Note

Eigenvalue problems of Nordhaus–Gaddum type

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Abstract

Let \( G \) be a graph with \( n \) vertices and \( m \) edges and let \( \mu_1(G) \geq \cdots \geq \mu_n(G) \) be the eigenvalues of its adjacency matrix. We discuss the following general problem. For \( k \) fixed and \( n \) large, find or estimate

\[
f_k(n) = \max_{v(G)=n} |\mu_k(G)| + |\mu_k(G^-)|.
\]

In particular, we prove that

\[
\frac{4}{3} n - 2 \leq f_1(n) < (\sqrt{2} - c)n
\]

for some \( c > 10^{-7} \) independent of \( n \). We also show that

\[
\sqrt{\frac{2}{3}} n - 3 < f_2(n) < \sqrt{\frac{2}{3}} n,
\]

\[
\sqrt{\frac{2}{3}} n - 3 < f_n(n) \leq \sqrt{\frac{3}{2}} n.
\]

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1. Introduction

Our notation is standard (e.g., see [1,2,6]); in particular, all the graphs are defined on the vertex set \( \{1, 2, \ldots, n\} = [n] \) and \( G(n, m) \) stands for a graph with \( n \) vertices and \( m \) edges. Given a graph \( G \) of order \( n \), we index the eigenvalues of the adjacency matrix of \( G \) as \( \mu(G) = \mu_1(G) \geq \cdots \geq \mu_n(G) \). As usual, \( G^- \) denotes the complement of a graph \( G \) and \( \omega(G) \) stands for the clique number of \( G \).

Nosal [10] showed that for every graph \( G \) of order \( n \),

\[
n - 1 \leq \mu(G) + \mu(G^-) < \sqrt{2} n.
\]
Considerable attention has been given to the second of these inequalities. In \cite{8} it was shown that 
\[
\mu(G) + \mu(G) \leq \sqrt{\left(2 - \frac{1}{\omega(G)} - \frac{1}{\omega(G)} \right) n(n - 1)},
\]
(2)
improving earlier results in \cite{3,4,7,11}. Unfortunately inequality (2) is not much better than (1) when both \(\omega(G)\) and \(\omega(G)\) are large enough. Thus, it is natural to ask whether \(\sqrt{2}\) in (1) can be replaced by a smaller absolute constant for \(n\) sufficiently large. In this note we answer this question in the positive but first we state a more general problem.

**Problem 1.** For every \(1 \leq k \leq n\) find
\[
f_k(n) = \max_{v(G)=n} |\mu_k(G)| + |\mu_k(G)|.
\]
It is difficult to determine \(f_k(n)\) precisely for every \(n\) and \(k\), so at this stage it seems more practical to estimate it asymptotically. In this note we show that
\[
\frac{4}{3}n - 2 \leq f_1(n) < (\sqrt{2} - c)n
\]
for some \(c > 10^{-7}\) independent of \(n\). For \(f_2(n)\) we give the following tight bounds:
\[
\frac{\sqrt{3}}{2}n - 3 < f_2(n) < \frac{\sqrt{3}}{2}n.
\]
(4)
We also show that
\[
\frac{\sqrt{3}}{2}n - 3 < f_n(n) \leq \frac{\sqrt{3}}{2}n.
\]
(5)
Finally for fixed \(k, 2 < k < n\), and \(n\) large, we prove that
\[
\left\lfloor \frac{n}{k} \right\rfloor - 1 \leq f_k(n) \leq \sqrt{\frac{2}{k}}n,
\]
\[
\left\lfloor \frac{n}{k} \right\rfloor + 1 \leq f_{n-k}(n) \leq \sqrt{\frac{2}{k}}n.
\]

**2. Bounds on \(f_1(n)\)**

Before stating the main result of this section, we shall recall two auxiliary results whose proofs can be found in \cite{9}. Given a graph \(G = G(n, m)\), let
\[
s(G) = \sum_{u \in V(G)} \left| d(u) - \frac{2m}{n} \right|.
\]

