Alternative approaches to asymptotic behaviour of eigenvalues of some unbounded Jacobi matrices

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Abstract

In this article we calculate the asymptotic behaviour of the point spectrum for some special self-adjoint unbounded Jacobi operators $J$ acting in the Hilbert space $l^2 = l^2(\mathbb{N})$. For given sequences of positive numbers $\lambda_n$ and real $q_n$ the Jacobi operator is given by $J = SW + WS^* + Q$, where $Q = \text{diag}(q_n)$ and $W = \text{diag}(\lambda_n)$ are diagonal operators, $S$ is the shift operator and the operator $J$ acts on the maximal domain. We consider a few types of the sequences $\{q_n\}$ and $\{\lambda_n\}$ and present three different approaches to the problem of the asymptotics of eigenvalues of various classes of $J$’s. In the first approach to asymptotic behaviour of eigenvalues we use a method called successive diagonalization, the second approach is based on analytical models that can be found for some special $J$’s and the third method is based on an abstract theorem of Rozenbljum.

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1. Introduction

In this work we calculate the point spectrum or the asymptotic behaviour of the point spectrum for some special unbounded Jacobi operators $J$ acting in the Hilbert space $l^2 = l^2(\mathbb{N})$. We shall present three different approaches to the problem of the asymptotics of eigenvalues of various classes of $J$’s. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis in $l^2$. $S$ stands for the shift operator on $l^2$ given on the basis vectors as follows $Se_n = e_{n+1}$, $n \geq 1$. For given sequences of positive numbers $\lambda_n$ and real $q_n$ one defines the Jacobi operator by $J = SW + WS^* + Q$, where $Q$ and $W$ are the associated diagonal operators, $Qe_n = q_n e_n$ ($Q = \text{diag}(q_n)$) and $We_n = \lambda_n e_n$ ($W = \text{diag}(\lambda_n)$) for $n \geq 1$ and the operator $J$ acts on the maximal domain. We call $\lambda_n$’s the weights of $J$. It turns that in the case $\lim \inf_{n \to \infty} q_n^2 (\lambda_n^2 + \lambda_{n-1}^2)^{-1} > 2$ and $\lim_{n \to \infty} |q_n| = +\infty$ the operator $J$ is self-adjoint and its spectrum is discrete [9].

As far as we know the problem of finding the asymptotics of eigenvalues of Jacobi matrices has not been systematically studied in a general situation. However, the density of eigenvalues of rational Jacobi matrices was analysed by Dehesa in [2,3]. Let us also mention that there are special Jacobi matrices which appear in quantum optics with known

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asymptotics of their eigenvalues [16]. In [11] the asymptotics of eigenvalues for a special class of Jacobi matrices was found. Moreover, the method proposed in [11] (so called successive diagonalization method) can be applied for a quite general set of \( J \)'s (Theorem 2.2, Section 2 of this paper). The method presented in Theorem 2.2 resembles the idea used by Rozenbljum in [14] in his studies of the asymptotic of eigenvalues of some pseudo-differential operators on the unit circle. By using his abstract result [14, Theorem 1] we can find asymptotics of eigenvalues of a certain \( J \) when Theorem 2.2 does not apply (see Section 4). Our application of this result is rather simple, and we hope to find more interesting one in the future.

The third approach we propose here concerns special classes of Jacobi matrices for which analytical model can be found. Then we deal with differential operator \( J \) (unitarily equivalent to \( J \)) of the first order acting in a Hilbert space of holomorphic functions. This method is rather limited to special \( J \)'s but the results give very sharp asymptotical formulae (exponential decays of errors). Let us mention that the method of successive diagonalization has been recently applied in [1] by Boutet de Monvel et al. to find the asymptotics of eigenvalues of a modified Jaynes–Cummings model.

This paper only deals with Jacobi matrices with discrete spectra. There is already a large literature devoted to infinite self-adjoint Jacobi matrices and their relation to the theory of orthogonal polynomials. In recent years several methods were found for studies of spectral properties of general Jacobi matrices in \( l^2(\mathbb{N}) \) and \( l^2(\mathbb{Z}) \), see [5,4,8–11]. Finally, let us also mention the recent monograph of Teschl [15].

The paper is organized as follows. In Section 2 the method of successive diagonalization is briefly described. Section 3 contains several natural examples of Jacobi matrices for which analytical models can be found and uses them to compute precise asymptotic behaviour of their eigenvalues. Finally, in Section 4 simple applications of the Rozenbljum abstract theorem are given.

2. Successive diagonalization method

The idea of successive diagonalization was introduced for a special class of Jacobi matrices in [11]. It is based on the following general lemma:

**Lemma 2.1** (Lemma 2.1, Janas and Naboko [11]). Let \( D \) be a self-adjoint diagonal operator in a Hilbert space \( H \) given by \( De_n = \mu_n e_n \), where \( \{e_n\} \) is an orthonormal basis in \( H \) and simple eigenvalues \( \mu_n \to \infty \) are ordered by \( |\mu_k| \leq |\mu_{k+1}| \). Assume that \( |\mu_i - \mu_k| \geq \epsilon_0 \) if \( i \neq k \). If \( R \) is compact (not necessary self-adjoint) operator in \( H \) then the operator \( T = D + R \) has discrete spectrum which consists of complex eigenvalues \( \lambda_n(T) \) and \( \lambda_n(T) = \mu_n + O(\|R e_n\|) \) for large \( n \).

Successive diagonalization method was applied in [11] to a special class of Jacobi operators. It was used there in three steps. Then every next step allows to obtain the asymptotics of the eigenvalues of Jacobi operator more precisely then the previous one. In this article we apply this method using only one step, but it gives (we hope) satisfactory explanation of the main idea of successive diagonalization.

