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Graphs and obstructions in four dimensions

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Abstract

For any graph $G = (V, E)$ without loops, let $\mathscr{C}_2(G)$ denote the regular CW-complex obtained from G by attaching to each circuit C of G a disc. We show that if G is the suspension of a flat graph, then $\mathscr{C}_2(G)$ has an embedding into 4-space. Furthermore, we show that for any graph G in the collection of graphs that can be obtained from K_7 and $K_{3,3,1,1}$ by a series of ΔY- and $Y\Delta$-transformations, $\mathscr{C}_2(G)$ cannot be embedded into 4-space.

Keywords: Planarity; Linking number; Intersection number; Minors; CW-complex

1. Introduction

For any graph $G = (V, E)$ without loops, let $\mathcal{C}_2(G)$ denote the regular CW-complex obtained from G by attaching to each circuit C of G a disc. So we view the graph here as a regular CW-complex; for the definition of regular CW-complex, see most books on algebraic topology. (A circuit in a graph can be seen as a subgraph homeomorphic to the 1-sphere.) We call a graph G 4-*flat* if $\mathcal{C}_2(G)$ can be embedded piecewise linearly in 4-space. This property can be viewed as the 4-dimensional analog of planarity and flatness of graphs. (A graph is planar if it can be embedded in the plane, see, for example, [4], and a graph is flat if is has an embedding in 3-space such that every circuit of G bounds a open disc in 3-space disjoint from the graph [10].) A 4-dimensional analog of planar graphs was also studied by Gillman [6]. In Theorem 1, we will show that any minor of a 4-flat graph is again 4-flat. So the class of all 4-flat graphs is closed under taking minors.

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In this paper we give a collection of graphs which are not 4-flat. Before introducing this collection, we need some definitions. A graph G' is obtained from graph G by a ΔY-transformation if G' is obtained by deleting the edges of a triangle in G and by adding a new vertex and edges connecting this vertex to all vertices of the triangle. A graph G' is obtained from G by a $Y\Delta$-transformation if G' is obtained by deleting a vertex v of degree 3 (and its incident edges) in G and by adding an edge between each pair of vertices of the set of neighbors of v. By K_7 we denote the graph with 7 vertices in which each pair of vertices is connected by an edge and by $K_{3,3,1,1}$ we denote the graph with 8 vertices in which we can partition the vertex set into four classes, two of size 3 and two of size 1, such that an edge connects two distinct vertices if and only if they belong to different class. See [4] for more about graph theory. We call the collection of all graphs that can be obtained from K_7 and $K_{3,3,1,1}$ by applying a series of ΔY- and $Y\Delta$-transformations the *Heawood family*. This family contains 78 graphs,[2] 20 of which are obtained from K_7 by applying ΔY- and $Y\Delta$-transformations, and 58 of which are obtained from $K_{3,3,1,1}$ by applying ΔY- and $Y\Delta$-transformations. We shall show that each graph in the Heawood family is not 4-flat; hence a graph containing any of these graph as a minor cannot be 4-flat.

The reason to introduce the concept of 4-flat comes from the graph invariant $\mu(G)$. This invariant is introduced in Colin de Verdière [2] and it characterizes the class of planar, and flat graphs as those class of graphs G with $\mu(G) \leqslant 3$, and 4, respectively. The question arises what class of graphs are characterized by $\mu(G) \leqslant 5$. Looking for analogy of the classes of planar, and flat graph leads us to conjecture that 4-flat graphs are characterized by $\mu(G) \leqslant 5$. In Section 4, we shall see that the graphs of the Heawood family are forbidden minors for $\mu(G) \leqslant 5$, providing support for the conjecture that 4-flat graphs are characterized by $\mu(G) \leqslant 5$.

We now describe in short how we will prove that the graphs of the Heawood family are not 4-flat. Let $\mathcal{C}$ be a regular CW-complex and let σ, τ be cells of $\mathcal{C}$. We say that σ, τ are *adjacent* if the smallest subcomplex of $\mathcal{C}$ containing σ and the smallest subcomplex of $\mathcal{C}$ containing τ have nonempty intersection. We say that σ is *incident* to τ if τ belongs to the smallest subcomplex of $\mathcal{C}$ containing σ. Let σ_1, σ_2 be cells of $\mathcal{C}$. We say that σ_1, σ_2 have τ in common if both σ_1 and σ_2 are incident to τ. Define $\mathcal{I}_4$ to be the class of all graphs G for which there exists a mapping f of $\mathcal{C}_2(G)$ into 4-space such that $\mathrm{I}_2(f(\sigma), f(\tau)) = 0$ for every pair σ, τ of nonadjacent 2-cells of $\mathcal{C}_2(G)$. Here $\mathrm{I}_2(f(\sigma), f(\tau))$ denotes the equivalence class of the intersection number of $f(\sigma)$ with $f(\tau)$ under congruence modulo 2. Clearly, 4-flat graphs belong to $\mathcal{I}_4$. Now, we shall show that if G' is obtained from a graph in $\mathcal{I}_4$ by a ΔY- or a $Y\Delta$-transformation, then G' belongs to $\mathcal{I}_4$ as well. Furthermore, we shall show that the graphs K_7 and $K_{3,3,1,1}$ do not belong to $\mathcal{I}_4$, and so we obtain that the graphs of the Heawood family cannot be 4-flat.

Similar to the way $\mathcal{I}_4$ is defined, we define the classes $\mathcal{I}_2$ and $\mathcal{I}_3$. We shall see in Section 6 that the class $\mathcal{I}_2$ is equal to the class of planar graphs and that $\mathcal{I}_3$ is equal to the class of flat graphs. This leads us to conjecture that the class of 4-flat graphs coincides with $\mathcal{I}_4$.

2. Preliminaries

In this paper all mappings are assumed to be piecewise linear. We denote the real line by E and the Euclidean k-space by E^k; mostly, we shall write k-space instead of Euclidean k-space. By E^0_+ and E_+ we denote the spaces of all $x \in E$ with $x \geqslant 0$ and of all $x \in E$ with $x > 0$, respectively;

[2] I thank Rudi Pendavingh for giving me the list of all these 78 graphs.

by E_-^0 and E_- we denote the spaces of all $x \in E$ with $x \leqslant 0$ and of all $x \in E$ with $x < 0$, respectively.

Let $S \subseteq E^n$ be a topological subspace of E^n, and let v be a point in E^n. The *cone* on S with vertex v in E^n is the topological subspace of E^n formed by all line segments with one end in S and the other equal to v.

In this paper all graphs are allowed to parallel edges, but not any loop. A *minor* of a graph G is a graph obtained from a subgraph of G by contracting a sequence of edges of the subgraph and removing any loop. A proper minor of H is a minor unequal to H. A *minor-closed* class of graphs is a class $\mathcal{G}$ of graphs such that each minor and each graph isomorphic to a graph in $\mathcal{G}$ belongs to $\mathcal{G}$. A graph H is an *excluded minor* of a minor-closed class $\mathcal{G}$ if H does not belong to $\mathcal{G}$, but every proper minor of H belongs to $\mathcal{G}$. The well-quasi-ordering theorem of Robertson and Seymour [11] tells us that for every minor-closed class of graphs, the collection of all its excluded minors is finite. Hence, by Theorem 1 there is a finite collection of graphs such that any graph which is not 4-flat contains a minor isomorphic to a graph in this collection.

The next lemma will be used to show that for certain classes of graphs, such as the class of graphs with $\mu(G) \leqslant 5$ (see Section 4) and $\mathcal{I}_4$ (see Section 6), the graphs of the Heawood family are some excluded minors for these classes.

A graph is obtained from graph G by *subdividing* an edge e if it is obtained from G by deleting edge $e = w_1w_2$ and by adding a new vertex v and edges connecting this vertex to w_1 and w_2.

Lemma 1. *Let $\mathcal{G}$ be a minor-closed class of graphs closed under taking ΔY- and $Y\Delta$-transformation, adding parallel edges, and subdividing edges. Let $\{H_1, \ldots, H_k\}$ be a collection of connected graphs, each of which does not belong to $\mathcal{G}$, but such that each proper minor of these graphs belongs to $\mathcal{G}$. Let $\mathcal{H}$ be the collection of all graphs that can be obtained from $H_1, \ldots, H_k$ by applying ΔY- and $Y\Delta$-transformations. Then each proper minor of a graph G in $\mathcal{H}$ belongs to $\mathcal{G}$.*

Proof. It suffices to show that each minor obtained from G by deleting or contracting an edge belongs to $\mathcal{G}$. We proceed by induction to the minimum number of ΔY- and $Y\Delta$-transformations that have to be applied to any of the graphs in $\{H_1, \ldots, H_k\}$ to get G. The case where this numbers is equal to zero is given by the statement of the theorem. If this number is greater than zero, then by induction G is obtained from G' by either a ΔY- or $Y\Delta$-transformation, and G' is a graph obtained from a graph in $\{H_1, \ldots, H_K\}$ by applying ΔY- and $Y\Delta$-transformations and each proper minor of G' belongs to $\mathcal{G}$. We now look to two cases.

