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Boundary regularity for a family of overdetermined problems for the Helmholtz equation

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Abstract

If a nonconstant solution u of the Helmholtz equation exists on a bounded domain with u satisfying overdetermined boundary conditions (u and its normal derivative both required to be constant on the boundary), then under certain assumptions the boundary of the domain is proved to be real-analytic. Under weaker assumptions, if a real-analytic portion of the boundary has a real-analytic extension, then that extension must also be part of the boundary. Also, an explicit formula for u is given and a condition (which does not involve u) is given for a bounded domain to have such a solution u defined on it. Both of these last results involve acoustic single- and double-layer potentials.

Keywords: Boundary regularity; Overdetermined; Helmholtz equation; Acoustic single-layer potential; Acoustic double-layer potential; Schiffer problem; Pompeiu problem; Pompeiu conjecture

1. Introduction

Throughout this paper, let Ω denote a nonempty bounded open connected subset of $\mathbf{R}^n$, with $n \geqslant 2$. Let $a, b \in \mathbf{R}$ and $\lambda \in \mathbf{C}$ be constants. We consider solutions u of the *Helmholtz equation*

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$$\Delta u + \lambda u = 0 \quad \text{in } \Omega, \tag{1}$$

subject to the overdetermined boundary conditions

$$u = a \qquad \text{on } \partial\Omega, \quad \text{and} \tag{2}$$

$$\frac{\partial u}{\partial n} = b \quad \text{on } \partial\Omega, \tag{3}$$

where Δ is the Laplacian operator, $\partial\Omega$ is the boundary of Ω, and $\partial/\partial n$ takes the (exterior) normal derivative on $\partial\Omega$. This two-parameter family (a and b being the parameters) of overdetermined problems is that studied in Willms and Gladwell [27] and Willms et al. [26]. If $b = 0$ (and $a \neq 0$), we get as a special case *Schiffer's problem* (see [27] and Yau [28]), which was shown to be the same as the *Pompeiu problem*. (See [26–28], Agranovsky [1], Aviles [2], Berenstein [3], Brown and Kahane [4], Brown, Schreiber and Taylor [5], Ebenfelt [9–11], Garofalo and Segala [12], Kobayashi [15], Ramm [18], Williams [24,25], and Zalcman [29, 30].) On the other hand, if $a = 0$ (and $b \neq 0$), we get as a special case what is called "*Serrin's problem*" in [27] (see also Serrin [20] and Weinberger [23]) and (apparently incorrectly; see above) *Schiffer's conjecture* in Chatelain et al. [6].

For any domain Ω there are constant solutions u of (1)–(3) (necessarily with $b = 0$ and either $a = 0$ or $\lambda = 0$). If, however, a *nonconstant* solution u of (1)–(3) exists on a domain Ω (for some constants a, b and λ), that makes a very strong statement about Ω. In fact, the following striking conjecture is quite reasonable to make ([27], with modifications; for $b = 0$ this is the *Pompeiu conjecture*, see references cited for the *Pompeiu problem* above):

Conjecture. *Let Ω be a* Lipschitz domain (*i.e., every x in $\partial\Omega$ has a neighborhood U_x such that $\partial\Omega \cap U_x$ is, after a possible rotation of the coordinate system, the graph of a Lipschitz continuous function*). *Assume that $\mathbf{R}^n \setminus \Omega$ (i.e., the complement of Ω in $\mathbf{R}^n$) is connected. Assume that there is a* nonconstant *solution u of* (1)–(3) *for some constants a, b and λ. Then Ω must be an n-dimensional ball.*

(See Remark 5 after Theorem 1 to see why the Lipschitz domain generality is appropriate and how the boundary conditions (2) and (3) should be interpreted in that case. For the rest of this paper, outside of Remark 5, the boundary values of u and $\partial u/\partial n$ in (2) and (3) should be interpreted classically as continuous extensions to $\partial\Omega$ of the values of u and $\partial u/\partial n$ in Ω.)

Section 2 derives an explicit formula for u and also gives a characterization of any domain Ω having a nonconstant solution u of (1)–(3) defined on it. Both of these involve *acoustic single- and double-layer potentials*, defined in Section 2.

Section 3 proves two results (Theorems 2 and 3) about the regularity of $\partial\Omega$. While they fall far short of proving the above conjecture, they provide at least some evidence in its favor.

