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Zadeh's influence on mathematics

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Abstract The purpose of this paper is to present Lotfi Zadeh's influence on mathematics. Mathematics rests on the foundation of logic and set theory. L.A. Zadeh's seminal paper on fuzzy sets laid the groundwork for fuzzy logic and thus the foundation of fuzzy mathematics.

1. Introduction

The purpose of this paper is to present Lotfi Zadeh's influence on mathematics. Mathematics rests on the foundation of logic and set theory. L.A. Zadeh's seminal paper "Fuzzy Sets" [1] laid the groundwork for fuzzy logic and thus the foundation of fuzzy mathematics. His work has inspired the writing of thousands of articles and hundreds of books involving fuzzy logic. There have been over 200 books on fuzzy logic published just in Springer-Verlag Studies in Fuzziness and Soft Computing alone. It is not our intention to credit the very long list of individuals who contributed to the development of fuzzy mathematics. Neither do we attempt to provide an extensive bibliography due to space constraints. The interested reader can see [2,3]. We concentrate only on topology and algebra and rely heavily on [2,3].

The classical mathematical theories, by which certain types of certainty can be expressed, are the classical set theory and the probability theory. In terms of set theory, uncertainty is expressed by any given set of possible alternatives in situations where only one of the alternatives may actually happen. Uncertainty expressed in terms of sets of alternatives results from the nonspecificity inherent in each set. Probability theory expresses uncertainty in terms of a classical measure on subsets of a given set of alternatives. The set theory, introduced by Zadeh [1], presents the notion that membership in a given subset is a matter of degree rather than that of totally in or totally out. This concept is captured in [1] by defining a fuzzy subset of a universal set, X, to be a function from X into the closed interval $[0, 1]$.

Another broad framework for dealing with uncertainty is the fuzzy measure theory founded by Sugeno. The fuzzy measure theory replaces the classical measure theory by replacing the additivity requirement with the weaker requirements of monotonicity, with respect to set inclusion and the continuity or semicontinuity of fuzzy measures [4–13].

2. History

L.A. Zadeh published his seminal paper "Fuzzy Sets" in 1965. This inspired mathematicians to fuzzify mathematical structures. The evolution of the fuzzification of mathematics can be broken into four stages:

1. Straightforward fuzzification during the sixties and seventies;
2. Exploration of numerous possible choices in the generalization process during the eighties;
3. Standardization, axiomatization and L-fuzzification in the nineties;
4. Deeper development of many areas of fuzzy mathematics in the 21st century.

As communicated to me by E. Kerre, the years 2001–2009 found an expansion of the frontier of many areas in fuzzy mathematics, such as axiomatics of structures and concepts, fuzzy logic in the narrow sense, interval valued fuzzy sets and

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fuzzification of mathematical disciplines, such as geometries, to name a few. One exception is fuzzy abstract algebra where the fuzzification of more algebraic structures took place and where new approaches were developed, but where the frontier did not noticeably expand.

3. Fuzzy set theory

Mathematics is based on the notion of a set. One of the most fundamental concepts is the notion of an ordered pair. There are different approaches to the development of set theory depending upon the underlying axiom system. For example, take set membership and the empty set as undefined symbols, then define a set as follows: A is a set if and only if $A = \emptyset$ or there exists $x \in A$. Define $\{x\} = A$ if and only if A is a set, $A \neq \emptyset$, and for all $y \in A$, $y = x$. Define $\{x, y\} = A$ if and only if A is a set, $A \neq \emptyset$, and for all $z \in A$, either $z = x$ or $z = y$. Then an ordered pair (x, y) is defined to be $\{\{x\}, \{x, y\}\}$. The Cartesian product, $A \times B$, of sets A and B is defined to be $\{(x, y)|x \in A, y \in B\}$. Then, a relation from A into B is a subset of $A \times B$.

A fuzzy subset of a set, X, is defined to be a function of X into the closed interval $[0, 1]$. Let $\mathcal{F}(X)$ denote the set of all fuzzy subsets of X. The notion of the Cartesian product of a fuzzy subset, $\tilde{A}$, of a set, X, and a fuzzy subset, $\tilde{B}$, of a set, Y, can be fuzzified as follows: for all $(x, y) \in X \times Y$, $(\tilde{A} \times \tilde{B})(x, y) = \min\{\tilde{A}(x), \tilde{B}(y)\}$. The Cartesian product of fuzzy subsets satisfies many of the properties that crisp Cartesian products satisfy. However there are some deviations. For example, $\tilde{A} \times \tilde{B} = \tilde{B} \times \tilde{A}$ does not imply that $\tilde{A} = \tilde{B}$. An extensive list can be found in [2].