**Proposition 2.** For every graph \(G = G(n, m)\),
\[
\frac{s^2(G)}{2n^2 \sqrt{2m}} \leq \mu_1(G) - \frac{2m}{n} \leq \sqrt{s(G)},
\]
(6)
and
\[
\mu_n(G) + \mu_n(G) \leq -1 - \frac{s^2(G)}{2n^3}.
\]
(7)
Decreasing the constant \(\sqrt{2}\) in (1) happened to be a surprisingly challenging task for the author. The little progress that has been made is given in the following theorem.
Theorem 3. There exists $c \geq 10^{-7}$ such that

$$
\mu_1(G) + \mu_1(\overline{G}) \leq (\sqrt{2} - c)n.
$$

for every graph $G$ of order $n$.

Proof. Assume the opposite: let $\varepsilon = 10^{-7}$ and let there exist a graph $G$ of order $n$ such that

$$
\mu_1(G) + \mu_1(\overline{G}) > (\sqrt{2} - \varepsilon)n.
$$

Writing $A(G)$ for the adjacency matrix of $G$, we have

$$
\sum_{i=1}^{n} \mu_i^2(G) = \text{tr}(A^2(G)) = 2e(G),
$$

implying that

$$
\mu_1^2(G) + \mu_n^2(G) + \mu_1^2(\overline{G}) + \mu_n^2(\overline{G}) \leq 2e(G) + 2e(\overline{G}) < n^2.
$$

From

$$
\mu_1^2(G) + \mu_n^2(\overline{G}) \geq \frac{1}{2} (\mu_1(G) + \mu_1(\overline{G}))^2 > \left(1 - \frac{\varepsilon}{\sqrt{2}}\right)^2 n^2 > \left(1 - \sqrt{2}\varepsilon\right)n^2
$$

we find that

$$
|\mu_n(G)| + |\mu_n(\overline{G})| \leq \sqrt{2(\mu_n^2(G) + \mu_n^2(\overline{G}))} < \sqrt{2\sqrt{2}\varepsilon n},
$$

and so, $\mu_n(G) + \mu_n(\overline{G}) > -2^{3/4}\varepsilon^{1/2}n$. Hence, by (7), we have $s^2(G) \leq 2^{7/4}\varepsilon^{1/2}n^4$. On the other hand, by (6) and in view of $s(G) = s(\overline{G})$, we see that

$$
\mu_1(G) + \mu_1(\overline{G}) \leq n - 1 + 2\sqrt{s(G)} < n + 2\sqrt{s(G)} \leq n + 2^{23/16}\varepsilon^{1/8}n,
$$

and, by (9), it follows that

$$(\sqrt{2} - \varepsilon)n < n + 2^{23/16}\varepsilon^{1/8}n.$$

Dividing by $n$, we obtain $(\sqrt{2} - 1) < \varepsilon + 2^{23/16}\varepsilon^{1/8}$, a contradiction for $\varepsilon = 10^{-7}$. $\square$

It is certain that the upper bound given by Theorem 3 is far from the best one. We shall give below a lower bound on $f_1(n)$ which seems to be tight.

Given $1 \leq r < n$, let $G$ be the join of $K_r$ and $\overline{K_{n-r}}$. $G$ satisfies (see, e.g. [5])

$$
\mu_1(G) + \mu_1(\overline{G}) = \frac{r - 1}{2} + \sqrt{nr - \frac{3r^2 + 2r - 1}{4}} + n - r - 1 = n - \frac{r + 3}{2} + \sqrt{nr - \frac{3r^2 + 2r - 1}{4}}.
$$

The right-hand side of this equality is increasing in $r$ for $0 \leq r \leq (n - 1)/3$ and we find that

$$
f_1(n) > \frac{4n}{3} - 2.
$$

This gives some evidence for the following conjecture.