2. An explicit formula for u and other results

By Trèves [21, pp. 257–259], a rotation-symmetric fundamental solution of $\Delta + \lambda$ is $\gamma_\lambda(|x-y|)$, where

$$\gamma_\lambda(r) = \frac{1}{4}\left(\frac{\sqrt{\lambda}}{2\pi r}\right)^{\beta} N_\beta\left(\sqrt{\lambda} r\right) \quad \text{for } r > 0,$$

where $\beta = (n-2)/2$, $\sqrt{\lambda}$ is either square root of λ, and N_β is the Neumann function of order β.

Following Colton and Kress [7, p. 38 for $n = 3$ and pp. 63–66 for $n = 2$], we define the *acoustic single-layer potential* $S_\lambda(y)$ and the *acoustic double-layer potential* $D_\lambda(y)$ (both with *density* 1) by

$$S_\lambda(y) = \int_{\partial\Omega} \gamma_\lambda(|x-y|)\, ds(x), \quad \text{for } y \in \mathbf{R}^n \setminus \partial\Omega,$$

and

$$D_\lambda(y) = \int_{\partial\Omega} \frac{\partial}{\partial n_x} \gamma_\lambda(|x-y|)\, ds(x), \quad \text{for } y \in \mathbf{R}^n \setminus \partial\Omega,$$

where $ds(x)$ indicates that integration is done using surface measure on $\partial\Omega$ with respect to the x variables (for each fixed y) and where $\partial/\partial n_x$ takes the (exterior) normal derivative with respect to the x variables.

In the proof of the following theorem, we will use *Green's first identity*

$$\int_\Omega v\Delta u + \nabla u \cdot \nabla v\, dx = \int_{\partial\Omega} v \frac{\partial u}{\partial n}\, ds \tag{4}$$

and *Green's second identity*

$$\int_\Omega u\Delta v - v\Delta u\, dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, ds \tag{5}$$

(see [7, p. 16] and Miranda [16, pp. 12–14]).

Theorem 1. *Assume that Ω is of class C^1. Assume that u is a solution on Ω of* (1)–(3). *Then*:

(a) $\lambda \int_\Omega |u|^2\, dx = \int_\Omega |\nabla u|^2\, dx - ab$ (*surface measure of* $\partial\Omega$),
(b) $u(y) \equiv -bS_\lambda(y) + aD_\lambda(y)$ *for* $y \in \Omega$,
(c) *for any* $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *with* $\Delta v + \lambda v = 0$ *in* Ω, *we have*

$$\int_{\partial\Omega} a\frac{\partial v}{\partial n} - bv\, ds = 0, \quad \textit{and} \tag{6}$$

(d) $0 \equiv -bS_\lambda(y) + aD_\lambda(y)$ *for* $y \in \mathbf{R}^n \setminus \overline{\Omega}$.

Proof. (a) follows by taking $v = \bar{u}$ (i.e., the complex conjugate of u) in (4). Letting $\delta_y(x)$ denote the *delta "function"* based at y, using the fact that $\gamma_\lambda(|x - y|)$ is a fundamental solution of $\Delta + \lambda$, using $\Delta_x\gamma_\lambda(|x - y|) \equiv \Delta_y\gamma_\lambda(|x - y|)$ for $x \neq y$ (Δ_x and Δ_y are the Laplacian operators with respect to the x and y variables, respectively), (1)–(3) and (5), we have *for* y *in* Ω that

$$\begin{aligned} u(y) &= \int_\Omega u(x)\delta_y(x)\,dx \\ &= \int_\Omega u\big[\Delta_x\gamma_\lambda(|x-y|) + \lambda\gamma_\lambda(|x-y|)\big] - \gamma_\lambda(|x-y|)[\Delta u + \lambda u]\,dx \\ &= \int_{\partial\Omega} a\frac{\partial}{\partial n_x}\gamma_\lambda(|x-y|) - \gamma_\lambda(|x-y|)b\,ds(x) \\ &= aD_\lambda(y) - bS_\lambda(y), \quad \text{proving (b).} \end{aligned}$$

Now let $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ with $\Delta v + \lambda v = 0$ in Ω. Then by (1)–(3) and (5) we have

$$0 = \int_\Omega u[\Delta v + \lambda v] - v[\Delta u + \lambda u]\,dx = \int_{\partial\Omega} a\frac{\partial v}{\partial n} - vb\,ds,$$

proving (c). For any fixed $y \in \mathbf{R}^n \setminus \overline{\Omega}$, we obtain (d) by taking $v(x) \equiv \gamma_\lambda(|x - y|)$ (for $x \in \overline{\Omega}$) in (c). □

Remarks.