Suppose first that G is obtained from G' by a ΔY-transformation. Let H be a graph obtained from G by contracting or deleting one edge e. If e does not belong to the Y, then H can be obtained from a graph in $\mathcal{G}$ by applying a ΔY-transformation, and hence itself belongs to $\mathcal{G}$. So suppose that e is one of the edges of the Y. Now, if H is obtained from G by contracting e, then H is a proper subgraph of G'. If H is obtained from G by deleting e, then H is obtained from a proper subgraph by subdividing one edge. So in both cases it follows that H belongs to $\mathcal{G}$.

Suppose next that G is obtained from G' by a $Y\Delta$-transformation. Let H be a graph obtained from G by contracting or deleting one edge e. Just as above, the case where e does not belong to the Δ is clear, so we suppose that e is one of the edges of the Δ. If H is obtained from G by deleting e, then H is obtained from G' contracting one of the edges of the Y. If H is obtained from G by contracting e, H is obtained from G' by contracting two edges of the Y and adding an edge parallel to the remaining edge. So in both cases it follows that H belongs to $\mathcal{G}$. □

3. A 4-dimensional version of flatness

Theorem 1. *Each minor of a* 4*-flat graph is* 4*-flat.*

Proof. Let G be a 4-flat graph. It suffices to show the theorem for the cases where the minor G' arises from G by either the deletion of a vertex or an edge, or the contraction of an edge.

As $\mathcal{C}_2(G')$ is a subcomplex of $\mathcal{C}_2(G)$ when G' arises from G by deletion of a vertex or an edge, it is clear that $\mathcal{C}_2(G')$ can be embedded into 4-space. So we are left with the case that G' arises from G by contraction of an edge e. Let $\mathcal{D}$ be the complex obtained from $\mathcal{C}_2(G)$ by deleting all 2-cells that are incident to the ends of e but not the edge e itself. Then $\mathcal{D}$ can be embedded in 4-space by ϕ. Take a small neighborhood B around $\phi(e)$ homeomorphic to the 4-ball such that the intersection of each 2-cell of $\mathcal{D}$ with ∂B is a curve in ∂B (and whose two ends are the intersection of the two edges adjacent to e with ∂B). We may assume that B is the unit ball in 4-space. Map the vertex v_e obtained from contracting e to the origin O, and take inside B the cone on $\partial B \cap \phi(\mathcal{D})$ with vertex O in 4-space. Leave every outside B the same. Then we have an embedding of $\mathcal{C}_2(G')$ in 4-space. □

An embedding of a graph G in 3-space is *flat* if every circuit of G bounds a open disc in 3-space disjoint from the graph. A graph is *flat* if it has a flat embedding. The Petersen family is the collection of all graphs that can be obtained from the Petersen graph by a series of ΔY- and $Y\Delta$-transformations. This is a family of graphs containing 7 graphs, one of which is the Petersen graph. Robertson et al. [10] show that a graph is flat if and only if it has no minor isomorphic to a graph in the Petersen family.

The *suspension* of a graph G is the graph obtained from G by adding a new vertex and connecting this vertex to all vertices of G. The next theorem gives an analog of the following: the suspension of a planar graph is flat.

Theorem 2. *Let G be a flat graph and let $S(G)$ be a graph obtained from G by adding a new vertex v and edges* (*possibly multiple*) *from this vertex to all vertices of G. Then $S(G)$ is* 4*-flat.*

Proof. Since G is a flat graph, there is a mapping ψ of $\mathcal{C}_2(G)$ into E^3, such that the restriction of ψ to G is an embedding, each 2-disc is embedded by ψ, and for each 2-disc σ, $\psi(\sigma)$ and $\psi(G)$ have common points only on the boundary of σ.

The embedding ϕ of $\mathcal{C}_2(S(G))$ into E^4 we construct will consist of two parts. In short, the 2-discs of $\mathcal{C}_2(G)$ are embedded into $E^3 \times E^0_-$, and the 2-discs of $\mathcal{C}_2(S(G))$ containing v are embedded into $E^3 \times E^0_+$. We shall first embed $\mathcal{C}_2(G)$ into $E^3 \times E^0_-$.

Choose for each 2-disc σ of $\mathcal{C}_2(G)$ a positive number a_σ, such that these numbers are mutually different. Define the restriction of ϕ to $\mathcal{C}_2(G)$ by $\phi(x) = (\psi(x), 0)$ if $x \in G$ and $\phi(x) = (\psi(x), -a_\sigma d(x))$ if x belongs to σ, where $d(x)$ denotes the distance in E^3 of $\psi(x)$ to $\psi(G)$. This is an embedding because if $\phi(x) = \phi(y)$, then $\psi(x) = \psi(y)$ and $a_\sigma d(x) = a_\tau d(y)$, and since $\psi(x) = \psi(y)$ implies $d(x) = d(y)$, we see that $\sigma = \tau$, which implies that $x = y$.

Let $\mathcal{D}$ be the subcomplex of $\mathcal{C}_2(S(G))$ consisting of all 2-cells whose boundaries contain v. We shall now embed $\mathcal{D}$ into $E^3 \times E^0_+$ such that $\phi^{-1}(E^3 \times \{0\}) = G$. For this, we first construct a cell-complex $\mathcal{K}$ as follows. For each edge e incident with v, let $w = w_e$ be the other end of e, and attach a copy of the unit interval, I, to w in G by identifying 0 of I with w. Denote this simplicial complex by $\mathcal{K}'$. (So identifying all 1's for all copies of the unit interval gives $S(G)$.) For each path P in G, let u_1 and u_2 be the ends of P, and attach to each copy of the unit interval at u_1 and

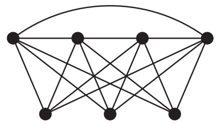

Fig. 1. $K_{3,4}$ with a circuit of size 4 on its color class of size 4.

each copy of the unit interval at u_2 a copy of $P \times I$ to $\mathcal{K}'$ by identifying $P \times \{0\}$ with the path P in G and identifying $u_1 \times I$ and $u_2 \times I$ with their corresponding copies in $\mathcal{K}'$. The resulting cell complex is denoted by $\mathcal{K}$.

Since each 2-cell of $\mathcal{K}$ is of the form $P \times I$ with P a path in G, each point in $\mathcal{K}$ is of the form (p, s) with p a point on a path in G and $s \in I$. So we can put a height function h on $\mathcal{K}$ by defining $h(q) = s$ if $q = (p, s)$. We denote by $\mathcal{K}_0$ and $\mathcal{K}_1$ the subcomplexes $h^{-1}(0)$ and $h^{-1}(1)$, respectively. Triangulate $\mathcal{K}$ such that each vertex of this triangulation belongs to $\mathcal{K}_0$ or $\mathcal{K}_1$.

We now show that $\mathcal{K}$ has an embedding ψ into $E^3 \times I$ with $E^3 \times \{0\} \cap \psi^{-1}(\mathcal{K}) = \mathcal{K}_0$, $E^3 \times \{1\} \cap \psi^{-1}(\mathcal{K}) = \mathcal{K}_1$. To this end, let ψ_1 be the mapping of $\mathcal{K}$ into $E^3 \times I$ defined by $\psi_1(p, s) = (\phi(p), x)$ for every point (p, s) in $\mathcal{K}$. Perturb ψ_1 a little, leaving G fixed, by putting the vertices in $\mathcal{K}_1$ in generic position, and let the resulting map of $\mathcal{K}$ in $E^3 \times I$ be ψ_2. Then $\psi_2(\mathcal{K})$ has only a finite number of self-intersections, and so we can find a $0 < t \leqslant 1$ such that $\psi_2(\mathcal{K})$ has no self-intersections between $E^3 \times \{0\}$ and $E^3 \times \{t\}$. The restriction of $\psi_2(\mathcal{K})$ in $E^3 \times [0, t]$ is homeomorphic to $\mathcal{K}$, hence $\mathcal{K}$ can be embedded into $E^3 \times [0, t]$, such that G is embedded into $E^3 \times \{0\}$ and each point (p, s) of $\mathcal{K}$ is mapped into $E^3 \times \{s\}$. Since $[0, t]$ is homeomorphic to I, there exists an embedding ψ of $\mathcal{K}$ in $E^3 \times I$ with the required property.