A list of operations on fuzzy subsets follows.

Let $\{\tilde{A}_i|i \in I\}$ be a collection of fuzzy subsets of X. Define the fuzzy subsets, $\cap\tilde{A}_i$ (intersection) and $\cup\tilde{A}_i$ (union), of X by for all x in X, $(\cap\tilde{A}_i)(x) = \inf\{\tilde{A}_i(x)|i \in I\}$ and $(\cup\tilde{A}_i)(x) = \sup\{\tilde{A}_i(x)| i \in I\}$.

Let $\tilde{A}$ and $\tilde{B}$ be fuzzy subsets of X. Then, $\tilde{A} \subseteq \tilde{B}$ if for all $x \in X$, $\tilde{A}(x) \leq \tilde{B}(x)$. If $\tilde{A} \subseteq \tilde{B}$ and there exists $x \in X$ such that $\tilde{A}(x) < \tilde{B}(x)$, then we write $\tilde{A} \subset \tilde{B}$.

Let $x \in X$ and $t \in (0, 1]$. For a subset, A of X, define the fuzzy subset, t_A, as follows: for all $x \in X$, $t_A(x) = t$, if $x \in A$ and $t_A(x) = 0$, otherwise. We sometimes write x_t for $t_{\{x\}}$. We call $t_{\{x\}}$ a fuzzy singleton in X.

Let Y and Z be sets. Let $\tilde{A}$ be a fuzzy subset of $X \times Y$ and $\tilde{B}$ be a fuzzy subset of $Y \times Z$. Define the composition, $\tilde{A} \circ \tilde{B}$, by for all $x \in X$ and $z \in Z$, $(\tilde{A} \circ \tilde{B})(x, z) = \sup\{\min\{\tilde{A}(x, y), \tilde{B}(y, z)\}|y \in Y\}$.

Let $\tilde{A}$ be a fuzzy subset of X and let $t \in [0, 1]$. Define $\tilde{A}_t = \{x \in X|\tilde{A}(x) \geq t\}$. Then $\tilde{A}_t$ is called a t-cut of $\tilde{A}$ or a t-level set of $\tilde{A}$. The support of $\tilde{A}$, written $\mathrm{Supp}(\tilde{A})$, is defined to be the set $\{x \in X|\tilde{A}(x) > 0\}$.

Let $\tilde{A}$ be a fuzzy subset of X. Then, $\tilde{A}$ is said to have the sup property if for every subset Y of X, there exists y_0 in Y, such that $\tilde{A}(y_0) = \sup\{\tilde{A}(y)|y \in Y\}$.

Let f be a function from a nonempty set, X, into a nonempty set, Y. Let $\tilde{B}$ be a fuzzy subset of Y. The pre-image of $\tilde{B}$ under f, written $f^{-1}(\tilde{B})$, is the fuzzy subset of X defined by $f^{-1}(\tilde{B})(x) = \tilde{B}(f(x))$ for all x in X. Let $\tilde{A}$ be a fuzzy subset of X. The image of $\tilde{A}$ under f, written $f(\tilde{A})$, is the fuzzy subset of Y defined by $f(\tilde{A})(y) = \sup\{\tilde{A}(z)|z \in f^{-1}(\{y\})\}$, if $f^{-1}(\{y\})$ is not empty, $f(\tilde{A})(y) = 0$ otherwise.

Krassimir Atanassov introduced the notion of the degree of nonmembership in the definition of a fuzzy subset. This idea is incorporated with the notion of membership in a fuzzy subset and the resulting structure is called an intuitionistic fuzzy set. In [14], the reader can find a fairly comprehensive and complete treatment on intuitionistic fuzzy set theory and applications in a variety of diverse fields.

Hohle shows that large parts of fuzzy set theory are subfields of sheaf theory [15,16]. Consequently, fuzzy set theory is closer to mainstream mathematics than one might think. Other papers on the foundation of fuzzy set theory can be found in [15,16].