1. From (a) above, we see that λ must be real. From Lemma 3 of [27] we see that the conjecture of Section 1 holds if $\lambda < 0$ (assuming Ω is of class $C^{2+\epsilon}$ for some $\epsilon > 0$). If $\lambda = 0$, a solution of (1) is harmonic on Ω so that it must attain its maximum and minimum values on $\partial\Omega$ (cf. Gilbarg and Trudinger [13, Theorem 3.1]). Thus by (2) it must be constant. Thus we may assume that $\lambda > 0$ in trying to prove the conjecture (so long as Ω is of class $C^{2+\epsilon}$).
2. (b) above gives an explicit formula for u.
3. (c) was first proved in [26]. They also proved the converse, that if (6) holds for some $\lambda > 0$ and for every $v \in C^2(\overline{\Omega})$ with $\Delta v + \lambda v = 0$, then (1)–(3) have a *nonconstant* solution u on Ω.
4. (d) gives an alternative (with no reference to u) to assuming that (1)–(3) have a nonconstant solution u on Ω. The alternative is to assume that $S_\lambda(y)$ and $D_\lambda(y)$ are linearly dependent on $\mathbf{R}^n \setminus \overline{\Omega}$ (for some real λ). In fact, if Ω is of class C^2 and if (d) holds for some a, b and λ, then u given by (b) satisfies

(1)–(3) (by [7, Theorem 3.1], with the real part of their $\Phi(x, y)$ equal to $-\gamma_\lambda(|x-y|)$ with $n = 3$ and $k = \sqrt{\lambda}$, by Courant and Hilbert [8, p. 496]).

5. Theorem 1 can be generalized to the case that Ω is a Lipschitz domain. (This is the appropriate setting now that a good theory of Dirichlet and Neumann boundary value problems for the Helmholtz equation and of single- and double-layer acoustic potentials is available for such domains; see Mitrea [17].) The results will only be summarized here, since the technical details would take too long to describe carefully (and the extra generality may not be of interest to everyone). Let $p > 1$ and $q > 1$ with $p^{-1} + q^{-1} = 1$. Equation (1) would be interpreted classically, as before. Equation (2) would hold in the sense that $\mathcal{N}(u) \in L^p(\partial\Omega, ds)$ (where $\mathcal{N}(u)$ is the inward nontangential maximal function of u [17]) and $u|_{\partial\Omega} = a$ in the sense of nontangential limits ds-a.e. Equation (3) would hold in the sense that $\mathcal{N}(\nabla u) \in L^q(\partial\Omega, ds)$ and $\partial u/\partial n|_{\partial\Omega} = b$, interpreted in a similar way. The key to the proof of the generalized theorem is the existence (see Verchota [22, Theorem 1.12]) of a sequence $\{\Omega_j\}_{j=1}^{\infty}$ of open subsets of Ω with each $\partial\Omega_j \in C^\infty$, with $\partial\Omega_j$ converging nontangentially and uniformly to $\partial\Omega$, with the unit exterior normal vectors n_j of $\partial\Omega_j$ converging ds-a.e. and in every $L^r(\partial\Omega, ds)$ (for $1 \leqslant r < \infty$) to the unit exterior normal n of $\partial\Omega$, and with the boundary measures ds_j for $\partial\Omega_j$ converging to ds as $j \to \infty$. If the constructions of the proof above are carried out for each Ω_j, taking the limit as $j \to \infty$ gives the conclusions as before. In part (a), some care is required to prove that our reinterpretations of (2) and (3) guarantee that $\int_\Omega |u|^2\,dx < \infty$. (From the limit process leading to the equation of (a), it then follows that $\int_\Omega |\nabla u|^2\,dx < \infty$.) In part (c), weaker assumptions can be assumed for v. For the same p and q as for u, assume that $v \in C^2(\Omega)$ with $\Delta v + \lambda v = 0$ in Ω, but with $\mathcal{N}(v) \in L^p(\partial\Omega, ds)$, $v|_{\partial\Omega} \in L^p(\partial\Omega, ds)$, $\mathcal{N}(\nabla v) \in L^q(\partial\Omega, ds)$, and $\partial v/\partial n|_{\partial\Omega} \in L^q(\partial\Omega, ds)$.

3. Two results on the regularity of $\partial\Omega$

For the remainder of this paper, for any $\delta > 0$ and any $x \in \mathbf{R}^n$, let $B_\delta(x)$ denote the open ball in $\mathbf{R}^n$ with center x and radius δ.