Now take the cone of $\psi(\mathcal{K}_1)$ with vertex $(0, 0, 0, 2)^T$ in 4-space. Altogether, we have an embedding ϕ of $\mathcal{D}$ into $E^3 \times E^0_+$ with $\phi^{-1}(0) = G$. Hence $\mathcal{C}_2(S(G))$ can be embedded into E^4. □

From Theorem 2 it follows

Lemma 2. *Any proper minor of K_7 or $K_{3,3,1,1}$ is 4-flat.*

Proof. *Case K_7*: If G arises from K_7 by deleting an edge, then it is the suspension of a flat graph, and hence it is 4-flat. If G arises from K_7 by contracting an edge, then by deleting a vertex we obtain a flat graph and hence G is 4-flat.

Case $K_{3,3,1,1}$: If G arises from $K_{3,3,1,1}$ by contracting an edge, then it is a suspension of a flat graph and hence is 4-flat. For the case where G arises from $K_{3,3,1,1}$ by deleting an edge, we distinguish two cases. Let v and w be the vertices of degree 7 in $K_{3,3,1,1}$. If G arises from $K_{3,3,1,1}$ by deleting an edge $e \neq vw$, then it is a subgraph of a suspension of a flat graph and hence is 4-flat. If G arises from $K_{3,3,1,1}$ by deleting edge vw, then it is a subgraph of a suspension of the graph obtained from $K_{3,4}$ by adding a circuit of size 4 to the color class of size 4 (see Fig. 1), which is flat, and hence G is 4-flat. □

In Section 7, we shall see that K_7 and $K_{3,3,1,1}$ are not 4-flat. Hence, by Lemma 2, these graphs are excluded minors for this class. However, we do not know if each graph in the Heawood family is an excluded minors for the class of 4-flat graphs. We make the following two conjectures.

Conjecture 1. *Any graph obtained from a 4-flat graph G by adding an edge in parallel to one of the edges of G is again 4-flat.*

Conjecture 2. *Any graph obtained from a* 4*-flat graph by a* ΔY*- or* $Y\Delta$*-transformation is again* 4*-flat.*

By Lemma 1, the truth of the Conjectures 1 and 2 implies that all graphs in the Heawood family are excluded minors for the class of 4-flat graphs.

4. The Colin de Verdière parameter

The Colin de Verdière parameter $\mu(G)$ was introduced in [2] (see [3] for the English translation). Its definition is in terms of matrices, but it turns out that it describes topological embeddability properties of the graph, as the following show:

- A graph G is planar if and only if $\mu(G) \leqslant 3$.
- A graph G is flat if and only if $\mu(G) \leqslant 4$.

Before giving the definition of $\mu(G)$ we need some other definitions. Let $G = (V, E)$ be a graph with n vertices and let O_G denote the collection of all symmetric $n \times n$ matrices $M = (m_{i,j})$ with $m_{i,j} < 0$ if $i \neq j$, and i and j are connected by an edge, and $m_{i,j} = 0$ if $i \neq j$, and i and j are not connected by an edge (so the entries on the diagonal may be any real number). A matrix $M \in O_G$ fulfills the Strong Arnol'd property if the only symmetric matrix $X = (x_{i,j})$ with $x_{i,j} = 0$ if $i = j$ or if i and j are adjacent, and satisfying $MX = 0$, is the all-zero matrix. The parameter $\mu(G)$ is defined as the largest corank of any $M \in O_G$, with exactly one negative eigenvalue, that fulfills the Strong Arnol'd Property. (The corank of M is $n - \operatorname{rank} M$.)

The parameter $\mu(G)$ is minor-monotone; that is, if G' is a minor of G, then $\mu(G') \leqslant \mu(G)$. Hence by the well-quasi-ordering theorem of Robertson and Seymour, the class of all graph G with $\mu(G) \leqslant k$ can be described in terms of a finite collection of excluded minors. For $k = 3$, the excluded minors are $K_{3,3}$ and K_5. For $k = 4$, the excluded minors are all graphs that can be obtained from K_6 by applying ΔY- and $Y\Delta$-transformations; that is, all graphs in the Petersen family. The reason that these graphs are excluded minors for $\mu(G) \leqslant 4$ follows from $\mu(K_6) = 5$ and the following theorems of Bacher and Colin de Verdière [1]. They state their theorems in a more general form; we do not need that here.

Theorem 3. *If G' is obtained from G by subdividing an edge, then $\mu(G') \geqslant \mu(G)$. If G is obtained from G' by suppressing a vertex of degree* 2 *and $\mu(G') \geqslant 4$, then $\mu(G) \geqslant \mu(G')$.*

Theorem 4. *If G' is obtained from G by a ΔY-transformation, then $\mu(G') \geqslant \mu(G)$. If G is obtained from G' by a $Y\Delta$-transformation and $\mu(G') \geqslant 5$, then $\mu(G) \geqslant \mu(G')$.*

It is shown by Lovász and Schrijver [9] that the graphs in the Petersen family are all excluded minors of the class $\mu(G) \leqslant 4$.

Another theorem we shall need is:

Theorem 5. *Let G' be the suspension of a graph G. Then $\mu(G') = \mu(G) + 1$ if G is not the complement of K_2.*

For more information and theorems on the Colin de Verdière parameter, we refer to [8]. We state here:

Theorem 6. *Each graph G in the Heawood family has $\mu(G) = 6$. Each proper minor H of such a graph G has $\mu(H) < 6$.*

Proof. Since $\mu(K_{3,3,1,1}) = 6$ and $\mu(K_7) = 6$, by the result of Bacher and Colin de Verdière, the graphs G of the Heawood family have $\mu(G) = 6$. To prove the second part of the theorem, it is, by Lemma 1, sufficient to show that each proper minor H of $K_{3,3,1,1}$ and K_7 has $\mu(H) < 6$. We leave the case of K_7 to the reader. The case where H is a obtained from $K_{3,3,1,1}$ by contracting one edge follows from the fact that after suppressing parallel edges H is isomorphic to a proper subgraph of K_7. The case where H is obtained from $K_{3,3,1,1}$ by deleting one edge follows the fact that H is a subgraph of a suspension of a flat graph. □

We do not know if the graphs in the Heawood family are all excluded minors for the class of graphs G with $\mu(G) \leqslant 5$. We conjecture that they are all excluded minors.

5. The classes $\mathcal{I}_2$, $\mathcal{I}_3$, and $\mathcal{I}_4$

For any nonnegative integer k, we denote by B^k the k-ball, and we denote by S^k the k-sphere. Let $\phi_1 : B^{k_1} \to E^n$ and $\phi_2 : B^{k_2} \to E^n$ be continuous mappings with $(\phi_1(\partial B^{k_1}) \cap \phi_2(B^{k_2})) \cup (\phi_1(B^{k_1}) \cap \phi_2(\partial B^{k_2})) = \emptyset$ and $k_1 + k_2 = n$. We say that ϕ_1 and ϕ_2 are *in general position* if $\phi_1(B^{k_1})$ and $\phi(B^{k_2})$ have a finite number of intersections and at these intersections they intersect transversely. If ϕ_1 and ϕ_2 are in general position, the *intersection number* mod 2 of ϕ_1 and ϕ_2, which we denote by $\mathrm{I}_2(\phi_1, \phi_2)$, is the equivalence class of $|\phi_1(B^{k_1}) \cap \phi(B^{k_2})|$ under congruence modulo two. If $\phi_1 : S^{k_1} \to E^n$ and $\phi_2 : S^{k_2} \to E^n$ are continuous mappings with $\phi_1(S^{k_1}) \cap \phi_2(S^{k_2}) = \emptyset$ and $k_1 + k_2 = n - 1$, we denote by $\mathrm{link}_2(\phi_1, \phi_2)$ the equivalence class of the linking number of ϕ_1 and ϕ_2 under congruence modulo two. (For the general definition of intersection and linking number, we refer to Dold [5, pp. 197–202]. For a definition as used in differential topology, see Hirsch [7].)

The intersection number and linking number are invariant under sufficiently small perturbations. That is, if $\phi_1 : B^{k_1} \to E^n$ and $\phi_2 : B^{k_2} \to E^n$ are continuous mappings in general position, with $(\phi_1(\partial B^{k_1}) \cap \phi_2(B^{k_2})) \cup (\phi_1(B^{k_1}) \cap \phi_2(\partial B^{k_2})) = \emptyset$ and $k_1 + k_2 = n$, and $\phi'_1 : B^{k_1} \to E^n$ and $\phi'_2 : B^{k_2} \to E^n$ are obtained from ϕ_1 and ϕ_2, respectively, by a small perturbation and ϕ'_1 and ϕ'_2 are in general position, then $\mathrm{I}_2(\phi'_1, \phi'_2) = \mathrm{I}_2(\phi_1, \phi_2)$. If $\phi_1 : S^{k_1} \to E^n$ and $\phi_2 : S^{k_2} \to E^n$ are continuous mappings with $\phi_1(S^{k_1}) \cap \phi_2(S^{k_2}) = \emptyset$ and $k_1 + k_2 = n - 1$, and $\phi'_1 : S^{k_1} \to E^n$ and $\phi'_2 : S^{k_2} \to E^n$ are obtained from ϕ_1 and ϕ_2, respectively, by a small perturbation, then $\mathrm{link}_2(\phi'_1, \phi'_2) = \mathrm{link}_2(\phi_1, \phi_2)$.

