)$, then $u \in \Gamma(X)$ by (ii). Since every fuzzy ultrafilter is also a fuzzy filter,

$$\Omega(X) \subset \Gamma(X) \cap \Phi(X).$$

Example 2 of 2.8 shows that $c(u) \in \Gamma(X) \cap \Phi(X)$ but $c(u) \notin \Omega(X)$. Consequently $\Omega(X)$ is properly contained in $\Gamma(X) \cap \Phi(X)$.

(iv) Suppose $u \not\subset G_1$ and $u \not\subset G_2$. There exist $\lambda, \mu \in u$ such that $\lambda \notin G_1$ and $\mu \notin G_2$. Then $\lambda \vee \mu \in u$, $\lambda \wedge \mu \notin G_1$, and $\lambda \wedge \mu \notin G_2$. Therefore, $\lambda \wedge \mu \notin G_1 \cup G_2$.

(v) and (vi) Suppose $\lambda \in I^X$ is such that $\operatorname{supp}\lambda = X$. By virtue of 2.3(i), $\lambda \in u$ for each $u \in \Omega(X)$. Thus, $\mathscr{S} = \{\lambda \in I^X\colon \operatorname{supp}\lambda = X\} \subset u$ $\forall u \in \Omega(X)$ and therefore $\mathscr{S} \subset \bigcap_{u \supset \mathscr{S}} u$. Consider $v_x \in I^X$ defined by

$$v_x(x) = \tfrac{2}{3}.$$

Set

$$\mathscr{F} = \{\mu \in I^X : \mu \geqslant \nu_x\}.$$

Then $\mathscr{F}$ is a fuzzy filter. Let $\rho \in \mathscr{P}$ be such that

$$\rho(x) = \tfrac{1}{3}$$

Here $\rho \notin \mathscr{F}$. Then $\mathscr{F}$ is also a fuzzy grill and $u \not\subset \mathscr{F}$ for any u.

3. Fuzzy Basic Proximity

3.1[1]. A fuzzy basic proximity on X is a binary relation Π on I^X which satisfies the following conditions:

(FP1) $\Pi = \Pi^{-1}$,

(FP2) $\lambda \vee \mu \in \Pi(\nu)$ iff $\lambda \in \Pi(\nu)$ or $\mu \in \Pi(\nu)$,

(FP3) if $\lambda \wedge \mu \neq 0$, then $\lambda \in \Pi(\mu)$,

(FP4) $0 \notin \Pi(\lambda)$ for every $\lambda \in I^X$.

Here $\Pi(\lambda) = \{\mu \in I^X : \langle \mu, \lambda \rangle \in \Pi\}$.

Remark (1). $\Pi(\lambda \vee \mu) = \Pi(\lambda) \cup \Pi(\mu)$.

Remark (2). $\lambda \geqslant \mu$ implies $\Pi(\lambda) \supset \Pi(\mu)$.

3.2[1]. A binary relation Π on I^X is a fuzzy basic proximity on X iff it satisfies the following conditions:

(FG1) $\Pi = \Pi^{-1}$,

(FG2) $\Pi(\lambda) \in \Gamma(X)$ for every $\lambda \in I^X$,

(FG3) $\Pi(\lambda) \supset \bigcup_{\lambda \in u} u$.

3.3[1]. Let Π be a fuzzy basic proximity on X. An element $\mu \in I^X$ is called a fuzzy proximal neighborhood of $\lambda \in I^X$ wrt Π iff $1 - \mu \notin \Pi(\lambda)$. The set of all fuzzy proximal neighborhoods of λ wrt Π is denoted by $\mathcal{N}(\Pi, \lambda)$.

3.4. Theorem. *Let Π be a fuzzy basic proximity on X, then*

(i) *$\mathcal{N}(\Pi, \lambda) = c(\Pi(\lambda))$ and hence is a fuzzy filter,*

(ii) *$\mathcal{N}(\Pi, \lambda \vee \mu) = \mathcal{N}(\Pi, \lambda) \cap \mathcal{N}(\Pi, \mu)$,*

(iii) *$\lambda \geqslant \mu$ implies $\mathcal{N}(\Pi, \lambda) \subset \mathcal{N}(\Pi, \mu)$,*

(iv) *$\Pi(\lambda) \subset d(\mathcal{N}(\Pi, \lambda))$,*

(v) *if $\mu \in \mathcal{N}(\Pi, \lambda)$, $\mu' \in \mathcal{N}(\Pi, \lambda')$, then $\mu \vee \mu' \in \mathcal{N}(\Pi, \lambda \vee \lambda')$.*

Proof. Conditions (i)–(iii) and (v) are known [1]. We shall be proving (iv) and giving alternative proofs of (ii), (iii), and (v).