Let \( \Sigma_{1/p} \) \( (p > 0) \) denote the set of compact operators such that its singular numbers \( s_k \) satisfy \( s_k = O(1/k^p) \). By \( \Sigma_{1/p}^b \) we mean the subset of \( \Sigma_{1/p} \) of operators in \( l^2 \) which possess a band type matrix form in the basis \( \{e_n\}_{n=1}^\infty \).

**Theorem 2.2.** Let

\[
J_1 = Q + SW + WS^*,
\]

where \( Q = \text{diag}(q_n) \), \( W = \text{diag}(\lambda_n) \) and assume that

(i) \( q_n = \rho n^2(1 + A_n), A_n \to 0, \rho \in \mathbb{R}\backslash\{0\} \);

\[
\lambda_n = n^\beta(1 + w_n), w_n \to 0;
\]

(ii) \( \beta \geq 0, \quad \alpha > 2\beta + 1; \)

(iii) \( A_{n+1} - A_n = o(1/n) \).

Then the eigenvalues \( \lambda_n(J_1) \) of \( J_1 \) satisfy \( \lambda_n(J_1) = q_n + O(1/n^2 - 2\beta - 1) \) for large \( n \).
Therefore, the operators \(A_1, A_2 \in \Sigma^{b}_{1/(\alpha-\beta-1)}\) are finite dimensional and belong to \(\Sigma^{b}_{1/p}\) by their definitions. Then the operators \(X := (I + SA_1 + A_2S^*)J_1 - Q(I + SA_1 + A_2S^*) \in \Sigma^{b}_{1/p}\) (2.2)
for some \(p\). If we use the notation \([A, B] = AB - BA\) for any operators \(A\) and \(B\) then we have
\[
X = [S, Q]A_1 + A_2[S^*, Q] + SW + WS^* + S1SA1SW + A2W + SA1WS^* + A2S^*WS^*.
\]
Note that \(SA1SW, A2W, SA1WS^*, A2S^*WS^* \in \Sigma^{b}_{1/(2\beta-1)}\) provided \(A_1, A_2\) are chosen as above.

Denote \((\Delta q)(n) := q_{n+1} - q_n\) then \([S, Q]e_n = -(\Delta q)(n)e_{n+1} = SDe_n\), where \(D = \text{diag}(-(\Delta q)(n))\) and similarly we have that \([S^*, Q] = -DS^*\). Next, note that
\[
(\Delta q)(n) = \rho(n + 1)^2(1 + \Delta_{n+1}) - \rho n^2(1 + \Delta_n)
= \rho(n + 1)^2 - \rho n^2 + \rho[(n + 1)^2 - n^2]A_{n+1} + \rho n^2(A_{n+1} - A_n)
\]
and \((n + 1)^2 - n^2 = \omega x^{-1} + O(n^{-2}).\) But \(n^2(A_{n+1} - A_n) = o(n^{-1})\) by (iii), so we have
\[
(\Delta q)(n) = \rho \omega x^{-1} + o(n^{-1}).
\]

There exists a finite dimensional diagonal operator \(Q_1\) such that \(D + Q_1\) is invertible and we can define operators \(A_1 := -(D + Q_1)^{-1}W\) and \(A_2 := W(D + Q_1)^{-1}\). The operators \(A_1\) and \(A_2\) are diagonal and belong to \(\Sigma^{b}_{1/(2\beta)}\) by their definitions. Then the operators
\[
[S, Q]A_1 + SW = S[(D + Q_1)A_1 + W] - SQ_1A_1 = -SQ_1A_1
\]
and
\[
A_2[S^*, Q] + WS^* = A_2Q_1S^*
\]
are finite dimensional and belong to \(\Sigma^{b}_{1/p}\) for every \(p\) and consequently \(X\) belongs to \(\Sigma^{b}_{1/(2\beta-1)}\). Denote \(K = I + SA1 + A2S^*\). We can choose \(Q_1\) such that \(K\) is invertible, then \(KJ_1 - QK = X\), i.e.
\[
J_1 = K^{-1}(Q + XK^{-1})K.
\]

Therefore, \(\sigma(J_1) = \sigma(Q + XK^{-1})\) and by Lemma 2.1
\[
\lambda_n(J_1) = \lambda_n(Q + XK^{-1}) = q_n + O(\|(K^{-1})^*X^*e_n\|) = q_n + O(1/(n^{2-2\beta-1})). \quad \square
\]

3. Special classes of Jacobi matrices with analytic models

In this section we shall study a few examples of unbounded Jacobi matrices in \(L^2\), which do not fall into the class considered in Theorem 2.2. It happens that all examples are defined by sequences of power like behaviour i.e. \(q_n = \delta n^\gamma, \lambda_n = \gamma n^\beta(1 + \Delta_n), \Delta_n \to 0, n \to 0\), with \(\alpha - \beta\) equals, respectively, to 1, \(\frac{1}{2}\), 0. Nevertheless, by using specific forms of their weights and diagonals, we shall find the asymptotic of the eigenvalues. This approach was used for example in [7]. In all three examples, unitarily equivalent operator models of \(J\)'s in some Hilbert spaces of holomorphic functions are used. These model operators turn out to be differential operators. The original equation for the eigenvalues \(Jf = \lambda f\) is transformed into the corresponding differential equation. However, analysis of these differential equations is not so trivial because they do not fall in general into any (known to us) classical classes describing special functions. In this section we investigate the spectra of Jacobi matrices given by (3.1), (3.12), (3.28).