Let $\mathcal{C}$ be a regular CW-complex and let $\phi : \mathcal{C} \to E^n$ be a continuous map. We say that ϕ is *in general position* if for each pair of open cells σ_1, σ_2 of $\mathcal{C}$ with $\sigma_1 \neq \sigma_2$ and $\dim \sigma_1 + \dim \sigma_2 < n$, $\phi(\sigma_1) \cap \phi(\sigma_2) = \emptyset$, and for each pair of open cells σ_1, σ_2 of $\mathcal{C}$ with $\sigma_1 \neq \sigma_2$ and $\dim \sigma_1 + \dim \sigma_2 = n$, $\phi(\sigma_1)$ and $\phi(\sigma_2)$ have a finite number of intersections and at these intersections they intersect transversely. Let $\phi : \mathcal{C} \to E^n$ be a continuous map in general position. For nonadjacent cells σ_1, σ_2 of $\mathcal{C}$ with $\dim \sigma_1 + \dim \sigma_2 = n$, $\mathrm{I}_2(\phi(\sigma_1), \phi(\sigma_2)) = 1$ if and only if the intersection number of $\phi(\sigma_1)$ and $\phi(\sigma_2)$ is odd. The following equality holds for nonadjacent cells σ_1, σ_2 of $\mathcal{C}$ with $\dim \sigma_1 + \dim \sigma_2 = n + 1$:

$$\mathrm{link}_2(\phi(\partial\sigma_1), \phi(\partial\sigma_2)) = \mathrm{I}_2(\phi(\partial\sigma_1), \phi(\sigma_2)) = \sum_{\tau} \mathrm{I}_2(\phi(\tau), \phi(\sigma_2)),$$

where the sum is over all cells τ with $\dim \tau = \dim \sigma_1 - 1$ belonging to the boundary of σ_1.

We now introduce weakenings of the classes of planar, flat, and 4-flat graphs. The class $\mathcal{I}_2$ is the class of all graphs G such that there exists a mapping ϕ in general position of G into 2-space, such that $\mathrm{I}_2(\phi(e_1), \phi(e_2)) = 0$ for every pair of nonadjacent edges e_1, e_2 of G. The class $\mathcal{I}_3$ is the class of graphs G, such that there exists a mapping ϕ in general position of $\mathcal{C}_2(G)$ into 3-space such that $\mathrm{I}_2(\phi(\sigma), \phi(e)) = 0$ for every edge e and 2-cell σ of $\mathcal{C}_2(G)$ with σ nonadjacent to e. The class $\mathcal{I}_4$ is the class of graphs G, such that there exists a mapping ϕ in general position of $\mathcal{C}_2(G)$ into 4-space such that $\mathrm{I}_2(\phi(\sigma), \phi(\tau)) = 0$ for every pair of nonadjacent 2-cells σ, τ of $\mathcal{C}_2(G)$.

It is clear that planar graphs belong to $\mathcal{I}_2$, that flat graphs belong to $\mathcal{I}_3$, and that 4-flat graphs belong to $\mathcal{I}_4$.

Lemma 3. *Let $k \in \{2, 3, 4\}$. If a graph belongs to $\mathcal{I}_k$, then each of its subgraphs belongs to $\mathcal{I}_k$.*

Lemma 4. *Let $k \in \{2, 3, 4\}$. If G is a subdivision of a graph in $\mathcal{I}_k$, then G belongs to $\mathcal{I}_k$. If G is a obtained from a graph in $\mathcal{I}_k$ by adding parallel edges, then G belongs to $\mathcal{I}_k$.*

The proofs of these lemmas are easy.

Proposition 5. *If a graph belongs to $\mathcal{I}_2$, then each of its minors belongs to $\mathcal{I}_2$.*

Proof. Let G be a graph belonging to $\mathcal{I}_2$. To prove the proposition, it suffices to show that if G' arises from G by deleting an edge or a vertex, or by contracting an edge e, then it belongs to $\mathcal{I}_2$. From Lemma 3 it follows that G' belongs to $\mathcal{I}_2$ if G' arises from G by deleting an edge or a vertex. We shall now consider the case where G' is obtained from G by contracting an edge e.

Let ϕ be a mapping in general position of G into 2-space, such that $\mathrm{I}_2(\phi(g), \phi(h)) = 0$ for every pair of nonadjacent edges h, g of G.

We may assume that $\phi(e)$ has no self-intersections. For, if this is the case then we do the following. Let v be one of the ends of e, and take the nearest self-intersection p of $\phi(e)$ when going along $\phi(e)$ from $\phi(v)$ to $\phi(w)$, where w is the other end of e. Take a small neighborhood around p and let P_1 and P_2 be the restriction $\phi(e)$ in this neighborhood. If the neighborhood is sufficiently small, P_1 and P_2 intersect in p only. We assume that P_1 is the nearest part to $\phi(v)$ when going along $\phi(e)$ from $\phi(v)$ to the other end of $\phi(e)$. Let C be the restriction of $\phi(e)$ between $\phi(v)$ and p. Take in an small neighborhood of C a 1-sphere disjoint from C. We may assume that this 1-sphere intersects P_2 in just two points. Delete the part of P_2 inside these two points and replace it by the part of the 1-sphere that encloses $\phi(v)$. Denote the new map by ϕ'. Then $\mathrm{I}_2(\phi'(e), \phi'(j)) = 0$ for every edge j nonadjacent to e, and we have removed the self-intersection p. Repeating this for every self-intersection, we obtain a new mapping ψ of G into 2-space, in which $\psi(e)$ has no self-intersection.

Furthermore, we may assume that $\phi(j)$ is disjoint from $\phi(e)$ for each edge $j \neq e$. For, if this is not the case, then we do the following. Let v be one of the ends of e, take the nearest intersection p of $\phi(e)$ with the image of an edge j, when going along $\phi(e)$ from $\phi(v)$ to the other end of $\phi(e)$. Let C be the part of $\phi(e)$ between $\phi(v)$ and p, and take in a small neighborhood of C a 1-sphere around C; we may assume that this 1-sphere does not intersect the images of any edges of G not adjacent to v, and that it intersects $\phi(j)$ in just two points. Delete the part of $\phi(j)$ inside these two points and replace it by the part of the 1-sphere that encloses $\phi(v)$. Repeating this for all intersection points, yields a new mapping ϕ' of G into 2-space, with $\mathrm{I}_2(\phi'(g), \phi'(h)) = 0$ for every pair of nonadjacent edges g, h of G, but such that $\phi'(e)$ is disjoint from $\phi'(j)$ for any edge j.

Contracting $\phi(e)$ in 2-space now gives a mapping ϕ' in general position of G' into 2-space, such that for every pair of nonadjacent edges g, h of G', $\mathrm{I}_2(\phi'(g), \phi'(h)) = 0$. □

Proposition 6. *If a graph belongs to $\mathcal{I}_3$, then each of its minors belongs to $\mathcal{I}_3$.*

Proof. Let G be a graph belonging to $\mathcal{I}_3$. To prove the proposition, it suffices to show that if G' arises from G by deleting an edge or a vertex, or by contracting an edge e, then it belongs to $\mathcal{I}_3$. From Lemma 3 the case follows where G' arises from G by deleting an edge or a vertex. So it remains to consider the case in which G' arises from G by contracting an edge e.

Let ϕ be a mapping in general position of $\mathcal{C}_2(G)$ into 3-space, such that $\mathrm{I}_2(\phi(\sigma), \phi(g)) = 0$ for every pair of nonadjacent 2-cell σ and edge g of $\mathcal{C}_2(G)$. We may assume that $\phi(e)$ has no points in common with the images of every 2-cell τ not incident to e. For, if this is not the case, then we do the following. Let v be one of the ends of e, take the nearest intersection p of a 2-cell τ not incident to e with $\phi(e)$ when going along $\phi(e)$ from $\phi(v)$ to the other end of $\phi(e)$. Take in a small neighborhood of the restriction, l, of $\phi(e)$ between $\phi(v)$ and p, a 2-sphere around l; we may assume that this sphere does not intersect the images of any edges of G not adjacent to v, and that it intersects $\phi(\tau)$ in a circle. Delete the part of $\phi(\tau)$ inside the circle and replace it by the part of the sphere that encloses $\phi(v)$. Repeating this for all intersection points, yields a new mapping ϕ' of $\mathcal{C}_2(G)$ into 3-space, with $\mathrm{I}_2(\phi'(\sigma), \phi'(g)) = 0$ for every pair of nonadjacent 2-cell σ and edge g of $\mathcal{C}_2(G)$, but for which $\phi(e)$ is disjoint from $\phi(\tau)$ for any 2-cell τ nonadjacent from e.