(ii) By Remark 3.1(1), we have $\Pi(\lambda \vee \mu) = \Pi(\lambda) \cup \Pi(\mu)$. Using Theorem 2.7(vi) and Theorem 3.4(i), we obtain $c(\Pi(\lambda \vee \mu)) = c(\Pi(\lambda)) \cap c(\Pi(\mu))$, i.e., $\mathscr{N}(\Pi, \lambda \vee \mu) = \mathscr{N}(\Pi, \lambda) \cap \mathscr{N}(\Pi, \mu)$.

(iii) It follows from Remark 3.1(2) and the fact that c is order reversing.

(iv) By Theorem 2.7(i) and (ii) and Theorem 3.4(i), we have $\Pi(\lambda) = c(c(\Pi(\lambda))) \subset d(\mathscr{N}(\Pi, \lambda))$.

(v) Let $\mu \in \mathscr{N}(\Pi, \lambda)$, $\mu' \in \mathscr{N}(\Pi, \lambda')$. Then $\mu \vee \mu' \in c(\Pi(\lambda))$ and $\mu \vee \mu' \in c(\Pi(\lambda'))$, by virtue of Theorem 3.4(i). By using Theorem 2.7(vi) and Remark 3.1(1), we have $\mu \vee \mu' \in c(\Pi(\lambda)) \cap c(\Pi(\lambda')) = c(\Pi(\lambda) \cup \Pi(\lambda')) = c(\Pi(\lambda \vee \lambda')) = \mathscr{N}(\Pi, \lambda \vee \lambda')$.

3.5. Definition. Let (X, Π) be a fuzzy basic proximity space. For $\mu \in I^X$, we define

$$c_\pi(\mu) = \bigvee_{\lambda_x \in \Pi(\mu)} \lambda_x.$$

3.6. Theorem. *Let (X, Π) be a fuzzy basic proximity space. Then the function $c_\pi: I^X \to I^X$ satisfies*

(FC1) $c_\pi(0) = 0$,

(FC2) $c_\pi(\mu \vee \nu) = c_\pi(\mu) \vee c_\pi(\nu)$,

(FC3) $\mu \leqslant c_\pi(\mu)$.

It follows that (X, c_π) is a fuzzy closure space (analogous to Čech's closure space).

Proof. (FC1) By definition, $c_\pi(0) = \bigvee_{\lambda_x \in \Pi(0)} \lambda_x = 0$.

(FC2) $c_\pi(\mu \vee \nu) = \bigvee_{\lambda_x \in \Pi(\mu \vee \nu)} \lambda_x = \bigvee_{\lambda_x \in \Pi(\mu) \cup \Pi(\nu)} \lambda_x$, by Remark 3.1.(1). It follows that $c_\pi(\mu \vee \nu) = c_\pi(\mu) \vee c_\pi(\nu)$.

(FC3) If $\mu \neq 0$, then there exists $y \in X$ such that $\mu(y) \neq 0$. Consider the fuzzy point $\mu_y^* \in I^X$. Since $\mu_y^* \wedge \mu \neq 0$, $\mu_y^* \in \Pi(\mu)$ follows from Definition 3.1.(FP3). Also $\mu = \bigvee_{\mu(y) \neq 0} \mu_y^*$. Hence, $c_\pi(\mu) = \bigvee_{\lambda_x \in \Pi(\mu)} \lambda_x \geqslant \bigvee_{\mu(y) \neq 0} \mu_y^* = \mu$. This completes the proof of Theorem 3.6.