Conjecture 4.

$$
f_1(n) = \frac{4n}{3} + O(1).
$$
We conclude this section with an improvement of the lower bound in (1). Using the first of inequalities (6) we obtain
\[ \mu_1(G) + \mu_1(G^c) \geq n - 1 + \frac{s^2(G)}{2n^2} \left( \frac{1}{\sqrt{2e(G)}} + \frac{1}{\sqrt{2e(G^c)}} \right) \\geq n - 1 + \sqrt{2} \frac{s^2(G)}{n^3}. \]

3. A class of graphs

In this section we shall describe a class of graphs that give the right order of \( f_2(G) \) and, we believe, also of \( f_n(G) \).

Let \( n \geq 4 \). Partition \([n]\) into four sets \( A, B, C, D \) so that \(|A| \geq |B| \geq |C| \geq |D| \geq |C| - 1 \). Join every two vertices inside \( A \) and \( D \), join each vertex in \( B \) to each vertex in \( A \cup C \), join each vertex in \( D \) to each vertex in \( C \). Denote the resulting graph by \( G(n) \).

Note that if \( n \) is divisible by 4, the sets \( A, B, C, D \) have equal cardinality and we see that \( G(n) \) is isomorphic to its complement.

Our main goal in this section is to estimate the eigenvalues of \( G(n) \). Write \( \text{ch}(A) \) for the characteristic polynomial of a matrix \( A \). The following general theorem holds.

**Theorem 5.** Suppose \( G \) is a graph and \( V(G) = \sqcup_{i=1}^k V_i \) is a partition into sets of size \( n \) such that

(i) for all \( 1 \leq i \leq k \), either \( e(V_i) = \binom{n}{2} \) or \( e(V_i) = 0 \);
(ii) for all \( 1 \leq i < j \leq k \), either \( e(V_i, V_j) = n^2 \) or \( e(V_i, V_j) = 0 \).

Let the sets \( V_1, \ldots, V_p \) be independent and \( V_{p+1}, \ldots, V_k \) induce a complete graph. Then the characteristic polynomial of the adjacency matrix of \( G \) is given by
\[ \text{ch}(A(G)) = x^{pn-p}(1-x)^{(k-p)n-(k-p)} \text{ch}(R), \]
where \( R = (r_{ij}) \) is a \( k \times k \) matrix such that
\[
 r_{ij} = \begin{cases} 
 0 & \text{if } i \neq j \text{ and } e(V_i, V_j) = 0, \\
 n & \text{if } i \neq j \text{ and } e(V_i, V_j) = n^2, \\
 0 & \text{if } i = j \text{ and } e(V_i) = 0, \\
 n-1 & \text{if } i = j \text{ and } e(V_i) = \binom{n}{2}. 
\end{cases}
\]

The proof of this theorem is a straightforward exercise in determinants, so we omit it.

If \( n \) is divisible by 4, say \( n = 4k \), for the characteristic polynomial of \( A(G(n)) \) we have by Theorem 5
\[
\text{ch}(A(G(n))) = x^{2k-2}(1-x)^{2k-2} \begin{bmatrix} 
 k-1-x & k & 0 & 0 \\
 k & -x & k & 0 \\
 0 & k & -x & k \\
 0 & 0 & k & k-1-x \\
\end{bmatrix}.
\]

By straightforward calculations, setting \( a = 1 - 1/k \) and \( y = x/k \), we see that
\[
\text{ch}(A(G(n))) = x^{2k-2}(1-x)^{2k-2}[(a-y)(y^2(a-y) + 2y - a) - (y^2 - ay - 1)]
= x^{2k-2}(1-x)^{2k-2}(y^2 - (1+a)y - (1-a))(y^2 + (1-a)y - (a+1)).
\]