Now let $\mathcal{D}$ be the subcomplex of $\mathcal{C}_2(G)$ obtained by deleting all cells incident to both ends of e but not e itself. Then the restriction of ϕ to $\mathcal{D}$ is a mapping in general position, such that $\mathrm{I}_2(\phi(\sigma), \phi(e)) = 0$ for every nonadjacent 2-cell σ and edge g of $\mathcal{D}$. Since for each 2-cell σ not containing both ends of e, $\phi(\sigma)$ is disjoint from $\phi(e)$, contracting $\phi(e)$ in 3-space gives a mapping ϕ' in general position of $\mathcal{C}_2(G')$ into 3-space such that for each nonadjacent 2-cell σ and edge g of $\mathcal{C}_2(G')$, $\mathrm{I}_2(\phi'(\sigma), \phi'(g)) = 0$. □

Proposition 7. *If a graph belongs to $\mathcal{I}_4$, then each of its minors belongs to $\mathcal{I}_4$.*

The proof goes along the same lines as the proof of Proposition 6.

Hence by the well-quasi-ordering theorem of Robertson and Seymour, the class $\mathcal{I}_k$, for $k = 2, 3, 4$, can be described by a finite collection of excluded minors. In the next section, we shall give all excluded minors of the classes $\mathcal{I}_2$ and $\mathcal{I}_3$; it will turn out that $\mathcal{I}_2$ equals the class of planar graphs and that $\mathcal{I}_3$ equals the class of flat graphs. In Section 7, we shall give some excluded minors of the class $\mathcal{I}_4$. However, we do not know whether these are all excluded minors.

Theorem 7. *Let $k \in \{3, 4\}$. Let $G = (V, E)$ be a graph and let $G' = (V', E')$ be obtained from G by a $Y\Delta$-transformation. If G belongs to $\mathcal{I}_k$, then G' belongs to $\mathcal{I}_k$.*

Proof. We consider only the $k = 3$ here, the case $k = 4$ can be done similarly.

Let v be the vertex of degree 3 on which we apply the $Y\Delta$-transformation, let e_1, e_2, e_3 be the edges incident to v, and let w_1, w_2, w_3 the endpoints of e_1, e_2, e_3 different from v. We shall denote the edges of the Δ by f_1, f_2, f_3, where f_i $(i = 1, 2, 3)$ connects w_{i+1} and w_{i+2} (indices read modulo three).

Since G belongs to $\mathcal{I}_3$, there exists a mapping ϕ in general position of $\mathcal{C}_2(G)$ into 3-space, such that $\mathrm{I}_2(\phi(\sigma), \phi(e)) = 0$ for each pair σ, e of nonadjacent 2-cell σ and edge e of $\mathcal{C}_2(G)$. For $i = 1, 2, 3$, take a curve d_i in 3-space, near $\phi(e_{i+1})$ and $\phi(e_{i+2})$, disjoint from $\phi(e_i)$ and disjoint from $\phi(\sigma)$ for any 2-cell σ that is nonadjacent to e_{i+1} and e_{i+2}. We shall map f_i to d_i for $i = 1, 2, 3$. Let τ_i ($i = 1, 2, 3$) be a 2-cell bounded by f_i, e_{i+1}, e_{i+2}, and map τ_i into 3-space so that it is bounded by d_i, $\phi(e_{i+1})$, and $\phi(e_{i+2})$, and that it is disjoint from $\phi(e)$ for any edge e nonadjacent to e_{i+1} and e_{i+2}; denote this mapping by $\psi(\tau_i)$.

We now define a mapping ϕ' of $\mathcal{C}_2(G')$ into 3-space. On $G \setminus \{e_1, e_2, e_3\}$ define $\phi' = \phi$. Define $\phi'(f_i) = d_i$ ($i = 1, 2, 3$). For each 2-cell σ_C of $\mathcal{C}_2(G')$ incident to at most one vertex of w_1, w_2, w_3, define $\phi'(\sigma_C) = \phi(\sigma_C)$. Define $\phi'(\sigma_\Delta) = \psi(\tau_1) \cup \psi(\tau_2) \cup \psi(\tau_3)$. For each 2-cell σ_C of $\mathcal{C}_2(G')$ incident to exactly one edge of f_1, f_2, f_3, say f_i, let C' be the circuit of G obtained from C by deleting the edge f_i from C and adding the edges e_{i+1} and e_{i+2}; define $\phi'(\sigma_C) = \phi(\sigma_{C'}) \cup \psi(\tau_i)$. For each 2-cell σ_C of $\mathcal{C}_2(G')$ incident to exactly two edges of f_1, f_2, f_3, say f_i, f_{i+1}, let C' be the circuit of G obtained from C by deleting the edges f_i, f_{i+1} and adding the edges e_i and e_{i+1}; define $\phi'(\sigma_C) = \phi(\sigma_{C'}) \cup \psi(\tau_i) \cup \psi(\tau_{i+1})$. Apply a small perturbation to put ϕ' in general position.

We claim that $\mathrm{I}_2(\phi'(\sigma), \phi'(g)) = 0$ for each pair of nonadjacent 2-cell σ and edge g of $\mathcal{C}_2(G')$. To see this we consider several cases. Let σ, g be a pair of nonadjacent 2-cell and edge of $\mathcal{C}_2(G')$. If g is one of the edges f_1, f_2, f_3, say $g = f_i$, then, as $\phi'(f_i)$ is near $\phi'(e_{i+1})$ and $\phi'(e_{i+2})$, $\mathrm{I}_2(\phi'(\sigma), \phi'(f_i)) = \mathrm{I}_2(\phi'(\sigma), \phi'(e_{i+1})) + \mathrm{I}_2(\phi'(\sigma), \phi'(e_{i+2})) = 0$. So we may assume that g is not equal to one of the edges f_1, f_2, f_3. If σ is incident to at most vertex of w_1, w_2, w_3, then clearly $\mathrm{I}_2(\phi'(\sigma), \phi'(g)) = 0$. If $\sigma = \sigma_\Delta$, then $\mathrm{I}_2(\phi'(\sigma), g) = \sum_{i=1}^{3} \mathrm{I}_2(\psi(\tau_i), g) = 0$. If $\sigma = \sigma_C$ is incident to exactly one edge of f_1, f_2, f_3, say f_i, then $\mathrm{I}_2(\phi'(\sigma_C), g) = \mathrm{I}_2(\phi(\sigma_{C'}), g) + \mathrm{I}_2(\psi(\tau_i), g) = 0$, where C' is the circuit of G obtained from C by deleting the edge f_i from C and adding the edges e_{i+1}, e_{i+2}. If $\sigma = \sigma_C$ is incident to exactly two edges of f_1, f_2, f_3, say f_i, f_{i+1}, then $\mathrm{I}_2(\phi'(\sigma_C), g) = \mathrm{I}_2(\phi(\sigma_{C'}), g) + \mathrm{I}_2(\psi(\tau_i), g) + \mathrm{I}_2(\psi(\tau_{i+1}), g) = 0$, where C' is the circuit of G obtained from C by deleting the edges f_i, f_{i+1} from C and adding the edges e_i and e_{i+1}. □

Theorem 8. *Let $k \in \{3, 4\}$. Let G be a graph and let G' be obtained from G by a ΔY-transformation. If G belongs to $\mathcal{I}_k$, then G' belongs to $\mathcal{I}_k$.*

Proof. We consider only the case $k = 4$ here, the case $k = 3$ can be done similarly. For convenience we set $\mathcal{C} := \mathcal{C}_2(G)$ and $\mathcal{C}' := \mathcal{C}_2(G')$. By Δ we denote the circuit bounding the triangle on which we apply the ΔY-transformation. Let the vertices of Δ be w_1, w_2, w_3 and let the edges of Δ be f_1, f_2, f_3, where f_i has ends w_{i+1} and w_{i+2} (indices read modulo 3). Remember that, for each circuit C, we denote by σ_C the 2-cell of $\mathcal{C}_2(G)$ bounded by C. So σ_Δ denotes the 2-cell of $\mathcal{C}_2(G)$ bounded by Δ.