3.7. Definition. For every $u \in \Omega(X)$ we define

$$\Pi(u) = \{\mu \in I^X : \mu \in \Pi(\lambda)\ \forall \lambda \in u\} = \bigcap_{\lambda \in u} \Pi(\lambda).$$

3.8. Theorem. *For every fuzzy basic proximity Π on X and every fuzzy ultrafilter $u \in \Omega(X)$ we have*

(1) $\Pi(u) \in \Gamma(X)$,

(2) $\Pi(u) \supset u$,

(3) $v \subset \Pi(u)$ *iff* $u \subset \Pi(v)$.

Proof. (1) Each $\Pi(\lambda)$ is a fuzzy stack. Therefore, $\bigcap_{\lambda \in u} \Pi(\lambda) = \Pi(u)$ is also a fuzzy stack. Suppose $\mu \vee \nu \in \Pi(u)$. Then, for every $\lambda \in u$, $\mu \vee \nu \in \Pi(\lambda)$. Equivalently, $\lambda \in \Pi(\mu \vee \nu) = \Pi(\mu) \cup \Pi(\nu)$. Hence, $u \subset \Pi(\mu) \cup \Pi(\nu)$. From Theorem 2.9(iv), it follows that either $u \subset \Pi(\mu)$ or $u \subset \Pi(\nu)$. Hence, either $\mu \in \Pi(u)$ or $\nu \in \Pi(u)$. Accordingly, $\Pi(u) \in \Gamma(X)$.

(2) For $\mu \in u$ and any $\lambda \in u$, $\mu \wedge \lambda \neq 0$ and therefore $\mu \in \Pi(\lambda)$. Hence $u \subset \Pi(u)$.

(3) Here $v \subset \Pi(u)$ is equivalent to $\mu \in \Pi(\lambda)$ for every $\mu \in v$ and every $\lambda \in u$. But $\mu \in \Pi(\lambda)$ iff $\lambda \in \Pi(\mu)$ and hence the required inclusion.

3.9. Definition. Let Π be a fuzzy basic proximity on X. A fuzzy grill G on X is called a fuzzy Π-clan on X if $\lambda, \mu \in G$ implies $\lambda \in \Pi(\mu)$.

3.10. Definition. A fuzzy Π-clan σ on X, which satisfies the additional condition; $\sigma \subset \Pi(\lambda)$ implies $\lambda \in \sigma$, is called a fuzzy Π-cluster on X.

3.11. *Remarks.* (i) If $\lambda \in \sigma$ and $\lambda \leqslant \mu$, then $\mu \in \sigma$.

(ii) If σ_1 and σ_2 are two fuzzy Π-clusters in a fuzzy basic proximity space (X, Π) such that $\sigma_1 \subset \sigma_2$, then $\sigma_1 = \sigma_2$.

Proof. (i) It is straightforward.

(ii) It is parallel to that of [7, Lemma 5.6].

3.12. Definition. Let Π be a fuzzy basic proximity on X. For $G \subset I^X$, define

$$b(\Pi, G) = \{\lambda \in I^X : c_\pi(\lambda) \in G\}.$$

A fuzzy Π-clan G which satisfies the additional condition $b(\Pi, G) = G$ is called a fuzzy Π-bunch.

3.13 *Remarks.* (i) If G is a fuzzy grill, then $b(\Pi, G)$ is a fuzzy grill and $b(\Pi, G) \supset G$.

(ii) If G_1 and G_2 are fuzzy grills such that $G_1 \supset G_2$, then $b(\Pi, G_1) \supset b(\Pi, G_2)$.

3.14. Theorem. *For a fuzzy Π-clan G, we have*

$$G \subset \bigcap_{\lambda \in G} \Pi(\lambda).$$

Proof. Suppose $\mu \notin \bigcap_{\lambda \in G} \Pi(\lambda)$. Then $\mu \notin \Pi(\lambda)$ for some $\lambda \in G$. Then G being a fuzzy Π-clan, $\mu \notin G$. Hence, $G \subset \bigcap_{\lambda \in G} \Pi(\lambda)$.

3.15. THEOREM. *For a fuzzy Π-clan σ the following are equivalent:*

(a) *σ is a fuzzy Π-cluster,*

(b) $\sigma = \bigcap_{\lambda \in \sigma} \Pi(\lambda)$.