Recall that the Hardy spaces \(H^2 = H^2(D)\) in the unit disc \(D = \{z \in \mathbb{C} : |z| < 1\}\) consists of all holomorphic functions \(f(z) = \sum_{n=0}^\infty q_n z^n\) in \(D\) such that \(\sup_{r < 1} \int_0^{2\pi} |f(re^{i\theta})|^2\) \(d\theta = \sum_{n=0}^\infty |a_n|^2 < \infty\).

Let \(A_0 = \text{diag}(n)_{n=1}^\infty\) be the diagonal operator determined by the sequence \(\{n\}_{n=1}^\infty\).
3.1. Analytic model for an example of a Jacobi matrix with \( \alpha - \beta = 1 \).

We start with analysis of the operator
\[
J_2 = A_0 + cS + cS^*.
\]  
(3.1)

Note that \( J_2 \) is formally in the class of Jacobi matrices defined in Theorem 2.2 with \( \alpha = 1 \) and \( \beta = 0 \). Therefore, it is clear that the assumptions of Theorem 2.2 are not satisfied. The spectrum of \( J_2 \) is discrete and consists of the eigenvalues only. Assume that \( c \neq 0 \) (the case \( c = 0 \) is trivial and \( \sigma(J_2) = \{1, 2, \ldots\} \)).

Recall the first type Bessel functions for \( k \in \mathbb{N} \) and \( x \in \mathbb{C} \) given by the power series \( J_k(x) = \sum_{n=0}^{\infty} (-1)^n 1/(n! (n + k)) (x^2/4)^k \) [17]. Using the Gamma function \( \Gamma(z) \) and taking \( 1/\Gamma(z) = 0 \) for \( z \in \{ -1, -2, \ldots \} \) the Bessel functions can be defined for complex indices by
\[
J_v(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! (n + v + 1)} \left( \frac{x^2}{4} \right)^{n + v}
\]
for \( v \in \mathbb{C} \), \( x \in \mathbb{R} \) if \( J_v(0) = \infty \) if \( \Re v < 0 \). Moreover; \( J_{-k}(x) = (-1)^k J_k(x) \) for \( k \in \mathbb{N} \) [17].

**Theorem 3.1.** Let \( J_2 \) be an operator on \( l^2 \) given by (3.1) then
\[
\sigma(J_2) = \left\{ \lambda \in \mathbb{R} : \lambda + c^2 - 1 = \sum_{k=2}^{\infty} \frac{c^k J_k(2c)}{(k - 2)! (k - \lambda)} \right\} \cup \{ k \in \mathbb{N} : J_k(2c) = 0 \}
\]
\[
= \{ \lambda_n(J_2) : n \geq k_0 \},
\]
where
\[
\lambda_n(J_2) = n - r_n, \quad r_n = O \left( \frac{c^{2n}}{n!(n-1)!} \right), \quad n \geq k_0
\]
for some \( k_0 \).

**Proof.** Let us consider the equation
\[
J_2 f = \lambda f,
\]  
(3.2)

where \( f = \{ f_n \}_{n=1}^{\infty} \). Although, (3.2) is the second order difference equation of the first order, we will change it to a differential equation by the method suggested in [7]. Take the coherent state for the shift operator \( S \) in \( l^2 \) [13] \( f_z = \sum_{n=1}^{\infty} z^{n-1} e_n \) and let \( f = \sum_{n=1}^{\infty} f_n e_n \). Then (3.2) is equivalent to the relation \( (f_z, J_2 f) = \lambda (f_z, f) \) for all \( z \in \mathbb{D} \).

In the other words \( (f_z, A_0 f) + c (f_z, S f) + c (f_z, S^* f) = \lambda (f_z, f) \) for all \( z \in \mathbb{D} \).

Denote by
\[
\Phi(z) := (f_z, f).
\]
(3.3)

Straightforward calculations prove that
\[
(f_z, A_0 f) = \Phi(z) + z\Phi'(z);
\]
\[
(f_z, S f) = z\Phi(z);
\]
\[
(f_z, S^* f) = \frac{1}{z} (\Phi(z) - \Phi(0))
\]
(3.4)

and we get the equation
\[
z\Phi'(z) + (1 + cz)\Phi(z) + \frac{1}{z} (\Phi(z) - \Phi(0)) = \lambda \Phi(z).
\]
(3.5)