Since G belongs to $\mathcal{I}_4$, there is a mapping ϕ in general position of $\mathcal{C}$ into 4-space such that $\mathrm{I}_2(\phi(\sigma), \phi(\tau)) = 0$ for each pair σ, τ of nonadjacent 2-cells of $\mathcal{C}$.

$$\text{We may assume that } \mathrm{I}_2(\phi(\sigma_\Delta), \phi(\sigma)) = 0 \text{ for every 2-cell } \sigma \neq \sigma_\Delta \text{ incident to } w_1. \tag{1}$$

We shall map $\mathcal{C}$ in 4-space such that (1) holds. Let P be the set of all intersection points of σ_Δ with cells σ that have only w_1 in common with σ_Δ. Let c be a simple curve in σ_Δ which starts in w_1 and ends in a point of P and which traverses all points in P. Let d be a simple curve which starts in w_1 and ends in w_3, and which goes along c to the last point c, then goes back to a point near w_1, and then goes along f_2 to w_3; see Fig. 2.

Map σ_Δ to the disc in $\phi(\sigma_\Delta)$ bounded by $\phi(f_1), \phi(f_3), d$. Map each 2-cell $\tau \neq \sigma_\Delta$ incident to f_2 to the union of $\phi(\tau)$ and the disc in $\phi(\sigma_\Delta)$ bounded by $\phi(f_2)$ and d. The mapping ϕ' defined this way still satisfies $\mathrm{I}_2(\phi'(\sigma), \phi'(\tau)) = 0$ for each pair σ, τ of nonadjacent 2-cells of $\mathcal{C}$.

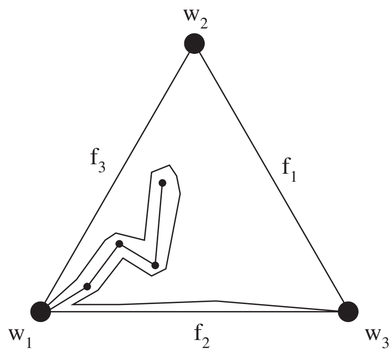

Fig. 2. Making ϕ satisfy (1).

Let d_1 be a simple curve in σ_Δ connecting w_2 and w_3, near f_2 and f_3 but disjoint from them, so that the ϕ-image of the part of σ_Δ bounded by d_1, f_2, f_3 is disjoint from $\phi(\sigma)$ for any 2-cell with $\sigma \neq \sigma_\Delta$. Choose a point v on d_1 different from w_2 and w_3, let e_2 be the part of d_1 between v and w_2, and let e_3 be the part of d_1 between v and w_3. Let e_1 be a curve connecting v and w_1 in the part of σ_Δ bounded by d_1, f_2, f_3, openly disjoint from f_2, f_3, e_2, e_3. Let τ_i $(i = 1, 2, 3)$ be the part of σ_Δ bounded by f_i, e_{i+1}, e_{i+2}.

The mapping ϕ induces a map ϕ' of G' into 4-space. We extend this map to all 2-cells so that $\mathrm{I}_2(\phi'(\sigma), \phi'(\tau)) = 0$ for each pair σ, τ of 2-cells of $\mathcal{C}'$ as follows. Each circuit C' of G' not containing v is also a circuit of G. We define $\phi'(\sigma_{C'}) = \phi(\sigma_{C'})$ for these circuits C'. For each circuit C' of G' containing v, let C_1 be the circuit obtained from C' by deleting the edges $e_i = w_i v$, $e_{i+1} = w_{i+1} v$ of C' incident to v and adding the edge f_{i+2}. Then C_1 is a circuit of G. We define $\phi'(\sigma_{C'}) = \phi(\sigma_{C_1}) \cup \phi(\tau_{i+2})$. Apply a small perturbation to put ϕ' into general position.

We need to show that, for each two nonadjacent 2-cells σ, τ of $\mathcal{C}'$, $\mathrm{I}_2(\phi'(\sigma), \phi'(\tau)) = 0$. This is clear if v does not belong to σ and τ, so we assume that at least one of σ, τ contains v; say σ contains v. Let C' be the circuit bounded by σ; then C' contains v. Let e_i and e_{i+1} be the edges of C' incident to v, and let C_1 be the circuit obtained by deleting v and adding edge f_{i+2}. Since $\phi'(\sigma_{C'})$ is close to $\phi(\sigma_{C_1}) \cup \phi(\tau_{i+2})$ and $\phi'(\tau) = \phi(\tau)$, we have $\mathrm{I}_2(\phi'(\sigma), \phi'(\tau)) = \mathrm{I}_2(\phi(\sigma_{C_1}), \phi(\tau)) + \mathrm{I}_2(\phi(\tau_{i+2}), \phi(\tau)) = 0$. □

Hence, if G is an excluded minor for $\mathcal{I}_4$, then any graph obtained from G by a series of ΔY- and $Y\Delta$-transformations will also be an excluded minor for $\mathcal{I}_4$.

6. Obstruction to embeddability

Let $\mathcal{C}$ be a finite regular cell complex (for example a simple graph or $\mathcal{C}_2(G)$ of a simple graph G). If n is a nonnegative integer, we denote by $P(\mathcal{C})_n$ the collection of all unordered nonadjacent pairs $\{\sigma_1, \sigma_2\}$ of cells in $\mathcal{C}$ with $\dim \sigma_1 + \dim \sigma_2 = n$. We say that $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_n$ is *incident* to $\{\tau_1, \tau_2\} \in P(\mathcal{C})_{n-1}$ if one of the following holds:

1. $\sigma_1 = \tau_1$ and σ_2 is incident to τ_2,
2. $\sigma_1 = \tau_2$ and σ_2 is incident to τ_1,

3. $\sigma_2 = \tau_2$ and σ_1 is incident to τ_1, or
4. $\sigma_2 = \tau_1$ and σ_1 is incident to τ_2.

Let $M = (m_{i,j})$ be the $P(\mathcal{C})_{n-1} \times P(\mathcal{C})_n$ matrix, with entries in $\mathbb{Z}_2$, defined by

$$m_{i,j} = \begin{cases} 0 & \text{if } j \text{ is not incident to } i, \text{ and} \\ 1 & \text{if } j \text{ is incident to } i. \end{cases} \tag{2}$$

For a mapping in general position ϕ of $\mathcal{C}$ into n-space, let $y(\phi) = y \in \mathbb{Z}_2^{P(\mathcal{C})_n}$ be the row vector with $y_{\{\sigma_1,\sigma_2\}} = \mathrm{I}_2(\phi(\sigma_1), \phi(\sigma_2))$ for $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_n$. Now, if ψ is any other mapping in general position, of $\mathcal{C}$ into n-space, then $y(\psi) - y(\phi)$ belongs to the row space of M. To see this informally, deform ϕ to ψ; we assume that the deformation is in general position. We can split the deformation into a series of small deformations where each such a small deformation is either a deformation in which new intersection points of the image of a cell σ_1 with the image of another cell σ_2 with $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_n$ appear, or a deformation in which the image of a cell σ_1 moves through the image of a cell σ_2 with $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_{n-1}$. (If none of these small deformations occur, then evidently $y(\phi) = y(\psi)$.) If new intersection points of the image of a cell σ_1 with the image of another cell σ_2 with $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_n$ appear, then an even number of new intersection points appear, and hence if ψ_1 is the mapping before the small deformation and ψ_2 is the mapping after the small deformation, then $y(\psi) = y(\phi)$. If the image of a cell σ_1 moves through the image of cell σ_2 with $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_{n-1}$, then $y(\psi_1) = y(\psi_2) + n_{\{\sigma_1,\sigma_2\}}$, where $m_{\{\sigma_1,\sigma_2\}}$ is the $\{\sigma_1, \sigma_2\}$th row of M, if ψ_1 is the mapping before the small deformation and ψ_2 is the mapping after the small deformation.

If there is a mapping ϕ of $\mathcal{C}$ into n-space, such that $\mathrm{I}_2(\phi(\sigma_1), \phi(\sigma_2)) = 0$ for every $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_n$, then $y(\phi) = 0$. Hence, for any other mapping ψ in general position of $\mathcal{C}$ into n-space, $y(\psi)$ belongs to the row space of N. So, if we can show that $y(\psi)$ does not belong to the row space of N for a mapping ψ, then there is no mapping ϕ of $\mathcal{C}$ into n-space with $\mathrm{I}_2(\phi(\sigma_1), \phi(\sigma_2)) = 0$ for each $\{\sigma_1, \sigma_2\} \in P(\mathcal{C})_n$.