Hence, we find that
\[
\mu_2(G) = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\left[\frac{n}{4}\right]^2 - \left[\frac{n}{4}\right]},
\mu_n(G) = -\frac{1}{2} - \sqrt{\frac{1}{4} + 2\left[\frac{n}{4}\right]^2 - \left[\frac{n}{4}\right]}.
\]
If $n$ is not divisible by 4, we will give some tight estimates of $\mu_2(G)$ and $\mu_n(G)$. Note first that $G(4\lceil n/4 \rceil)$ is an induced graph of $G(n)$ which in turn is an induced graph of $G(4\lfloor n/4 \rfloor)$. Thus, the adjacency matrix of $G(4\lfloor n/4 \rfloor)$ is a principal submatrix of the adjacency matrix of $G(n)$ which in turn is a principal submatrix of the adjacency matrix of $G(4\lfloor n/4 \rfloor)$. Since the eigenvalues of a matrix and its principal submatrices are interlaced [6, Theorem 4.3.15], we obtain

$$\begin{align*}
-\frac{1}{2} + \sqrt{\frac{1}{4} + 2 \left\lfloor \frac{n}{4} \right\rfloor^2 - \left\lfloor \frac{n}{4} \right\rfloor} &\leq \mu_2(G) \leq -\frac{1}{2} + \sqrt{\frac{1}{4} + 2 \left\lfloor \frac{n}{4} \right\rfloor^2 - \left\lfloor \frac{n}{4} \right\rfloor}, \\
-\frac{1}{2} - \sqrt{\frac{1}{4} + 2 \left\lceil \frac{n}{4} \right\rceil^2 - \left\lceil \frac{n}{4} \right\rceil} &\leq \mu_n(G) \leq -\frac{1}{2} - \sqrt{\frac{1}{4} + 2 \left\lceil \frac{n}{4} \right\rceil^2 - \left\lceil \frac{n}{4} \right\rceil}.
\end{align*}$$

(10) (11)

4. The asymptotics of $f_2(n)$

In this section we shall prove Inequalities (4). Since $G$ is isomorphic to its complement, from (10) we readily have

$$f_2(n) \geq -1 + 2 \sqrt{\frac{1}{4} + 2 \left\lfloor \frac{n}{4} \right\rfloor^2 - \left\lfloor \frac{n}{4} \right\rfloor} > \sqrt{2}n - 3,$$

so all we need to prove is that $f_2(n) \leq n/\sqrt{2}$.

By (8) we have

$$\mu_1^2(G) + \mu_2^2(G) + \mu_n^2(G) + \mu_1^2(\overline{G}) + \mu_2^2(\overline{G}) + \mu_n^2(\overline{G}) \leq n(n - 1).$$

(12)

By Weyl’s inequalities [6, p. 181], for every graph $G$ of order $n$, we have

$$\mu_2(G) + \mu_n(\overline{G}) \leq \mu_2(K_n) = -1.$$

Thus, using $\mu_2 \geq 0$ and $\mu_n \leq -1$, we obtain

$$\mu_2^2(G) \leq \mu_n^2(\overline{G}) + 2\mu_n(\overline{G}) + 1 < \mu_n^2(\overline{G}).$$

Hence, from (12) and $\mu_1(G) + \mu_1(\overline{G}) \geq n - 1$, we find that

$$\frac{(n - 1)^2}{2} + 2\mu_2^2(G) + 2\mu_2^2(\overline{G}) \leq \mu_1^2(G) + \mu_2^2(G) + \mu_n^2(G) + \mu_1^2(\overline{G}) + \mu_2^2(\overline{G}) + \mu_n^2(\overline{G})$$

$$\leq n(n - 1).$$

After some calculations, we deduce that

$$\mu_2(G) + \mu_2(\overline{G}) \leq \frac{\sqrt{2}}{2}n,$$

completing the proof of Inequalities (4).