Note that (3.2) has a solution \( f \in l^2 \setminus \{0\} \) if and only if (3.5) has solution \( \Phi \in H^2 \setminus \{0\} \). Assume that we have \( \lambda \in \mathbb{R} \) such that (3.5) has a solution \( \Phi \in H^2 \). If \( \Phi(0) = 0 \) then by taking the limit at 0 in (3.5) we get \( c\Phi'(0) = 0 \). Because
\( c \neq 0 \) we have \( \Phi(0) = f_1 = 0 \) and \( \Phi'(0) = f_2 = 0 \). By the difference (3.2) we have \( f = 0 \), so \( \lambda \not\in \sigma_p(J) \). Therefore, instead of (3.5) we can consider the equation
\[
z^2 \Phi'(z) + [cz^2 + (1 - \lambda)z + c] \Phi(z) = c, \quad \Phi(0) = 1. \tag{3.6}
\]
We will look for all parameters \( \lambda \in \mathbb{R} \) for which (3.6) has non trivial solution in \( H^2 \). It is clear that the point spectrum of \( J_2 \) consists of all such numbers \( \lambda \). Let \( \Phi \) be in the form \( \Phi(z) = e^{-cz} \varphi(z) \). Of course, \( \Phi \in H^2 \iff \varphi \in H^2 \). Then (3.5) is equivalent to
\[
z^2 \varphi'(z) + [(1 - \lambda)z + c] \varphi(z) = ce^{cz}, \quad \varphi(0) = 1. \tag{3.7}
\]
Let \( \varphi \) have the power series
\[
\varphi(z) = \sum_{n=0}^{\infty} a_n z^n.
\]
Then (3.7) is satisfied when
\[
a_0 = 1, \\
a c a_1 + (1 - \lambda) a_0 = c^2, \\
(n - \lambda) a_n - 1 + c a_n = \frac{e^{n+1}}{n!}, \quad n \geq 2. \tag{3.8}
\]
Denote \( d_k = (\lambda - k) \ldots (\lambda - 2)(-1)^{k-1}, \quad k \geq 2 \), then we calculate the solution of (3.8)
\[
a_1 = c^{-1}(\lambda + c^2 - 1)
\]
and
\[
a_n = c^{-n} \sum_{k=0}^{n} \frac{e^{2k}}{k!} (\lambda - n) \ldots (\lambda - (k + 1)), \quad n \geq 2. \tag{3.9}
\]
For \( \lambda \not\in \{2, 3, \ldots\} \) we can rewrite (3.9) as
\[
a_n = \frac{(-1)^{n-1} d_n}{e^{n-1}} \left( a_1 + c^{-1} \sum_{k=2}^{n} \frac{(-1)^{k-1} e^{2k}}{d_k k!} \right), \quad n \geq 2. \tag{3.10}
\]
We should answer the question: for which \( \lambda \in \mathbb{R} \) the sequence \( \{a_n\} \in l^2 \).
Consider the case \( \lambda \in \{0, 1, 2, 3, \ldots\} \). Using (3.9)
\[
a_n = c^{-n} \sum_{k=\lambda}^{n} \frac{e^{2k}}{k!} (\lambda - n) \ldots (\lambda - (k + 1)) = \frac{(-1)^{n-\lambda}(n - \lambda)!}{e^{n-2\lambda}} \sum_{k=0}^{n-\lambda} \frac{(-1)^k e^{2k}}{(k + \lambda)!k!}, \quad n > \lambda.
\]
Now, notice that in this case
\[
\{a_n\} \in l^2 \iff \sum_{k=0}^{\infty} \frac{(-1)^k e^{2k}}{(k + \lambda)!k!} = 0 \iff J_{\lambda}(2c) = 0 \iff J_{-\lambda}(2c) = 0.
\]
For \( \lambda \notin \{0, 1, 2, \ldots\} \), using (3.10), the following relations hold:

\[
\{a_n\} \in \ell^2 \iff a_1 = c^{-1} \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!}
\]

\[
\iff c - 1 + e^2 = \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!}
\]

\[
\iff \sum_{k=0}^{\infty} \frac{(-1)^k c^{2k}}{\Gamma(1+k-\lambda)}k! = 0
\]

\[
\iff J_{-\lambda}(2c) = 0.
\]

Because

\[
1 = \sum_{j=2}^k \frac{(-1)^j}{(k-j)!(j-\lambda)}
\]

then we have

\[
\sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!} = \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{d_k k!} \sum_{j=2}^{k} \frac{(-1)^j}{(k-j)!(j-\lambda)} \frac{1}{j-\lambda}
\]

\[
= \sum_{k=2}^{\infty} \frac{(-1)^k c^{2k}}{(k-2)!} \left[ \sum_{j=0}^{\infty} \frac{(-1)^j c^{2(j+k)}}{(j+k)!} \right] \frac{1}{k-\lambda}
\]

\[
= \sum_{k=2}^{\infty} \frac{c^{2k}}{(k-2)!} \left[ \sum_{j=0}^{\infty} \frac{(-1)^j c^{2j}}{(j+k)!} \right] \frac{1}{k-\lambda}
\]

\[
= \sum_{k=2}^{\infty} \frac{c^{2k} J_{k}(2c)}{(k-2)!} \frac{1}{k-\lambda}.
\]

So, by the facts given above and the properties of the Bessel functions, we obtain

\[
\sigma_p(J_2) = \{ \lambda \in \mathbb{R} : J_{-\lambda}(2c) = 0 \}
\]

\[
= \left\{ \lambda \in \mathbb{R} : c - 1 + c^2 = \sum_{k=n+1}^{\infty} \frac{c^{2k} J_{k}(2c)}{(k-2)!} \frac{1}{k-\lambda} \right\} \cup \{ k \in \mathbb{N} : J_k(2c) = 0 \}.
\]

(3.11)

Because \( J_n(2c) > 0 \) for \( n \geq c^2 \), there exists a number \( \lambda \in \sigma_p(J_2) \cap (n-1, n) \) if \( n > c^2 \). Let \( \lambda = n - r_n \in \sigma_p(J_2) \), \( r_n \in (0, 1) \), \( n > c^2 \) and denote \( A_k = c^{2k} J_{k}(2c)/(k-2)! \) then

\[
n - r_n + c^2 - 1 = \sum_{k=2}^{n-1} \frac{A_k}{k-n+1} + \frac{A_n}{r_n} + \sum_{k=n+1}^{\infty} \frac{A_k}{k-n+r_n} \leq \sum_{k=2}^{\infty} |A_k| + \frac{A_n}{r_n},
\]

so \( n + c^2 - 2 - \sum_{k=2}^{\infty} |A_k| \leq A_n/r_n \).

There exists \( n_1 > c^2 \) such that \( \frac{1}{2} n \leq A_n/r_n \) for \( n \geq n_1 \). So, we get \( r_n \leq 2A_n/n, n \geq n_1 \), where \( A_n \leq c^{2n}/(n-2)! \).

\[
\sum_{j=0}^{\infty} c^{2j}/(j+n)! \leq c^{2n}/(n-2)! n!.
\]

Finally, using the above estimation it can be proved that for large \( n \geq n_0 > n_1 \) there exists exactly one number \( \lambda = n - r_n \in (n-1, n) \cap \sigma_p(J_2) \).