This can be given more flavor of algebraic topology; see [12–16], and see any book on algebraic topology for the definition of cycles, cocycles, etc. Let $\mathcal{C}$ be a finite regular cell complex. The *deleted product* $\mathcal{C}^*$ of $\mathcal{C}$ is defined as the subcomplex of $\mathcal{C} \times \mathcal{C}$ consisting of all cells $\sigma \times \tau$ with σ and τ nonadjacent. On $\mathcal{C}^*$ we put an antipodal map T defined by $T(x, y) = (y, x)$, $(x, y) \in \mathcal{C}^*$. The complex $\overline{\mathcal{C}^*}$ is obtained from $\mathcal{C}^*$ by identifying (x, y) with $(y, x) = T(x, y)$ for each $(x, y) \in \mathcal{C}^*$. By $\sigma \star \tau$ we denote the image of $\sigma \times \tau$ after identification of (x, y) with (y, x) for all $(x, y) \in \mathcal{C}^*$.

For a mapping ϕ of $\mathcal{C}$ into n-space, define the n-cochain $\vartheta[\phi]$ by $\vartheta[\phi](\sigma \star \tau) := \mathrm{I}_2(\phi(\sigma), \phi(\tau))$ for each n-cell $\sigma \star \tau$ of $\overline{\mathcal{C}^*}$; this is well-defined as $\mathrm{I}_2(\phi(\sigma), \phi(\tau)) = \mathrm{I}_2(\phi(\tau), \phi(\sigma))$. The n-cochain $\vartheta[\phi]$ is a n-cocycle since, for any $(n+1)$-cell $\sigma_1 \star \sigma_2$ of $\overline{\mathcal{C}^*}$:

$$\begin{aligned} \delta\vartheta[\phi](\sigma_1 \star \sigma_2) &= \vartheta[\phi](\partial\sigma_1 \star \sigma_2 + \sigma_1 \star \partial\sigma_2) \\ &= \mathrm{link}_2(\phi(\partial\sigma_1), \phi(\partial\sigma_2)) + \mathrm{link}_2(\phi(\partial\sigma_1), \phi(\partial\sigma_2)) \\ &= 0. \end{aligned}$$

If ψ is another mapping of $\mathcal{C}$ into n-space, then $\vartheta[\psi] - \vartheta[\phi]$ is equal to the coboundary of a $(n-1)$-cochain, and hence $\vartheta[\psi]$ and $\vartheta[\phi]$ belong to the same cohomology class of $H^n(\overline{\mathcal{C}^*}, \mathbb{Z}_2)$. We shall denote this class by $\vartheta^n_{\mathcal{C}}$.

If G belongs to $\mathcal{I}_2$, then there is a mapping ϕ of G into 2-space such that $\mathrm{I}_2(\phi(g), \phi(h)) = 0$ for every pair of nonadjacent edges $\{g, h\}$ of G. Hence $\vartheta^2_G = \vartheta[\phi] = 0$. Thus, $\vartheta^2_G \neq 0$ implies that G does not belong to $\mathcal{I}_2$. Now $\vartheta^2_G \neq 0$ if and only if there is a 2-cycle d of $\overline{G^*}$, such that

$\vartheta^2_G(d) = 1$. Hence the existence of a 2-cycle d, such that $\vartheta^2_G(d) = 1$ implies that G does not belong to $\mathcal{I}_2$. Conversely, if $\vartheta^2_G = 0$, then there exists a mapping ϕ of G into 2-space such that $\vartheta[\phi] = 0$; that is, $\mathrm{I}_2(\phi(g), \phi(h)) = 0$ for every pair of nonadjacent edges $\{g, h\}$ of G (see [14, Theorem 7]).

The same argument can be used for $\mathcal{I}_3$ and $\mathcal{I}_4$. These are the classes of graphs G with $\vartheta^3_{\mathcal{C}_2(G)} = 0$ and $\vartheta^4_{\mathcal{C}_2(G)} = 0$, respectively.

The graphs in $\mathcal{I}_2$ are easy to describe.

Lemma 8. *The graphs K_5 and $K_{3,3}$ do not belong to $\mathcal{I}_2$.*

Proof. We shall show this only for $K_{3,3}$, the proof for K_5 is analogous. Let $d = \sum e \star f$ be the 2-chain of $\overline{K^*_{3,3}}$ where the sum is over all unordered pairs of nonadjacent edge e, f of $K_{3,3}$. It is easy to see that d is a 2-cycle of $\overline{K^*_{3,3}}$. Since there is a mapping of $K_{3,3}$ into the 2-space which has exactly one unordered pair of nonadjacent edges with odd intersection number, we see that $\vartheta^2_{K_{3,3}}(d) = 1$. □

Theorem 9. *A graph belongs to $\mathcal{I}_2$ if and only if it is planar.*

Proof. If a graph is planar, then evidently it belongs to $\mathcal{I}_2$. For the converse, let G be a nonplanar graph. Then G has a subgraph homomorphic to K_5 or $K_{3,3}$. By Lemmas 3, 4 and 8, G does not belong to $\mathcal{I}_2$. □

Proposition 9. *A graph belongs to $\mathcal{I}_2$ if and only if its suspension belongs to $\mathcal{I}_3$.*

Proof. Let G be a graph in $\mathcal{I}_2$. Then, by Theorem 9, G is planar. Since the suspension of a planar graph is flat and any flat graph belongs to $\mathcal{I}_3$, we have proved one direction.

Conversely, let G be a graph not belonging to $\mathcal{I}_2$; let $S(G)$ be its suspension and v the suspended vertex. Then there is a 2-cycle $d = \sum e \star f$ of $\overline{G^*}$ such that $\vartheta^2_G(d) = 1$. For each edge e of G, denote by σ_e the 2-cell of $\mathcal{C}_2(S(G))$ whose boundary is the triangle formed by e and v. Let d' be the 3-chain $\sum(\sigma_e \star f + e \star \sigma_f)$, where the sum is over all unordered pairs $\{e, f\}$, such that $e \star f$ has nonzero coefficient in d. Then

$$\partial d' = \sum(\sigma_e \star \partial f + \partial e \star \sigma_f),$$

which is equal to zero since d is a 2-cycle. Hence d' is a 3-cycle. Let ϕ be a mapping in general position of G into 2-space, where we view the 2-space as $E^2 \times \{0\}$, and let ϕ' be a mapping in general position of $\mathcal{C}_2(S(G))$ into 3-space such that the suspended vertex of $S(G)$ is mapped into $E^2 \times E_+$, $V(G)$ is embedded into $E^2 \times \{0\}$, the interior of each edge of G is mapped into $E^2 \times E_-$, such that the projection of ϕ' to the plane $E^2 \times \{0\}$ is ϕ. Each edge e incident with the suspended vertex v is mapped on the line segment between $\phi(v)$ and $\phi(w)$, where w is the other end of e. The union of all line segments between $\phi'(p)$ and $\phi(p)$, and between $\phi(v)$ and $\phi(p)$, for all points p of e, forms a 2-disc onto which we map σ_e. Since $\mathrm{I}_2(\phi(e), \phi(f)) = \mathrm{I}_2(\phi'(e), \phi'(\sigma_f)) + \mathrm{I}_2(\phi'(\sigma_e), \phi'(f))$, we have $\vartheta[\phi'](d') = \vartheta[\phi](d) = 1$. Hence $S(G)$ does not belong to $\mathcal{I}_3$. □

Proposition 10. *None of the graphs in the Petersen family belongs to $\mathcal{I}_3$.*

Proof. Since each graph in the Petersen family can obtained from K_6 by a series of ΔY- or $Y\Delta$-transformations, it suffices to show that K_6 does not belong to $\mathcal{I}_3$, by Theorems 7 and 8. Now, this follows from Proposition 9. □

It is possible to prove that the suspension of a graph in $\mathcal{I}_2$ belongs to $\mathcal{I}_3$ without using Theorem 9. It is then interesting to notice that, from the facts that K_5 does not belong to $\mathcal{I}_2$ (so K_6 does not belong to $\mathcal{I}_3$) and that $K_{3,3,1}$ belongs to the Petersen family, we can deduce that $K_{3,3}$ does not belong to $\mathcal{I}_2$.

Theorem 10. *A graph belongs to $\mathcal{I}_3$ if and only if it is flat.*

Proof. If a graph is flat then, evidently, it belongs to $\mathcal{I}_3$. For the converse, use Propositions 6 and 10 to show that a graph does not belong to $\mathcal{I}_3$ if it is not flat. □

7. Some excluded minors for $\mathcal{I}_4$

A collection $\mathcal{D}$ of pairs of disjoint circuits of G is *even* if for each pair of nonadjacent edges e, f of G, there is an even number of pairs $(C, D) \in \mathcal{D}$ with $e \in E(C)$ and $f \in E(D)$, or $e \in E(D)$ and $f \in E(C)$.