Definition. A nonempty subset S of $\mathbf{R}^n$ is an $(n-1)$-*dimensional real-analytic surface* if for each $x \in S$ there is a real-analytic one-to-one map of the open unit ball $B_1(0)$ (centered at the origin 0) onto an open neighborhood U_x of x such that the inverse map is also real-analytic and such that $B_1(0) \cap \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n;\ x_n = 0\}$ maps onto $U_x \cap S$.

A theorem very similar to the following (but in the context of the Pompeiu problem) was proved in [24]:

Theorem 2. *Assume that Ω is of class C^1 (this includes the assumption that Ω is locally on one side of $\partial\Omega$ near any point of $\partial\Omega$) and that u is a* nonconstant *solution on Ω of* (1)–(3). *Assume that S and W are nonempty connected* $(n-1)$-*dimensional real-analytic surfaces with $S \subseteq \partial\Omega$ and $S \subseteq W$. Assume that W is orientable. Then $W \subseteq \partial\Omega$.*

Proof. Select a point $y^0 \in S$. Let n be the exterior unit normal vector to $\partial\Omega$ at y^0. Since W is connected, smooth and orientable, there is a unique extension of n to a smooth unit normal field (also denoted by n) on W. At each point x on W, use the Cauchy–Kovalevskaya theorem (Renardy and Rogers [19, pp. 46–58]) to solve the equation $\Delta U + \lambda U = 0$ on some open neighborhood U_x of x subject to the initial data $U = a$ and $\partial U/\partial n = b$ on $W \cap U_x$ ($\partial/\partial n$ is taken in the direction of the normal field n described above). Using the Holmgren uniqueness theorem [19, pp. 61–65] and the uniqueness of analytic continuation, we may piece together these local solutions to obtain a real analytic function U defined on an open set N^* containing W, with $\Delta U + \lambda U = 0$ on N^* and with $U = a$ and $\partial U/\partial n = b$ on W. We consider two cases.

Case 1. If $b \neq 0$, then clearly (since ∇U is then a nonzero normal vector at each point of W) for each $x \in W$ there is an open ball $B_x \subseteq N^*$ centered at x such that $z \in B_x$ and $U(z) = a$ imply that $z \in W$.

Case 2. If $b = 0$, then $a \neq 0$ (otherwise $u \equiv 0$ on Ω by (b) of Theorem 1) and $\lambda \neq 0$ (otherwise $u \equiv$ constant on Ω by Remark 1 above), so for each $x \in W$ we have $\Delta U(x) = -\lambda a \neq 0$, so there is an i with $1 \leqslant i \leqslant n$ such that $(\partial^2 U/\partial x_i^2)(x) \neq 0$. Thus, by the implicit function theorem, there is an open ball $B_x \subseteq N^*$ centered at x such that $\{z \in B_x;\ (\partial U/\partial x_i)(z) = 0\}$ is contained in the graph of a continuously differentiable function of $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. But $\nabla U \equiv 0$ on W (in this $b = 0$ case), so $\partial U/\partial x_i \equiv 0$ on W. Thus (decreasing the radius of B_x if necessary but keeping x as the center), $z \in B_x$ and $\nabla U(z) = 0$ imply that $z \in W$.

For either Case 1 or Case 2 above, let N be the union over all $x \in W$ of the open sets B_x. Clearly $W \subseteq N \subseteq N^*$. Let $y^0 \in S$ be the point selected above. There is clearly an open ball $G \subseteq N$ centered at y^0 so that $G \setminus \partial\Omega$ consists of precisely two nonempty components G_1 and G_2, with $G_1 \subseteq \Omega$ and $G_2 \subseteq \mathbf{R}^n \setminus \overline{\Omega}$ and such that $U \equiv u$ on G_1 (by the Holmgren uniqueness theorem). Let C be the component of $\Omega \cap N$ which contains G_1. By uniqueness of analytic continuation, $U \equiv u$ on C.

Let $W^* = W \cap \partial\Omega \cap \overline{C}$. Since $y^0 \in W^*$, W^* is not empty. Since $\partial\Omega \cap \overline{C}$ is closed in $\mathbf{R}^n$, W^* is closed in the relative topology of W. Since W is connected, once it is proved that W^* is open relative to W we have $W = W^*$, so that $W \subseteq \partial\Omega$ and the theorem is proved.