Proof. Suppose (a) holds. If $\mu \in \bigcap_{\lambda \in \sigma} \Pi(\lambda)$, then, for every $\lambda \in \sigma$, $\mu \in \Pi(\lambda)$. Equivalently, $\sigma \subset \Pi(\mu)$. Since σ is a fuzzy Π-cluster, $\mu \in \sigma$. Accordingly, $\bigcap_{\lambda \in \sigma} \Pi(\lambda) \subset \sigma$. Thus, (b) follows by combining it with Theorem 3.14.

Conversely, suppose (b) holds. Let $\sigma \subset \Pi(\mu)$. Then, for every $\lambda \in \sigma$, $\mu \in \Pi(\lambda)$. Equivalently, $\mu \in \bigcap_{\lambda \in \sigma} \Pi(\lambda) = \sigma$. Hence, σ is a fuzzy Π-cluster.

3.16. THEOREM. *For a fuzzy Π-cluster σ, we have*

$$\sigma = \bigcap_{\Pi(\lambda) \supset \sigma} \Pi(\lambda).$$

Proof. Let σ be a fuzzy Π-cluster. Set

$$\mathscr{R} = \{\lambda \in I^X : \sigma \subset \Pi(\lambda)\}.$$

Clearly, $\sigma \subset \bigcap_{\lambda \in \mathscr{R}} \Pi(\lambda)$. As σ is a fuzzy Π-cluster, for every $\lambda \in \sigma$, $\sigma \subset \Pi(\lambda)$, and therefore $\sigma \subset \mathscr{R}$. Hence, by Theorem 3.15(b),

$$\bigcap_{\lambda \in \mathscr{R}} \Pi(\lambda) \subset \bigcap_{\lambda \in \sigma} \Pi(\lambda) = \sigma,$$

and the result follows.

3.17. THEOREM. *If G is a fuzzy Π-clan, then there exists a maximal fuzzy Π-clan containing G. Every fuzzy Π-cluster is a maximal fuzzy Π-clan.*

Proof. Existence of a maximal fuzzy Π-clan containing a given fuzzy Π-clan follows from Zorn's lemma. Let σ be a fuzzy Π-cluster and Σ be a maximal fuzzy Π-clan containing σ. We have, for every $\lambda \in \Sigma$, $\sigma \subset \Sigma \subset \Pi(\lambda)$. But σ being a fuzzy Π-cluster, it follows that $\lambda \in \sigma$. Accordingly, $\Sigma \subset \sigma$ and therefore $\sigma = \Sigma$.

3.18. THEOREM. *The relation $u \Delta_\pi v$ iff $u \subset \Pi(v)$ is a reflexive and symmetric relation on $\Omega(X)$.*

Proof. An application of Theorem 3.8(2) and (3) yields the result.

3.19. THEOREM. *Let Δ be an arbitrary reflexive and symmetric relation on $\Omega(X)$. Set*

$$\Delta(u) = \bigcup_{v \Delta u} v.$$

Then a relation Π_Δ on I^X defined by

$$\Pi_\Delta(\lambda) = \bigcup_{\lambda \in u} \Delta(u)$$

is a fuzzy basic proximity on X.

Proof. Condition 3.2 (FG2) follows from Theorem 2.9(ii). Since Δ is reflexive on $\Omega(X)$, for every $u \in \Omega(X)$, $u \subset \Delta(u)$. Hence $\bigcup_{\lambda \in u} u \subset \bigcup_{\lambda \in u} \Delta(u) = \Pi_\Delta(\lambda)$. Therefore, 3.2 (FG3) is also satisfied. Finally, suppose that $(\mu, \lambda) \in \Pi_\Delta$. Then $\mu \in \Pi_\Delta(\lambda)$. Accordingly, $\mu \in v$ for some $v \in \Omega(X)$ such that $v \Delta u$ and $\lambda \in u \in \Omega(X)$. By the symmetry of Δ, it is obtained that $\lambda \in u$, where $u \Delta v$ and $\mu \in v$. Hence, $\lambda \in \Pi_\Delta(\mu)$. This proves 3.2 (FG1).

It follows from 3.2 that Π_Δ is a fuzzy basic proximity on X.