Let us mention that it is not difficult to prove the following fact:
Remark 3.2. If $A = \text{diag}(n)^{\pm \infty}_{n=-\infty}$ is a diagonal operator and $S$ is a shift operator in $l^2(\mathbb{Z})$ then $J_2 = A + cS + cS^*$ considered in $l^2(\mathbb{Z})$ has the spectrum equal to $\mathbb{Z}$. \hfill \Box

3.2. Analytic model for an example of a Jacobi matrix with $z - \beta = 0$.

The Jacobi matrix $J = Q + SW + WS^*$ in $l^2(\mathbb{Z})$ with diagonals given by $Q = \text{diag}(\delta n)^{\pm \infty}_{n=-\infty}$, $W = \text{diag}(\sqrt{(n + c)^2 + b^2})^{\pm \infty}_{n=-\infty}$ has been considered in [12] and in [6]. There have been found the formulae for eigenvalues of this operator. Here and in Section 4.1 we investigate Jacobi matrix $J_3$ with the same entries as $J$ but acting on $l^2(\mathbb{N})$. It turns out that spectral analysis of $J_3$ looks differently than that of $J$. Our approach is again based on analytical model of $J_3$ in $H^2$.

This section is devoted to the operator

$$J_3 = \delta A_0 + S(A_0 + cI) + (A_0 + cI)S^*, \quad (3.12)$$

in $l^2(\mathbb{N})$. Assume that $\delta > 2, c > 0$, for such parameters $\sigma(J_3) = \sigma_p(J_3)$ and the spectrum is discrete.

Let us consider the relation $J_3 f = \lambda f$. Again take the coherent state $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$ and $f = \sum_{n=1}^{\infty} f_n e_n$ then using $(f_z, J_3 f) = \lambda (f_z, f)$ we get

$$(f_z, S A_0 f) + c(f_z, S f) + (f_z, A_0 S^* f) + c(f_z, S^* f) + \delta(f_z, A_0 f) = \lambda (f_z, f),$$

for $|z| < 1$. Let $\Phi(z) = (f_z, f)$ then we have

$$(f_z, S A_0 f) = z^2 \Phi'(z) + z \Phi(z),$$

$$(f_z, A_0 S^* f) = \Phi'(z),$$

$$(f_z, S^* f) = \frac{1}{z} (\Phi(z) - \Phi(0))$$

and by (3.4)

$$z^2 \Phi'(z) + z \Phi(z) + cz \Phi(z) + \Phi'(z) + c \frac{1}{z} (\Phi(z) - \Phi(0)) + \delta(\Phi(z) + z \Phi'(z)) = \lambda \Phi(z)$$

(3.14)

and so

$$z(z^2 + \delta z + 1) \Phi'(z) = - ((1 + c)z^2 + (\delta - \lambda)z + c) \Phi(z) + c \Phi(0),$$

for $z \in \mathbb{D}$. As in the previous situation, this differential equation has a solution in $H^2 \setminus \{0\}$ exactly when $J_3 f = \lambda f$ has a solution $f \in l^2(\mathbb{N})$.

Notice that $z^2 + \delta z + 1 = (z - z_1)(z - z_2)$, where $z_1 = - (\delta + \sqrt{\delta^2 - 4})/2, z_2 = - (\delta - \sqrt{\delta^2 - 4})/2$ and $z_1 < -1 < z_2 < 0$.

We consider the following two cases of the parameter $c$.

(I) $c = 0$.

In this case, (3.14) is equivalent to

$$\frac{\Phi'(z)}{\Phi(z)} = - \frac{z + (\delta - \lambda)}{z^2 + \delta z + 1},$$

because

$$\frac{z + (\delta - \lambda)}{z^2 + \delta z + 1} = \frac{A}{z - z_1} + \frac{B}{z - z_2},$$

where $A = (z_1 - \lambda + \delta)/(z_1 - z_2)$ and $B = (\lambda - \delta - z_2)/(z_1 - z_2)$, so $\ln \Phi(z) = - A \ln(z - z_1) - B \ln(z - z_2) + C$, $\ln \Phi(z) = C + (z - z_1)^{-A}(z - z_2)^{-B}, \Phi(z) = C_1(z - z_1)^{-A}(z - z_2)^{-B}$. Therefore, $\Phi \in H^2$ if and only if $(z - z_2)^{-B} \in H^2$, but this condition is equivalent to the fact that $-B = k$, where $k = 0, 1, 2, \ldots$. It follows that the spectrum of $J_3$ consists
We look for an analytic in 

\[ \lambda_k(J_3) = k\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \sqrt{\delta^2 - 4 + \delta} \right) \]  

(3.15)

for \( k = 0, 1, 2, \ldots \)

(II) \( c \in (0, 1] \).

We can assume that \( \Phi(0) = 1 \). Indeed, suppose that \( \Phi \) is an analytic function in some open set that contains 0, \( \Phi(0) = 0 \) and satisfies (3.14) in this set. Then taking the limit at 0 of both sides of (3.14) we get \( \Phi'(0)(c + 1) = 0 \) and so \( \Phi'(0) = 0 \) (because \( c + 1 \neq 0 \)). Thus \( f_1 = \Phi(0) = 0, f_2 = \Phi'(0) = 0 \) and using the relation \( J_3 f = \lambda f \), we have \( f_n = 0, n \geq 1 \) and \( \lambda \) is not the eigenvalue of \( J_3 \).