To prove that W^* is open relative to W, let x^0 be any point of W^*. For some $\epsilon > 0$, we have $B_\epsilon(x^0) \subseteq N$ with $B_\epsilon(x^0) \setminus W$ consisting of precisely two components, C_1 and C_2. Since C is open and $x^0 \in \overline{C}$, C has nonempty intersection with one of these components, say C_1. We claim now that $C_1 \subseteq C$. If not, there would be a $z \in C_1$ with $z \in \partial C$. Since $C_1 \subseteq N$, z would have to be in $\partial\Omega$. Since $U \equiv u$ on C we have $U(z) = a$ in Case 1 above and $\nabla U(z) = 0$ in Case 2 above. Since N is the union of the balls B_x, there is an $x \in W$ such that $z \in B_x$. Thus in either Case 1 or Case 2 above we conclude that $z \in W$, contradicting $z \in C_1 \subseteq B_\epsilon(x^0) \setminus W$. Therefore the claim that $C_1 \subseteq C$ is established. (Note that thus $C_1 \subseteq \Omega$.)

We claim now that $C_2 \cap C$ is empty. If not, then by replacing C_1 by C_2 in the above argument, we also have $C_2 \subseteq C$, so that $C_1 \cup C_2 \subseteq \Omega$. Then x^0 would not be in $\partial\Omega$ (since Ω is locally on only one side of $\partial\Omega$ near any point of $\partial\Omega$), a contradiction. Thus $C_2 \cap C$ is empty. We claim now that $B_\epsilon(x^0) \cap W \subseteq \partial\Omega$. If not, there is a $z^0 \in B_\epsilon(x^0) \cap W$ with $z^0 \notin \partial\Omega$. Since $z^0 \in \overline{C_1} \subseteq \overline{C}$, we have $z^0 \in \overline{\Omega}$, so $z^0 \in \Omega$, in which case $C_2 \cap C$ is nonempty, a contradiction. Thus $B_\epsilon(x^0) \cap W \subseteq W^*$, proving that W^* is open in the relative topology of W and completing the proof. $\square$

Remark. Suppose Ω is of class C^1 with a nonconstant solution u of (1)–(3) on it. Suppose that $\partial\Omega$ contains a nonempty, relatively open portion of an $(n-1)$-dimensional hyperplane. Then Theorem 2 tells us that the entire hyperplane is contained in $\partial\Omega$, contradicting the boundedness of Ω. Thus no such portion of a hyperplane can be part of $\partial\Omega$.

A theorem somewhat similar to the following (but in the context of the Pompeiu problem) was proved in [25]:

Theorem 3. *Assume that Ω is of class C^1. Assume that there is a* nonconstant *solution $u \in C^2(\overline{\Omega})$ of* (1)–(3). *Then $\partial\Omega$ is an $(n-1)$-dimensional real-analytic surface.*

Proof. Apply Theorem 2 of Kinderlehrer and Nirenberg [14] with their u equal to our $u - a$, their $F(x, u, Du, D^2u) \equiv \Delta u + \lambda u + a\lambda$, and their $g(x, p_1, \ldots, p_n) \equiv p_1^2 + \cdots + p_n^2 - b^2$. They do everything locally about a general point of $\partial\Omega$, assumed to be the origin 0. If $b \neq 0$, $|\operatorname{grad} u| = |b| \neq 0$ clearly holds at the origin. (If $b = 0$, the conclusion of Theorem 3 is proved in [25].) Setting up our coordinate system so the x_n-direction is the exterior normal direction at the origin 0, $\partial g/\partial p_n = 2p_n$, so $\partial g/\partial p_n(0, (\operatorname{grad} u)(0)) = 2(\partial u/\partial n)(0) = 2b \neq 0$. Since F and g are real-analytic, so is $\partial\Omega$. $\square$

Remarks.

1. The assumption in Theorem 3 that $u \in C^2(\overline{\Omega})$ is strong and hard to verify in general. If we are willing to assume that Ω is of class $C^{2+\epsilon}$ (for some $\epsilon > 0$) however, then Theorem 6.14 of [13] insures the stronger result that $u \in C^{2+\epsilon}(\overline{\Omega})$.
2. The analysis in [14] is local about any point of the boundary (assumed to be the origin 0). Thus if Ω is of class C^1 and if the nonconstant solution u of (1)–(3) has second derivatives that extend continuously to $\partial\Omega \cap B_\epsilon(0)$ (for some $\epsilon > 0$), then $\partial\Omega \cap B_\epsilon(0)$ is real analytic.

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