4. Fuzzy logic

Fuzzy logic was first invented as a representation scheme and calculus for uncertain or vague notions. It is an infinite-valued logic that allows more human-like interpretation and reasoning. Plato posited the laws of thought. One such law was the Law of the Excluded Middle. Parminedes stated that statements could be both true and not true at the same time. Plato proposed a third region between true and false. Jan Lukasciewicz was the first to propose a systematic alternative to the bi-valued logic of Aristotle and described the 3-valued logic, with the third being "possible". Zadeh, in his theory of fuzzy sets, proposed using a membership function, μ, where μ was a function from the set of interest into the closed interval $[0, 1]$. He proposed new operations on the calculus of logic and showed that fuzzy logic was a generalization of classical and Boolean logic. He also proposed fuzzy numbers as a case of fuzzy sets, as well as the corresponding rules for consistent mathematical operations (fuzzy arithmetic).

With fuzzy set theory, one obtains a logic in which statements may be true or false to different degrees rather than the bivalent situation of being true or false; consequently, certain laws of bivalent logic to not hold, e.g. the law of the excluded middle and the law of contradiction. This results in an enriched scientific methodology.

Most mathematical fuzzy logic is based on t-norm logic. The notion of a t-norm is used for the concept of conjunction in fuzzy logic and intersection in fuzzy set theory. The notion of a co-norm, in particular maximum, is used for the concept of disjunction in fuzzy logic and union in fuzzy set theory. There are several definitions for the use of negation in fuzzy logic and complement in fuzzy set theory. Some of the most important propositional fuzzy logics are monodial t-norm based fuzzy logic and basic propositional fuzzy logic, where implication is defined as residuum of the t-norm, Lukasiewicz fuzzy logic, Godel fuzzy logic, product fuzzy logic, and Pavelka's logic. These fuzzy logics have been extended to predicate fuzzy logics by adding universal existential quantifiers.

Some of the remaining discussions on fuzzy logic is taken from [17]. Fuzzy propositional calculus generalizes classical propositional calculus by using the truth set, $[0, 1]$, instead of $\{0, 1\}$. Let V be a set of symbols representing atomic or elementary propositions. The set of formulas, F, is built up from V using the logical connectives $\wedge, \vee, \prime$. A truth evaluation is obtained by taking any function, $t : V \to [0, 1]$, and extending it to $\widetilde{t} : F \to [0, 1]$ by replacing each element, $a \in V$, which appears in the formula by its value, $t(a)$. This gives an expression in elements of $[0, 1]$ and the connectives, $\wedge, \vee, \prime$. The expression is evaluated by letting $\wedge, \vee, \prime$ be defined as in fuzzy set theory. If we define two formulas in F to be equivalent, if they have the same truth evaluation for all $\widetilde{t}$, then we obtain an equivalence relation on F. The set of equivalence classes of logically equivalent formulas forms a Kleene algebra. With each formula, associate the fuzzy subset, $[0, 1]^V \to [0, 1]$, of

$[0, 1]^V$ given by $t \to t(u)$. Then we have a function from F into $\mathcal{F}([0, 1]^V)$. This induces a one-to-one function from $F/\equiv$ into the set of functions from $[0, 1]^V$ into $[0, 1]$, that is into the set of fuzzy subsets of $[0, 1]^V$. This one-to-one function associates fuzzy logical equivalence with the equality of fuzzy subsets.

We next give a short discussion of interval-valued fuzzy logic. There is a school of thought that says assigning an exact number to an expert's opinion is too restrictive. Assigning an interval of values is more realistic. Hence, rather than map a set U into $[0, 1]$, it would be more realistic to map U into $[0, 1]^{[2]}$, where $[0, 1]^{[2]} = \{(a, b)|a, b \in [0, 1], a \leq b\}$. In order to construct the propositional calculus whose truth values are elements of $[0, 1]^{[2]}$, we need the appropriate algebra of these truth values. Let $\wedge$, $\vee$ be functions of $[0, 1]^{[2]} \times [0, 1]^{[2]}$ into $[0, 1]^{[2]}$, defined as follows:

$\forall(a, b), (c, d) \in [0, 1]^{[2]}$,

$$(a, b) \wedge (c, d) = (a \wedge b, c \wedge d),$$

and:

$$(a, b) \vee (c, d) = (a \vee b, c \vee d).$$

Define the function $'$ of $[0, 1]^{[2]}$ into $[0, 1]^{[2]}$ by $\forall(a, b) \in [0, 1]^{[2]}$, $(a, b)' = (b', a')$.

The propositional calculus, $F/\equiv$, is a DeMorgan algebra, but not a Kleene algebra. A more complete discussion can be found in [17].