4. FUZZY *LO*-PROXIMITY

4.1[9]. A fuzzy *LO*-proximity on a nonempty set X is a binary relation Π on I^X satisfying the following axioms:

(FLP1) $\Pi = \Pi^{-1}$,

(FLP2) $\mu \vee \rho \in \Pi(\nu)$ iff $\mu \in \Pi(\nu)$ or $\rho \in \Pi(\nu)$,

(FLP3) $\mu \wedge \rho \neq 0$ implies $\mu \in \Pi(\rho)$,

(FLP4) $0 \notin \Pi(\rho)$ for every $\rho \in I^X$,

(FLP5) $\rho \in \Pi(\mu)$ and $\lambda_x \in \Pi(\nu)$ for every $\lambda_x \leqslant \mu$ together imply $\rho \in \Pi(\nu)$.

The pair (X, Π) is called a fuzzy *LO*-proximity space.

4.2[9]. Let (X, Π) be a fuzzy *LO*-proximity space. Then the map $\mu \mapsto c_\pi(\mu) = \bar{\mu} = \bigvee_{\lambda_x \in \Pi(\mu)} \lambda_x$ is a fuzzy closure operator on I^X.

4.3[9]. If (X, Π) is a fuzzy *LO*-proximity space, then:

(i) $\bar{\rho} \in \Pi(\mu)$ iff $\rho \in \Pi(\mu)$,

(ii) $\bar{\rho}(x) \neq 0$ implies $\bar{\rho}(x) = 1$,

(iii) if there exists a $\mu_x \in I^X$ such that $(\rho, \mu_x) \in \Pi$ and $(\mu_x, \nu) \in \Pi$, then $(\rho, \nu) \in \Pi$.

4.4. LEMMA. *Let (X, Π) be a fuzzy LO-proximity space and $\sigma \subset I^X$ be such that*

(a) $0 \notin \sigma$,

(b) $\mu \in \sigma$ *iff* $\bar{\mu} \in \sigma$,

(c) $\mu \vee \rho \in \sigma$ *iff* $\mu \in \sigma$ *or* $\rho \in \sigma$.

If $\mu_0 \in \sigma$, then there exists a fuzzy ultrafilter Φ on X with $\mu_0 \in \Phi \subset \sigma$.

Proof. Let $\Omega = \{\omega \subset I^X$: (i) $\mu_0 \in \omega$; (ii) if $\mu_1, \mu_2, ..., \mu_n \in \omega$, then $\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_n \in \sigma\}$. By Zorn's lemma, Ω possesses a maximal element say ω_0 (wrt set inclusion). We shall show that ω_0 is a fuzzy filter on X. Clearly, $\mu_0 \in \omega_0 \subset \sigma$.

Let $\mu_1, \mu_2 \in \omega_0$. Then $\mu_1 \wedge \mu_2 \in \sigma$. Since $\omega_0 \cup \{\mu_1 \wedge \mu_2\} \in \Omega$, by the maximality of ω_0, we obtain $\mu_1 \wedge \mu_2 \in \omega_0$. Next, suppose $\mu_1 \geqslant \mu \in \omega_0$. Then $\mu_1 \in \sigma$. For $\mu_i \in \omega_0$, $i = 2, 3, ..., k$; $\bigwedge_{i=1}^{k} \mu_i \geqslant \mu \wedge (\bigwedge_{i=2}^{k} \mu_i) \in \sigma$ and hence, $\omega_0 \cup \{\mu_1\} \in \Omega$. Therefore, $\mu_1 \in \omega_0$. Let Φ be a fuzzy ultrafilter which contains ω_0. We proceed to show that $\Phi \subset \sigma$. On the contrary suppose that there exists a $\mu \in \Phi$ such that $\mu \notin \sigma$. By (b) $\bar{\mu} \notin \sigma$. Since $\bar{\mu} \wedge (1 - \bar{\mu}) = 0$, $1 - \bar{\mu} \notin \Phi$ and therefore $1 - \bar{\mu} \notin \omega_0$. As $1 = \bar{\mu} \vee (1 - \bar{\mu}) \in \sigma$, it follows from (c) that $1 - \bar{\mu} \in \sigma$. If $\rho_1, \rho_2, ..., \rho_n \in \omega_0$, then $\bigwedge_{i=1}^{n} \rho_i \in \omega_0$. Suppose $(1 - \bar{\mu}) \wedge \rho \in \sigma$ for each $\rho \in \omega_0$. It follows that $(1 - \bar{\mu}) \wedge (\bigwedge_{i=1}^{n} \rho_i) \in \sigma$ for $\rho_i \in \omega_0$, $i = 1, 2, ..., n$. Accordingly, $\omega_0 \cup \{1 - \bar{\mu}\} \in \Omega$ and therefore $(1 - \bar{\mu}) \in \omega_0$. Hence, there exists a $\rho \in \omega_0$ such that $\rho \wedge (1 - \bar{\mu}) \notin \sigma$. As $\bar{\mu} \notin \sigma$, $\bar{\mu} \wedge \rho \notin \sigma$. Therefore, $\rho = 1 \wedge \rho = [\bar{\mu} \vee (1 - \bar{\mu})] \wedge \rho = (\bar{\mu} \wedge \rho) \vee [(1 - \bar{\mu}) \wedge \rho] \notin \sigma$, by (c). This contradicts $\omega_0 \subset \sigma$, and completes the proof.