Now, we will look for the function \( \Phi \) of the form \( \Phi(z) = (z - z_1)^A \varphi(z) \) for some real number \( A \). Notice that \( \Phi \in H^2 \) if and only if \( \varphi \in H^2 \), because \( |z_1| > 1 \).

We have the new equation:

\[ z(z - z_1)^{A+1}(z - z_2)\varphi'(z) = -[(1 + c + A)z^2 + (\delta - \lambda - z_2 A)z + c](z - z_1)^A \varphi(z) + c. \]

One can choose \( A \) such that \((1 + c + A)z^2 + (\delta - \lambda - z_2 A)z + c = (z - z_1)(pz + q)\), for some \( p, q \). It follows that \( pz + q = (A + 1 + c)z + \delta - \lambda + z_1(1 + c) + A(z_1 - z_2) \) and \( A = (\delta(c - 1) - z_1 + \lambda)/(z_1 - z_2) \). Hence

\[ (z - z_2)(z\varphi'(z)) + [(A + 1 + c)z - \varphi'(z)A] \varphi(z) = \frac{c}{(z - z_1)^{A+1}}. \]  

(3.16)

The function of the right hand side of (3.16) is analytic in \( \mathbb{D} \),

\[ \frac{c}{(z - z_1)^{A+1}} = \sum_{n=0}^{\infty} b_n z^n, \]

(the power series converge in \( |z| < |z_1| \)) and

\[ b_0 = c(-z_2)^{A+1}, \]
\[ b_n = c(-1)^n (-z_2)^{A+1+n} \frac{(A + 1)(A + 2) \ldots (A + n)}{n!}, \quad n \geq 1. \]  

(3.17)

We look for an analytic in \( \mathbb{D} \) function \( \varphi(z) = \sum_{n=0}^{\infty} a_n z^n \) satisfying (3.16). Thus the coefficients of \( \varphi \) must satisfy

\[ a_0 = \frac{b_0}{-c z_2} = (-z_2)^A; \]
\[ a_1 = \frac{b_1 - (A + 1 + c)a_0}{-(c + 1)z_2} = \frac{(-z_2)^A}{(c + 1)z_2} [(A + 1)(cz_2^2 + 1) + c]; \]  

(3.18)

and the recurrence relation

\[ (n + A + c)a_{n-1} - (c + n)z_2 a_n = b_n, \quad n \geq 2. \]

This is the first order linear difference equation. We can solve it:

\[ a_n = g_n a_1 - \sum_{k=1}^{n} \frac{b_k}{(c + k)z_2} \prod_{l=k+1}^{n} \frac{l + A + c}{(c + l)z_2} \]  

(3.19)

or

\[ a_n = g_n \left[ a_1 - \sum_{k=2}^{n} \frac{b_k}{(c + k)z_2 g_k} \right], \quad n \geq 2. \]  

(3.20)
if \( l + A + c \neq 0 \) for all \( l \geq 1 \), where

\[
g_n = \prod_{k=2}^{n} \frac{k + A + c}{(c + k)z_2}
\]

Notice that if \( c \notin \{-1, -2, \ldots\} \) then \((c + k)z_2 \neq 0\) for every \( k \in \mathbb{N}\).

**Proposition 3.3.**

1. If \( k + A + c \neq 0 \) for all \( k \in \{2, 3, \ldots\} \) then

\[
\{a_n\} \in l^2 \iff a_1 = \sum_{k=2}^{\infty} \frac{b_k}{z_2(c + k)g_k}.
\]

2. If \( c \in (0, 1] \) and \( k_0 + A + c = 0 \) for some \( k_0 \in \{2, 3, \ldots\} \) then \( \{a_n\} \notin l^2 \).

**Proof.** Ad.1 ⇒ Because \( |g_n| \rightarrow \infty \) it must be that the second term of the product in (3.20) must be convergent to 0.

⇐ In this situation we have (see the definition of \( g_k \))

\[
a_n = g_n \sum_{k=n+1}^{\infty} \frac{b_k}{z_2(c + k)g_k} = \frac{b_{n+1}}{n + 1 + A + c} + \sum_{k=n+2}^{\infty} \frac{b_k}{k + A + c} \prod_{l=n+1}^{k-1} \left( \frac{z_2 - z_2A}{l + A + c} \right).
\]

By (3.17) \( |b_n| \leq C h^n \) and \( |z_2 - z_2A/(n + A + c)| \leq h \) for large \( n \), where \( h = 1/|z_1| + c = |z_2| + c < 1 \), and we have \( |a_n| \leq C h^{n+1} + C \sum_{k=n+2}^{\infty} h^k h^{k-n-2} \leq C (h^{n+1} + h^{n+2}1/(1-h)) < C' h^{n+1} \) for large \( n \), this implies \( \{a_n\} \in l^2 \).