Lemma 11. *A collection $\mathcal{D}$ of pairs of disjoint circuits of G is even if and only if the number*

$$\sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\phi(C), \phi(D)) \tag{3}$$

is independent of the embedding ϕ of G in 3*-space.*

Proof. Let ϕ, ψ be embeddings of G into 3-space. There exists a series of embeddings $\phi = \phi_1, \ldots, \phi_n = \psi$, where ϕ_{i+1} is obtained from ϕ_i by moving an edge e through an edge f. The difference

$$\sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\phi_{i+1}(C), \phi_{i+1}(D)) - \sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\phi_i(C), \phi_i(D)) \tag{4}$$

is equal to the number of pairs $(C, D) \in \mathcal{D}$ with $e \in E(C)$ and $f \in E(D)$, or $e \in E(D)$ and $f \in E(C)$. Hence, if $\mathcal{D}$ is even, then

$$\sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\phi(C), \phi(D)) = \sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\psi(C), \psi(D))$$

and conversely, if (3) is independent of the embedding, then $\mathcal{D}$ is even. □

Lemma 12. *A graph G does not belong to $\mathcal{I}_3$ if and only if there is a collection $\mathcal{D}$ of pairs of disjoint circuits of G, such that*

$$\sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\phi(C), \phi(D)) = 1 \tag{5}$$

for every embedding ϕ of G into 3*-space.*

Proof. Let d be a 3-cycle of $\overline{\mathcal{C}_2(G)^*}$ for which $\vartheta^3_{\mathcal{C}_2(G)}(d) = 1$. Let ϕ be a mapping of $\mathcal{C}_2(G)$ into 3-space in general position.

For each 2-cell σ, let C_σ be the set of all edges f for which $\sigma \star f$ has nonzero coefficient in d. Since d is a 3-cycle of $\overline{\mathcal{C}_2(G)^*}$, we have that C_σ is a cycle of G. Furthermore, $C_\sigma \cap C_\tau \neq \emptyset$ implies $\sigma = \tau$. Let F be the set of all 2-cells σ for which there is an edge e, such that $\sigma \star e$ has nonzero coefficient in d. We can write

$$d = \sum_{\sigma \in F, e \in C_\sigma} \sigma \star e$$

and hence we have

$$\begin{aligned}\vartheta^3_{\mathcal{C}_2(G)}(d) &= \sum_{\sigma \in F, e \in C_\sigma} \mathrm{I}_2(\phi(\sigma), \phi(e)) \\ &= \sum_{\sigma \in F} \mathrm{link}_2(\phi(\partial\sigma), \phi(C_\sigma)).\end{aligned}$$

Since $\vartheta^3_{\mathcal{C}_2(G)}(d) = 1$, we have $\sum_{\sigma \in F} \mathrm{link}_2(\phi(\partial\sigma), \phi(C_\sigma)) = 1$, and because each cycle is a sum of circuits, we have proved one direction of the theorem.

Conversely, let $\mathcal{D}$ be a collection of pairs (C, D) of circuits of G, such that (5) holds for every embedding ϕ of G into 3-space. By Lemma 11, $\mathcal{D}$ is even. Let

$$d := \sum_{(C,D) \in \mathcal{D}} \sum_{e \in E(D)} \sigma_C \star e.$$

We can write

$$\begin{aligned}\sum_{(C,D) \in \mathcal{D}} \mathrm{link}_2(\phi(C), \phi(D)) &= \sum_{(C,D) \in \mathcal{D}} \sum_{e \in E(D)} \mathrm{I}_2(\phi(\sigma_C), \phi(e)) \\ &= \vartheta[\phi](d) \\ &= 1.\end{aligned}$$

Since

$$\partial d = \sum_{(C,D) \in \mathcal{D}} \sum_{f \in E(C), e \in E(D)} f \star e$$

and, since $\mathcal{D}$ is even, $\partial d = 0$. Hence d is a cycle. Hence, we can write (6) as $\vartheta^3_{\mathcal{C}_2(G)}(d) = 1$. □

Proposition 13. *A graph belongs to $\mathcal{I}_3$ if and only if its suspension belongs to $\mathcal{I}_4$.*

Proof. Let G be a graph belonging to $\mathcal{I}_3$. Then G is flat, by Theorem 10. By Theorem 2, its suspension $S(G)$ is 4-flat, which implies that $S(G)$ belongs to $\mathcal{I}_4$.

Conversely, let G be a graph not belonging to $\mathcal{I}_3$. By Lemma 12, there is a collection $\mathcal{D}$ of pairs of disjoint circuits (C, D) such that for every embedding ϕ of G into 3-space,

$$\sum_{(C,D) \in \mathcal{D}} \mathrm{link}_2(\phi(C), \phi(D)) = 1.$$

From Lemma 11, it follows that $\mathcal{D}$ is even.

Let v be the suspended vertex of the suspension $S(G)$ of G. For any e of G, we denote by σ_e the 2-cell of $\mathcal{C}_2(S(G))$ whose boundary is the triangle spanned by e and v.

Let

$$d := \sum_{(C,D)\in\mathcal{D}} \sigma_C \star \sigma_D + \sum_{e\in E(C),(C,D)\in\mathcal{D}} \sigma_e \star \sigma_D + \sum_{e\in E(D),(C,D)\in\mathcal{D}} \sigma_C \star \sigma_e.$$

Then

$$\partial d = \sum_{e\in E(C),(C,D)\in\mathcal{D}} \sigma_e \star \partial\sigma_D + \sum_{e\in E(D),(C,D)\in\mathcal{D}} \partial\sigma_C \star \sigma_e.$$

Because $\mathcal{D}$ is even, $\partial d = 0$, and hence d is a 4-cycle.

Let ψ be a mapping of $\mathcal{C}_2(S(G))$ into 4-space such that G is embedded into $E^3 \times \{0\}$, v and each edge connecting v to a vertex of G is mapped into $E^3 \times E_+$, the interior of each σ_e is mapped into $E^3 \times E_+$, and the interior of each 2-cell σ_C is mapped into $E^3 \times E_-$. Then

$$\vartheta[\phi](d) = \sum_{(C,D)\in\mathcal{D}} \mathrm{I}_2(\phi(\sigma_C), \phi(\sigma_D)) + \sum_{e\in E(C),(C,D)\in\mathcal{D}} \mathrm{I}_2(\phi(\sigma_e), \phi(\sigma_D)) + \sum_{e\in E(D),(C,D)\in\mathcal{D}} \mathrm{I}_2(\phi(\sigma_C), \phi(\sigma_e)).$$

Since $\phi(\sigma_e) \cap \phi(\sigma_D) = \emptyset$ for $e \in E(C), (C, D) \in \mathcal{D}$ and $\phi(\sigma_C) \cap \phi(\sigma_e) = \emptyset$ for $e \in E(D), (C, D) \in \mathcal{D}$, we get

$$\begin{aligned}\vartheta[\phi](d) &= \sum_{(C,D)\in\mathcal{D}} \mathrm{I}_2(\phi(\sigma_C), \phi(\sigma_D)) \\ &= \sum_{(C,D)\in\mathcal{D}} \mathrm{link}_2(\phi(C), \phi(D)) \\ &= 1.\end{aligned}$$

Hence $\vartheta^4_{\mathcal{C}_2(S(G))}(d) = \vartheta[\phi](d) = 1$. □

Corollary 13.1. *K_7 and $K_{3,3,1,1}$ do not belong to $\mathcal{I}_4$.*

From Theorems 7, 8 and Corollary 13.1 it follows

Corollary 13.2. *None of the graphs of the Heawood family belongs to $\mathcal{I}_4$.*

Corollary 13.3. *None of the graphs of the Heawood family is 4-flat.*

Lemma 14. *Every proper minor of a graph of the Heawood family belongs to $\mathcal{I}_4$.*

Proof. Since each proper minor of $K_{3,3,1,1}$ or K_7 is 4-flat, by Lemma 2, we have that each proper minor of $K_{3,3,1,1}$ or K_7 belongs to $\mathcal{I}_4$. By Lemma 1, the lemma follows. □

Corollary 14.1. *The graphs of the Heawood family are (some) excluded minors for $\mathcal{I}_4$.*

8. Conclusion

In Section 6, we saw that the class of planar graphs and the class of flat graphs coincide with $\mathcal{I}_2$ and $\mathcal{I}_3$, respectively. We conjecture that the graphs in $\mathcal{I}_4$ are exactly the 4-flat graphs.

Since each graph G in the Heawood family has $\mu(G) > 5$, we make the following conjecture.

Conjecture 3. *A graph G has $\mu(G) \leqslant 5$ if and only if it is 4-flat.*

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