4.5. THEOREM. *Let (X, Π) be a fuzzy LO-proximity space. Then, for each $\mu \in \Pi(\lambda)$, there exists a fuzzy ultrafilter u on X such that $\mu \in u \subset \Pi(\lambda)$. Thus,*

$$\Pi(\lambda) = \bigcup_{u \subset \Pi(\lambda)} u.$$

Proof. Then $\Pi(\lambda)$ satisfies all the conditions of Lemma 4.4 and Theorem 4.5 follows.

4.6. THEOREM. *For a fuzzy LO-proximity Π on X and $\lambda \in I^X$,*

$$\Pi(\lambda) = \bigcup_{\lambda \in u} u.$$

Proof. Set

$$\mathscr{C} = \{u \in \Omega(X): \lambda \in u\}.$$

Let $u \in \mathscr{C}$. For λ in u, $\Pi(u) \subset \Pi(\lambda)$. Hence, $\bigcup_{\lambda \in u} \Pi(u) \subset \Pi(\lambda)$. Suppose $\mu \in \Pi(\lambda)$ but $\mu \notin \bigcup_{\lambda \in u} \Pi(u)$. Then, for each u in $\mathscr{C}$, $\mu \notin \Pi(u)$. This implies that $\mu \notin \Pi(\nu)$ for some $\nu \in u$. Equivalently, $\nu \notin \Pi(\mu)$ for some $\nu \in u$. Accordingly, $u \not\subset \Pi(\mu)$. Hence, for each u in $\mathscr{C}$, $u \not\subset \Pi(\mu)$. It follows that $\lambda \notin \Pi(\mu)$. For if $\lambda \in \Pi(\mu)$, then, by Theorem 4.5, there exists a $u \in \Omega(x)$ with $\lambda \in u \subset \Pi(\mu)$. This contradicts the assumption that $\mu \in \Pi(\lambda)$. Thus, $\Pi(\lambda) \subset \bigcup_{\lambda \in u} \Pi(u)$ and the proof is complete.

4.7. THEOREM. *If $v \subset \Pi(\lambda)$, then there exists a u satisfying $\lambda \in u \subset \Pi(v)$.*

Proof. Let $v \subset \Pi(\lambda)$. For every $\mu \in v$, we have $\mu \in \Pi(\lambda)$. Equivalently, we have, for every $\mu \in v$, $\lambda \in \Pi(\mu)$. Accordingly, $\lambda \in \bigcap_{\mu \in v} \Pi(\mu) = \Pi(v)$. $\Pi(v)$ satisfies the requirements of Lemma 4.4. Hence Theorem 4.7 follows.

4.8. THEOREM. *Let (X, Π) be a fuzzy LO-proximity space. Then every fuzzy Π-cluster is a fuzzy Π-bunch.*

Proof. Let G be a fuzzy Π-cluster. By Remark 3.13(i), $b(\Pi, G)$ is a fuzzy grill and $G \subset b(\Pi, G)$.