Ad.2. Let \( A = -k_0 - c \), then \( A + l + c \neq 0 \) for all \( l > k_0 \) and using (3.19) and (3.17) for \( n > k_0 \), we have

\[
a_n = \sum_{k=k_0}^{n} \frac{b_k}{z_2(k + c)} \prod_{l=k+1}^{n} \frac{A + l + c}{(c + l)z_2} = c(-z_2)^{A+1} \sum_{k=k_0}^{n} \frac{z_k^2 (-k_0 - c + 1) \cdots (-k_0 - c + k)}{k!} \prod_{l=k+1}^{n} \frac{l - k_0}{c + l}.
\]

Hence we can write (for some constant \( M(c, z_2, k) \))

\[
a_n = M(c, z_2, k_0)z_1^{n+1} \sum_{k=k_0}^{n} \frac{z_2^{2k} (1 - k_0 - c) \cdots (k - k_0 - c)}{k!} \prod_{l=k+1}^{n} \frac{l - k_0}{l + c} = M(c, z_2, k_0) \frac{z_2^{2k_0}}{k_0!} z_1^{n+1} \prod_{l=k+1}^{n} \frac{l - k_0}{l + c} \left[ 1 + \sum_{k=1}^{n-k_0} \frac{z_2^{2k}}{(k_0 + 1) \cdots (k_0 + k)} \prod_{l=k+1}^{n} \frac{l + c}{l - k_0} \right]
\]

and so there is a constant \( M_1(c, z_2, k) > 0 \) such that

\[
|a_n| \geq M_1(c, z_2, k_0)z_1^{n+1} \prod_{l=k_0+1}^{n} \frac{l - k_0}{l + c} \rightarrow +\infty, \quad n \rightarrow \infty. \quad \Box
\]

Let us come back to (3.21). Using (3.17), after some calculations, we obtain

\[
\sum_{k=2}^{\infty} \frac{b_k}{z_2(c + k)g_k} = c(-z_2)^{A+1} \sum_{k=2}^{\infty} \frac{z_2^{2k-2} \gamma_k A + k}{k} \prod_{l=0}^{k-2} \left( \frac{A + 1 + l}{A + 2 + l + c} \right),
\]

where

\[
\gamma_2 = 1, \quad \gamma_k = \prod_{l=2}^{k-1} \left( 1 + \frac{c}{l} \right), \quad k > 2.
\]

(3.22)
Then condition (3.21) from Proposition 3.3 can be transformed to
\[
- \frac{1}{c(c + 1)} [(A + 1)(cz^2 + 1) + c] = \sum_{k=2}^{\infty} z_{2k}^{2k} \gamma_k \frac{A + k}{k} \prod_{l=1}^{k-1} \left( 1 - \frac{c + 1}{A + 1 + l + c} \right).
\]

Denote by \( x = A + 1 + c \), then we have the condition for \( \lambda \) to belong to the spectrum of \( J_3 \):
\[
\lambda \in \sigma_p(J_3) \iff \lambda = (x - 1 - c)(z_1 - z_2) + z_1 - \delta(c - 1),
\]
where \( x \) is a real number such that
\[
- \frac{1}{c(c + 1)} [(x - c)(cz^2 + 1) + c] = \sum_{k=2}^{\infty} z_{2k}^{2k} \gamma_k \frac{x - 1 - c + k}{k} \prod_{l=1}^{k-1} \left( 1 - \frac{c + 1}{x + l} \right).
\]

The computations given below will allow to localize the point \( \lambda \) very precisely. Notice that
\[
\prod_{l=1}^{k-1} \left( 1 - \frac{c + 1}{x + l} \right) = 1 + \sum_{p=1}^{k-1} A_{kp} \frac{x + p}{x + p},
\]
where
\[
A_{21} = -(c + 1),
\]
\[
A_{k1} = -(c + 1) \prod_{l=1}^{k-2} \left( 1 - \frac{c + 1}{l} \right), \quad A_{k,k-1} = -(c + 1) \gamma_k(k - 1); \quad k > 2,
\]
\[
A_{kp} = -(c + 1) \gamma_{p+1} \prod_{l=1}^{k-p-1} \left( 1 - \frac{c + 1}{l} \right); \quad k > 2, \quad 2 \leq p \leq k - 2.
\]
Denote \( d_k = z_{2k}^{2k} \gamma_k(x - 1 - c + k)/k \), then
\[
P(x) = \sum_{k=2}^{\infty} d_k + \sum_{k=2}^{\infty} d_k \sum_{j=1}^{k-1} \frac{A_{kj}}{x + j},
\]
\[
P_1 = \sum_{k=2}^{\infty} \left( \sum_{l=k}^{\infty} d_l A_{l(k-1)} \right) \frac{1}{x + (k - 1)}
\]
\[
= \sum_{k=1}^{\infty} \left( \sum_{l=k+1}^{\infty} z_{2l}^{2l} \gamma_l \frac{x + l - 1 - c}{l} A_{lk} \right) \frac{1}{x + k}
\]
\[
= \sum_{k=1}^{\infty} \left( \sum_{l=k+1}^{\infty} z_{2l}^{2l} \frac{A_{lk}}{l} \right) + \sum_{k=1}^{\infty} \left[ -k \left( \sum_{l=k+1}^{\infty} z_{2l}^{2l} \gamma_l \frac{A_{lk}}{l} \right) + \sum_{l=k+1}^{\infty} z_{2l}^{2l} \gamma_l \frac{l - 1 - c}{l} A_{lk} \right] \frac{1}{x + k}
\]
\[
= \sum_{k=1}^{\infty} \left( \sum_{l=k+1}^{\infty} z_{2l}^{2l} \frac{A_{lk}}{l} \right) + \sum_{k=1}^{\infty} \left[ \sum_{l=k+1}^{\infty} z_{2l}^{2l} \gamma_l \frac{l - k - 1 - c}{l} A_{lk} \right] \frac{1}{x + k}.
\]
Denote
\[
 a = - \left( c z_2^2 + \frac{1}{c(c+1)} + \sum_{k=2}^{\infty} \frac{z_2^{2k} \gamma_k}{k} \right)
\]
and
\[
 b = \frac{c z_2^2}{1+c} - \sum_{k=2}^{\infty} \frac{z_2^{2k} \gamma_k}{k} \frac{k-1-c}{k} - \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \frac{z_2^{2l} \gamma_l}{l} \frac{A_{lk}}{l}.
\]
Notice that \( a < 0 \) because \( c > 0 \) and \( \gamma_k > 0 \). By the above calculations we look for \( x \in \mathbb{R} \) such that
\[
a x + b = \sum_{k=1}^{\infty} \frac{z_2^{2k+2} \gamma_{k+1}}{\gamma_k} \frac{-c A_{k+1} + \sum_{l=k+2}^{\infty} \frac{2(l-k-1) \gamma_l}{\gamma_{k+1}} \frac{A_{lk}}{l} (l-k-1-c)}{x+k} \\
= \sum_{k=1}^{\infty} \frac{z_2^{2k+2} \gamma_{k+1}}{\gamma_k} \frac{-c A_{k+1} + \sum_{j=2}^{\infty} \frac{2(j-1) \gamma_{j+k}}{\gamma_{k+1}} A_{k+j,k} \frac{j-1+c}{k+j} (j-1-c)}{x+k}.
\]
Therefore,
\[
a x + b = c(c+1) \sum_{k=1}^{\infty} \frac{z_2^{2k+2} \gamma_{k+1}^2}{\gamma_k} \gamma_{k+1} \frac{1}{k+1} + \sum_{j=2}^{\infty} \frac{z_2^{2(j-1)} \gamma_{j+k}}{\gamma_{k+1}} \frac{1}{s} \frac{1-c+1}{s} \frac{j-1-c}{j+k} \frac{1}{x+k}.
\]
(3.24)