5. Bounds on $f_n(n)$

In this section we shall prove Inequalities (5). From (11), as above, we have

$$f_n(n) > \frac{\sqrt{2}}{2}n - 3.$$

We believe that, in fact, the following conjecture is true.

Conjecture 6.

$$f_n(G) = \frac{\sqrt{2}}{2}n + O(1).$$

However, we can only prove that $f_n(G) < (\sqrt{3}/2)n$ which is implied by the following theorem.
Theorem 7. For every graph $G$ of order $n$,
$$\mu_n^2(G) + \mu_n^2(\overline{G}) \leq \frac{3}{8} n^2.$$ 

**Proof.** Indeed, suppose $(u_1, \ldots, u_n)$ and $(w_1, \ldots, w_n)$ are eigenvectors to $\mu_n(G)$ and $\mu_n(\overline{G})$. Let
$$U = \{i : u_i > 0\}, \quad W = \{i : w_i > 0\}.$$ 
Setting $V = [n]$, we clearly have $\mu_n^2(G) \leq E_G(U, V \setminus U)$ and $\mu_n^2(\overline{G}) \leq E_{\overline{G}}(W, V \setminus W)$. Since $E_G(U, V \setminus U) \cap E_{\overline{G}}(W, V \setminus W) = \emptyset$, we see that the graph $G' = (V, E_G(U, V \setminus U) \cup E_{\overline{G}}(W, V \setminus W))$ is at most 4-colorable and hence $G'$ contains no 4-cliques. By Turán’s theorem (e.g., see [1]), we obtain $e(G') \leq (3/8)n^2$, completing the proof. □

6. Bounds on $f_k(n)$, $2 < k < n$

In this section we shall give simple bounds on $f_k(n)$ for $2 < k < n$. Denote by $T_k(n)$ the Turán graph of order $n$ with $k$ classes. Recall that $T_k(n)$ is a complete $k$-partite graph whose vertex classes differ by at most 1 in size. We assume that $k$ is fixed and $n$ is large enough. Since $\mu_k(T_k(n)) = 0$ and $\mu_{n-k}(T_k(n)) \leq -\lceil n/k \rceil$ for $n$ large, we immediately have
$$f_k(n) \geq \lceil n/k \rceil - 1,$$ 
$$f_{n-k}(n) \geq \lceil n/k \rceil + 1.$$ 

We next turn to upper bounds on $f_k(n)$.

Theorem 8. For any fixed $k$ and any graph $G$ of sufficiently large order $n$,
$$|\mu_k(G)| + |\mu_k(\overline{G})| < \sqrt{\frac{2}{k} n}$$ 
and
$$|\mu_{n-k}(G)| + |\mu_{n-k}(\overline{G})| < \sqrt{\frac{2}{k} n}.$$ 

**Proof.** Set $e(G) = m$. Our first goal is to prove that $|\mu_k(G)| \leq \sqrt{2e(G)/k}$. If $\mu_k(G) \geq 0$, we have in view of (8)
$$k \mu_k^2(G) \leq \sum_{i=1}^{n} \mu_i^2(G) = 2m.$$ 
If $\mu_k(G) < 0$ and $|\mu_k(G)| > \sqrt{2m/k}$ then
$$\sum_{i=1}^{n} \mu_i^2(G) \geq (n-k)\mu_k^2(G) > 2m \frac{n-k}{k} > 2m,$$ 
a contradiction. Hence, $|\mu_k(G)| \leq \sqrt{2e(G)/k}$, and, by symmetry, $|\mu_k(\overline{G})| \leq \sqrt{2e(\overline{G})/k}$. Now
$$|\mu_k(G)| + |\mu_k(\overline{G})| \leq \sqrt{2e(G)/k} + \sqrt{2e(\overline{G})/k} \leq \sqrt{\frac{2}{k} n(n-1)} < \sqrt{\frac{2}{k} n},$$ 
proving inequality (13). The proof of inequality (14) goes along the same lines, so we omit it. □
References