Let $\lambda \in b(\Pi, G)$ and let μ be an arbitrary element of G. Then, G being a fuzzy Π-clan,

$$c_\pi(\lambda) \in G \Rightarrow c_\pi(\lambda) \in \Pi(\mu) \Rightarrow \lambda \in \Pi(\mu),$$

by virtue of 4.3(i). Since $\Pi = \Pi^{-1}$, we have $\mu \in \Pi(\lambda)$ and $G \subset \Pi(\lambda)$. Since G is a fuzzy Π-cluster, it follows that $\lambda \in G$. Hence, $b(\Pi, G) \subset G$. Thus, $b(\Pi, G) = G$.

The following shows that the converse of Theorem 4.8 *is not true*:

4.9. EXAMPLE. Let X be an infinite set. Define a relation Π on I^X as $(\lambda, \mu) \in \Pi$ iff either

(i) supp λ or supp μ is infinite, $\lambda \neq 0$, $\mu \neq 0$; or

(ii) both supp λ and supp μ are finite ($\neq \varnothing$) and $\lambda \wedge \mu \neq 0$.

It can be verified that Π is a fuzzy LO-proximity on X. Define $\sigma \subset I^X$ as

$$\sigma = \{\lambda \in I^X : \text{supp}\, \lambda \text{ is infinite}\}.$$

Then σ as defined is a fuzzy Π-bunch but not a fuzzy Π-cluster.

4.10. THEOREM. *In a fuzzy LO-proximity space (X, Π), if u is a fuzzy ultrafilter, then $b(\Pi, u)$ is a fuzzy Π-bunch.*

Proof. According to Theorem 2.9(ii), u is a fuzzy grill. An application of Remark 3.13(i) yields that $b(\Pi, u)$ is a fuzzy grill and $b(\Pi, u) \supset u$. Suppose λ, $\mu \in b(\Pi, u)$. Then both $c_\pi(\lambda)$ and $c_\pi(\mu)$ belong to u. Hence, $c_\pi(\lambda) \wedge c_\pi(\mu) \neq 0$. Accordingly, $c_\pi(\lambda) \in \Pi(c_\pi(\mu))$. Using 4.3(i) and 4.1 (FLP1), we obtain $\lambda \in \Pi(\mu)$. It follows that $b(\Pi, u)$ is a fuzzy Π-clan. It can be easily seen that $b(\Pi, b(\Pi, u)) = b(\Pi, u)$. Hence, $b(\Pi, u)$ is a fuzzy Π-bunch.

4.11. *Remark* (i) Suppose Π is a fuzzy LO-proximity on X. For each fuzzy point λ_x, $\sigma_{\lambda_x} = \{\mu \in I^X : \mu \in \Pi(\lambda_x)\}$ is a fuzzy Π-cluster. This follows from 4.3(iii).

(ii) If σ is any fuzzy Π-cluster in a fuzzy LO-proximity space (X, Π), then $\lambda \in \sigma$ iff $\bar{\lambda} \in \sigma$. This follows from Definition 3.10, Remark 3.11(i), and 4.3(i).

(iii) In a fuzzy LO-proximity space (X, Π), if $\mathscr{B}$ is a fuzzy Π-bunch and $\lambda_x \in \mathscr{B}$, then $\mathscr{B} = \sigma_{\lambda_x}$

4.12. THEOREM. *Let (X, Π) be a fuzzy LO-proximity space. If G is a fuzzy grill satisfying the additional condition $c_\pi(\mu) \in G$ iff $\mu \in G$, then the following statements are equivalent*:

(a) *G is a fuzzy Π-bunch*;

(b) *$u \subset G$ implies $G \subset \Pi(u)$*;

(c) $G \subset \bigcap_{u \subset G} \Pi(u) = \bigcap_{\lambda \in G} \Pi(\lambda)$;

(d) $u, v \subset G$ implies $u \subset \Pi(v)$.

Proof. Suppose (a) holds and $u \subset G$. Then, for every $\lambda \in u$, $G \subset \Pi(\lambda)$. Equivalently, $G \subset \Pi(u)$. Thus, (a) implies (b). Now suppose (b) holds and let λ, $\mu \in G$. Then, by Lemma 4.4, there exists a $u \in \Omega(X)$ such that $\mu \in u \subset G$. Hence, $\lambda \in G \subset \Pi(u) \subset \Pi(\mu)$, which shows that G is fuzzy Π-clan and therefore a fuzzy Π-bunch. Accordingly, (a) is equivalent to (b). Clearly, (b) is equivalent to (c) and (b) implies (d). Using Lemma 4.4, we have $G = \bigcup_{u \subset G} u$. It follows from (d) that $v \subset \Pi(u)$ for all $v \subset G$. Hence, $G \subset \Pi(u)$. Consequently, (b) follows from (d). This completes the proof of Theorem 4.12.