In 2006, Fuzzy Sets and Systems devoted four issues, 5, 6, 14 and 15, to themes concerning logic. Issue 5 dealt with the question, "What is fuzzy logic?" The papers in this issue concerned mathematical fuzzy logic, fuzzy logic to fuzzy mathematics, interval-valued fuzzy and machine intelligence. Issue 6 considered topology and category theory. In 2008, Fuzzy Sets and Systems devoted two issues, 9 and 19, to fuzzy logic. One issue was concerned with algebraic aspects of fuzzy and many-valued logics, the other with lattice-valued topology. An interesting account concerning fuzzy logic and probability can be found in [18].

In [19], Zadeh published a position paper concerning an extended fuzzy logic. Extended fuzzy logic adds to fuzzy logic a capability to reason imprecisely with imperfect information. This capability is exercised when precise reasoning is infeasible, excessively costly or unneeded. In [20], Zadeh discussed the equality of fuzzy logic and computing with words. Zadeh states that computing with words is a methodology in which words are used in place of numbers for computing and reasoning. Computing with words is a necessity when the available information is too imprecise to justify the use of numbers and when there is a tolerance for imprecision that can be exploited to achieve tractability, robustness, and better rapport with reality.

In 2010, Fuzzy Sets and Systems published an editorial concerning the 26th Linz Seminar on Fuzzy Set Theory in 2005. The seminar was devoted to the topic, "Fuzzy Logics and Related Structures".

5. Topology

Chang introduced the notion of a fuzzy topology of a set in 1968 [21]. A fuzzy topology on a set, X, is a set of fuzzy subsets, T of X, such that:

1. 1_X and 0_X are in T;
2. if $\tilde{A}, \tilde{B}$ are in T, then, $\tilde{A} \cap \tilde{B}$ is in T;
3. if $\{\tilde{A}_i | i \in I\}$ is a subset of T, then $\cup_i \tilde{A}_i$ is in T.

The pair (X, T) is called a fuzzy topological space and each element of T is called an open fuzzy set.

Fuzzifications of the concepts of a closed set, the interior of a set, the closure of a set, and the neighborhood of a point are followed in a straightforward manner. However, a straightforward fuzzification of the classical neighborhood axioms failed to characterize a fuzzy topology. References for this discussion can be found in [2].

Let (X, T) and (Y, U) be fuzzy topological spaces. Let f be a function of X into Y. Then f is said to be F-continuous if $f^{-1}(\tilde{B})$ is open in X for every open fuzzy subset, $\tilde{B}$ of Y. It was shown that f is F-continuous if and only if $f^{-1}(\tilde{B})$ is closed in X for every closed fuzzy subset, $\tilde{B}$ of Y. Twenty-six different forms of fuzzy continuity were introduced in the eighties. In 1991, a research group led by Kerre obtained all these notions of fuzzy continuity as particular cases of the central notion of fuzzy $\varphi\psi$ continuity, where φ and ψ denote operations on (X, T), i.e. mappings of $\mathcal{F}(X)$ into $\mathcal{F}(X)$, satisfying the condition that int $(\tilde{A}) \subseteq \varphi(\tilde{A})$ for all $\tilde{A} \in \mathcal{F}(X)$. An account of this unification process can be found in [2]. The group used the concept of an operation to unify several existing notions of separation, compactness, and open and closed mappings [2]. The group also considered smooth topological spaces and modifications of crisp, as well as fuzzy continuity of mappings. The fuzzy topology defined by Chang [21], the L-fuzzy topology defined by Goguen [22], and the stratified fuzzy topology defined by Lowen [23] are such that a fuzzy subset may only be either open or not open. The research group headed by the Kerre group considered a fuzzification of topology in that a fuzzy subset may assume a partial or intermediate degree of openness. These structures were called smooth topologies. References to some papers on smooth topological spaces can be found in [2].

Let (X, T) and (Y, U) be fuzzy topological spaces and let f be an F-continuous function of X onto Y. If X is compact, then it can be shown that Y is compact.

In the eighties, the concepts of normality and neighborhood in a fuzzy topology were considered. Hutton [24] introduced a definition of normality by a straightforward generalization of Urysohn's characterization in 1975. For crisp subsets, $A \cap B = \emptyset$ if and only if A is a subset of B^c. However, for fuzzy subsets, $\tilde{A} \cap \tilde{B} = 1_\emptyset$ only implies $\tilde{A}$ is a subset of $\tilde{B}^c$. The definitions of normal, weakly normal, and completely normal for fuzzy topologies were influenced by this fact. The definition of complete normality did not allow for the extension of Tietze's characterization theorem of complete normality.