4.13. THEOREM. *A fuzzy basic proximity Π on X is a fuzzy LO-proximity if and only if*

$$b(\Pi, \Pi(\lambda)) = \Pi(\lambda)$$

for every λ in I^X.

Proof. The only if part of Theorem 4.13 is straightforward.

For the if part, suppose Π is a fuzzy basic proximity on X and

$b(\Pi, \Pi(\lambda)) = \Pi(\lambda)$ for every λ in I^X. Suppose $\rho \in \Pi(\mu)$ and $\lambda_x \in \Pi(\nu)$ for every $\lambda_x \leqslant \mu$. Assume $c_\pi(\mu) \in \Pi(\lambda)$. It follows from Definition 3.12 that $\mu \in b(\Pi, \Pi(\lambda)) = \Pi(\lambda)$ for every λ in I^X. By virtue of Definition 3.5, we have $c_\pi(\nu) = \bigvee_{\mu_x \in \Pi(\nu)} \mu_x \geqslant \bigvee_{\lambda_x \leqslant \mu} \lambda_x = \mu$. But $\rho \in \Pi(\mu)$ implies $\mu \in \Pi(\rho)$ and therefore $c_\pi(\nu) \in \Pi(\rho)$ by 3.1 (FP2). Hence, $\nu \in \Pi(\rho)$ or $\rho \in \Pi(\nu)$, by 3.1 (FP1). Accordingly, 4.1 (FLP5) is satisfied. It follows that Π is a fuzzy *LO*-proximity on X.

4.14. THEOREM. *Let Π be a fuzzy LO-proximity on X and let $\lambda \in \Pi(\mu)$. Then there exists a fuzzy Π bunch containing λ and μ.*

Proof. Applying Theorem 4.5 to $\Pi(\mu)$, for each $\lambda \in \Pi(\mu)$ there is a fuzzy ultrafilter u_λ on X such that $\lambda \in u_\lambda \subset \Pi(\mu)$. It follows that $\mu \in \Pi(u_\lambda)$. An application of Lemma 4.4 to $\Pi(u_\lambda)$ yields a fuzzy ultrafilter u_μ with $\mu \in u_\mu \subset \Pi(u_\lambda)$. Equivalently, $u_\lambda \subset \Pi(u_\mu)$. Hence, $G = u_\lambda \cup u_\mu$ is a fuzzy Π-clan containing λ and μ. There exists a maximal fuzzy Π-clan G^* containing G, by Theorem 3.17.

Then G^* contains λ and μ and it is sufficient to show that G^* is a fuzzy Π-bunch. By Theorem 4.13, for each $\rho \in I^X$, $b(\Pi, \Pi(\rho)) = \Pi(\rho)$. Let $\rho \in G$. If $\eta \in G$, then $\eta \in \Pi(\rho)$ and therefore $G \subset \Pi(\rho)$. It follows from Remark 3.13(ii) that $b(\Pi, G) \subset b(\Pi, \Pi(\rho)) = \Pi(\rho)$.

Take $\xi \in b(\Pi, G)$. Then $c_\pi(\xi) \in G$ and therefore $c_\pi(\xi) \in \Pi(\rho)$ or $\xi \in \Pi(\rho)$, by 4.3(i). Equivalently, $\rho \in \Pi(\xi)$. Consequently, $G \subset \Pi(\xi)$ for every $\xi \in b(\Pi, G)$. For each $\xi \in b(\Pi, G)$, $b(\Pi, G) \subset b(\Pi, \Pi(\xi)) = \Pi(\xi)$. Hence, $b(\Pi, G)$ is a fuzzy Π-clan.

Also $G \subset G^* \subset b(\Pi, G^*)$. Thus, $b(\Pi, G^*)$ is a fuzzy Π-clan containing G, and G^* is the maximal fuzzy Π-clan containing G. Accordingly $b(\Pi, G^*) = G^*$ and G^* is a fuzzy Π-bunch, by Definition 3.12.

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