In [22,25,26], three types of membership relation involving fuzzy singletons were defined. These membership relations led to five different definitions of the notions of a neighborhood. (See [2,3] for references.) The approaches of Pu–Liu and Ghanim–Kerre–Mashour totally induce a complete formal parallelism with ordinary topology. These different approaches were then linked. The Mashhour neighbourhoods of a fuzzy singleton could be expressed in terms of the Pu neighborhoods and in terms of the Kerre neighborhoods. A lattice theoretical study of different approaches to the neighborhood concept has led to the introduction of several subclasses of fuzzy topological spaces. In [2], one can find more details concerning the relations between these concepts.

Much of the early work in fuzzy topology was based on its similarity with another branch of topology on a lattice – locale theory. Some researchers used ideas and methods of locale theory to study problems not involving "points" in fuzzy topology. Many problems on fuzzy topological spaces must involve the notion of "point", such as the separation and embedding theory. A summary of the work on points can be found in the book by Liu and Luo [27]. Almost all Kelley's General Topology has been generalized in the fuzzy framework and synthesized in [27].

A development of the theory of metric spaces of normal, upper semicontinuous fuzzy convex fuzzy sets with closed support sets, mainly on the base space $\mathbb{R}^n$, where $\mathbb{R}$ denotes the set of real numbers, can be found in [28]. The book contains much of the work by Diamond and Kloeden on the characterization of compactness in metric spaces of fuzzy sets.

If a completely distributive lattice, L, has an order-reversing involution, $' : L \to L$, then it is called an F-lattice. Let X be a nonempty set, L an F-lattice, and $\delta \subseteq L^X$. Then δ is called an L-fuzzy topology on X, and (L^X, δ) is called an L-fuzzy topological space or an L-fts if δ satisfies the following conditions:

1. $0_X, 1_X \in \delta$;
2. $\forall \mathcal{A} \subseteq \delta, \cup \mathcal{A} \in \delta$;
3. $\forall \tilde{A}, \tilde{B} \in \delta, \tilde{A} \cap \tilde{B} \in \delta$.

If $L = [0, 1]$, then (L^X, δ) or simply (X, δ) is called an F-topological space or an F-ts. Consideration of L-stratified spaces, the convergence theory and properties related to cardinals can be found in [27]. Separation properties are presented in [24,27]. The pointwise characterization of L-fuzzy complete regularity and its use in establishing the embedding theorem can be found in [27]. The problem of the L-fuzzy form of the Tietze Extension Theorem was raised in [29] and positively solved by Kubiak [30]. Results on L-fuzzy complete regularity can be found in [31]. In [27], the lattice-valued Hahn-Dieudonne-Tong Insertion Theorem is proved, as well as Kubiak's Fuzzy Insertion Theorem. Notions of compactness in fuzzy topological spaces are compared in [32]. Results concerning paracompactness can be found in [27]. Fuzzy versions of the uniformity theory have been established. A study of metrics in Hutton's sense and Erceg's sense can be found in [33,34]. Rodabaugh gave the first systematic on L-fuzzy topological spaces and locales [35]. Warner investigated the membership relation between points and L-fuzzy subsets in L-fuzzy topology by applying the notion of point in locales [12]. The relation between L-fts and locales has been investigated by Luo. Rodabaugh considered categorical frameworks for Stone Representation Theories [36]. Some other interesting papers are [37–41].

In [42,43], Pultr and Rodebaugh examined category theoretic aspects of chain-valued frames. Other papers on category theory and topology can be found in the same issue. In 2008, Fuzzy Sets and Systems published an issue devoted to lattice-valued topology. The interested reader can find many papers on fuzzy topology that cover a wide variety of topics in Fuzzy Sets and Systems alone.

In 2010, Fuzzy Sets and Systems published an issue concerning the 29th Linz Seminar on Fuzzy Set Theory in 2009. The theme was "Foundations of Lattice-Valued Mathematics with Applications to Algebra and Topology".

6. Algebra

In 1971, Rosenfeld introduced the notions of a fuzzy subgroupoid and subgroup [44]. The closure property in the crisp case yields the key to defining fuzzy substructures. Let X be a nonempty set and $*$ is a binary operation on X. A subset, Y, is closed under $*$ if for all $x, y \in Y, x * y \in Y$. The closure property for a fuzzy subset, $\tilde{A}$ of X, is that for all $x, y \in X, \tilde{A}(x * y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$. Hence if $\tilde{A}(x) = \tilde{A}(y) = 1$, then $\tilde{A}(x * y) = 1$. That is, if x and y are definitely in $\tilde{A}$, the $x * y$ is definitely in $\tilde{A}$. For groups, we have the following situation. Let $(G, *)$ be a group and H a nonempty subset of G. Then H is a subgroup of G if and only if for all $x, y \in H, x * y^{-1} \in H$. This leads to the following definition of a fuzzy subgroup of a group. Let $(G, *)$ be a group and $\tilde{A}$ a fuzzy subset of G. Then $\tilde{A}$ is a fuzzy subgroup of G if for all x, y in $G, \tilde{A}(x * y^{-1}) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$.

This technique is used to define fuzzy substructures of a variety of algebraic structures. Let $\tilde{X}$ be a fuzzy subset of an algebraic structure, A. It is common to define $\langle\tilde{X}\rangle$ to be the intersection of all fuzzy substructures of A of a certain type, and then show that $\langle\tilde{X}\rangle$ is a fuzzy substructure of the same type. One then proceeds to characterize $\langle\tilde{X}\rangle$.

Since Rosenfeld's paper, hundreds of papers fuzzifying various algebraic structures have been published. Some of the early work on fuzzy subgroups considered cosets, Lagrange's Theorem, free fuzzy subgroups, fuzzy subgroups of Dedekind groups solvable groups, Hall subgroups, nilpotent groups, fuzzy quasi-normal subgroups, generalized fuzzy subgroups, fully invariant and characteristic fuzzy subgroups, nilpotent fuzzy subgroups, commutator fuzzy subgroups, solvable fuzzy subgroups, lattices of fuzzy subgroups, infinite abelian groups, including structure results, divisible and pure fuzzy subgroups, invariants of fuzzy subgroups, basic and p-basic fuzzy subgroups, and fuzzy direct products, as related to a group being a direct product of its normal subgroups, fuzzy cyclic p-subgroups. A number of inequivalent fuzzy subgroups of certain finite Abelian groups has been determined. A notable early paper on fuzzy subgroups was by Anthony and Sherwood. They replaced min by a t-norm in the definition of a fuzzy subgroup. They examined two types of fuzzy subgroups: subgroup generated and function generated. They applied their results to abstract pattern recognition. One very important paper on fuzzy subgroups was by Tom Head [45,46]. The interested reader should see [47] for another paper along these lines. Head presented a method for deriving fuzzy theorems from crisp versions. Dib introduced fuzzy groups by means of fuzzy binary operations. Using a different approach, Demirci defined and studied vague and smooth groups via certain fuzzy binary operations involving fuzzy equalities. Using the notion of the quasi-coincidence of a fuzzy singleton with a fuzzy subset, Bhakat and Das defined a more general fuzzy subgroup than the one introduced by Rosenfeld. The notion of the quasi-coincidence used the standard fuzzy complement. These ideas were extended by replacing the standard complement by an arbitrary complement with an equilibrium. This notion was extended further to $(s, t]$-fuzzy subgroups for $0 \leq s < t \leq 1$ by others. Pertinent references can be found in [48].

The main contributor to the development of the theory of fuzzy subsemigroups was Kuroki. He developed theories concerning fuzzy ideals, fuzzy bi-ideals fuzzy interior ideals, fuzzy quasi-ideals and fuzzy semiprime ideals of semigroups. He also studied various types of fuzzy regular subsemigroups of a semigroup and fuzzy congruences on a semigroup. Fuzzy Rees congruences on a semigroup, prime fuzzy ideals, quasi-prime ideals,weakly quasi-prime ideals of semigroups and fuzzy ideal extensions of semigroups have been examined. Fuzzy subsemigroups were used to study fuzzy codes. Pertinent references can be found in [49].

The first paper concerning fuzzy ideal theory in rings was done by Liu [50]. Let R be a ring and let $\tilde{A}$ be a fuzzy subset of R. Then $\tilde{A}$ is called a fuzzy ideal of R if for all $x, y \in R$, $\tilde{A}(x - y) \geq \min\{\tilde{A}(x), \tilde{A}(y)\}$ and $\tilde{A}(xy) \geq \max\{\tilde{A}(x), \tilde{A}(y)\}$. A fuzzy ideal, $\tilde{P}$, of a ring, R, is called prime if for all fuzzy ideals

$\widetilde{A}$ and $\widetilde{B}$ of R, $\widetilde{A}\widetilde{B} \subseteq \widetilde{P}$ implies either $\widetilde{A} \subseteq \widetilde{P}$ or $\widetilde{B} \subseteq \widetilde{P}$. The notion of a fuzzy prime ideal of a ring was characterized. It was shown that a nonconstant fuzzy ideal, $\widetilde{P}$, of a ring, R, is prime if and only if $\mathrm{Im}(\widetilde{P}) = \{t, 1\}, 0 \leq t < 1$, and the level ideal, $\widetilde{P}_1$, is a prime ideal of R. A primary representation theory of fuzzy ideals of a ring was introduced and then the notion of a fuzzy algebraic variety of a fuzzy ideal. The characterization of a fuzzy irreducible algebraic variety followed, i.e. a nonconstant fuzzy algebraic variety, $\widetilde{V}$, is irreducible if and only if $\mathrm{Im}(\widetilde{V}) = \{0, t\}, 0 < t$, and the level variety, $\widetilde{V}_t$, is irreducible [51, 52]. These results and those of fuzzy ideals in local rings were applied to fuzzy intersection equations. The existence and uniqueness properties of fuzzy ideals and fractionary fuzzy ideals have been studied. Head, Golan, and Weinberger wrote important papers for deriving fuzzy theorems from crisp versions. Pertinent references can be found in [48].

The notion of a fuzzy subspace of a vector space was first considered by Katsaras and Liu [53]. It was shown that a fuzzy subspace with the sup property has a basis. A problem which went unsolved for a few years was whether or not any fuzzy subspace of an infinite dimensional vector space has a basis. The problem was answered negatively by Abdukhaliv Tulanbaev, and Umirbaev. Pertinent references can be found in [48].

The concept of a fuzzy submodule of a module was introduced by Negoita and Ralescu [54]. Free fuzzy submodules and their bases, primary fuzzy submodules, prime fuzzy submodules, exact sequences of L-modules, projective L-modules, simple and semisimple L-modules, fuzzy projective and injective fuzzy submodules, and homological and categorical aspects of fuzzy submodules have been studied.

Mordeson was the main contributor to the development of the concepts of fuzzy subfields, fuzzy algebraic varieties, and fuzzy subgroup algebras. The notions of algebraic and transcendental fuzzy field extensions, separable and inseparable fuzzy field extensions, composites, linear disjointness, separability, and the modularity of fuzzy field extensions have been studied [48]. The notion of neutrally closed fuzzy field extensions: a concept unique to fuzzy field extensions, has also been introduced. Other ideas have involved distinguished fuzzy subfields and the splitting of fuzzy field extensions. Treatment of a finite and infinite fuzzy Galois theory can be found in [48], as can a connection between fuzzy subgroups and fuzzy field extensions via fuzzy group subalgebras.

The areas of semirings and near-rings have seen only a modest interest. Fuzzy k-ideals in semirings and various aspect of fuzzy ideals in semirings have been examined. Some work has been done on fuzzy ideals and fuzzy R-subgroups of near-rings, hypernear-rings, and gamma near-rings. There has been recent interest in fuzzy hyper substructures, such as fuzzy hypergroups and semigroups, with application to fuzzy automata theory.

The study of BCK-algebras was initiated by K. Iseki in 1966, as a generalization of the concept of set difference and propositional calculus. He introduced a new algebra, called a BCI-algebra. The notion of BCI-algebras is a generalization of BCK-algebras and there is no proper class of commutative, positive implicative and implicative BCI-algebras [55]. Groups from Korea and Pakistan have been doing the bulk of the work on fuzzy substructures of BCH and BCK algebras and related algebras. A great number of papers have appeared concerning fuzzy ideals of these algebras. Y.B. Jun has been a major contributor.

Some important early work was undertaken by Chinese researchers. A major contributor was Yu Yandong. Much of Yu's work concerned TL-fuzzy substructures, where min is replaced with a t-norm and $[0, 1]$ is replaced with a complete lattice L. Lately, work on fuzzy algebraic structures has tailed off. However, there is currently an interest in fuzzy hypersubstructures of algebraic hyperstructures.

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