Denote by
\[
 A_k = c(c+1) z_2^{2(k+1)} \gamma_{k+1}^2 B_k.
\]
(3.25)

Notice that
\[
 \frac{\gamma_{j+k}}{\gamma_{k+1}} = \prod_{s=k}^{j+k-1} \left( 1 + \frac{c}{s} \right) \leq \gamma_j,
\]
\[
 |B_k| \leq 1 + \sum_{j=2}^{\infty} \frac{z_2^{2(j-1)} \gamma_j}{\gamma_{j+k}} \prod_{s=2}^{j-1} \left( 1 - \frac{c}{s} \right) (1+c) =: M_1
\]
and let
\[
 M_2 := c(c+1) M_1 \sum_{k=1}^{\infty} \frac{z_2^{2(k+1)} \gamma_{k+1}^2}{\gamma_k}.
\]
(3.26)

Denote \( f(x) = \sum_{k=1}^{\infty} A_k / (x+k) \). Note two simple facts.

**Fact 1.** Because \( A_k > 0 \), then \( f \) is continuous and decreasing in every interval \((-n-1, -n)\), where \( n = 1, 2, 3, \ldots \) and in \((-1, +\infty)\); moreover \( \lim_{x \to -\infty} f(x) = \pm \infty \). This implies that in \((-n-1, -n)\) there exists exactly one \( x \) such that \( a x + b = f(x) \) (for \( n \in \{1, 2, \ldots \} \)).

Remember that we denoted by \( L(x) \) the left side of Eq. (3.23) and by \( P(x) \) the right side of this equation. We can check that \( L(0) > 0 \) and \( P(0) \leq 0 \) (because we have assumed that \( c \in [0, 1] \)), so it is trivial that \( L(0) > P(0) \), but this implies that \( b > f(0) \). Therefore there exist exactly two points in \((-1, +\infty)\) satisfying equation \( a x + b = f(x) \). Finally \( \{ x \in \mathbb{R} : a x + b = f(x) \} = \{ x_n = -n + r_n : n = 0, 1, \ldots \} \) for some \( r_n \in (0, 1) \), when \( n \geq 2 \).

**Fact 2.** If \( f(x) = a x + b \) and \( x + n = r_n \in (0, 1) \), then there exists \( n_1 \) such that \( |r_n| \leq 2 A_n / |a| n \) for \( n \geq n_1 \).
Theorem 3.4. Let $J_3 = \delta A_0 + S(A_0 + c) + (A_0 + c)S^*$ be a Jacobi operator in $l^2$ and $\delta > 2$ then

$$\sigma(J_3) = \{\lambda_n(J_3) : n = 0, 1, 2, \ldots\},$$

where

1. in the case $c = 0$

$$\lambda_n(J_3) = n\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \sqrt{\delta^2 - 4 + \delta} \right);$$

2. in the case $c \in (0, 1]$

$$\lambda_n(J_3) = n\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \sqrt{\delta^2 - 4 + \delta} \right) + c \left( \sqrt{\delta^2 - 4 - \delta} - r_n\sqrt{\delta^2 - 4} \right),$$

where $|r_n| \leq C(c, \delta)z_2^{2(n+1)}c_{2/n}$ for $n > n_1$ for some large $n_1$.

(Note that $z_2 = -(\delta - \sqrt{\delta^2 - 4})/2 \in (-1, 0)$ and $\gamma_k = \prod_{j=2}^{k-1} (1 + \frac{c}{j})$.)

The spectrum of $J_3$ for the parameter $c > 1$ will be considered in Section 4.1.

3.3. Analytic model for an example of a Jacobi matrix with $\alpha - \beta = \frac{1}{2}$.

In this section we calculate the point spectrum for

$$J_4 = A_0 + \gamma S A_0^{1/2} + \gamma A_0^{1/2} S^*,$$

where $A_0 = \text{diag}(n)_{n=1}^{\infty}$ and $\gamma \in \mathbb{R}$. The reason for presenting this here is twofold. Firstly, the analytic model of $J_4$ uses the Bargmann space. Secondly, knowledge of $\sigma(J_4)$ can be used to compute asymptotic of the eigenvalues of some perturbations of $J_4$. It is easy to check that $J_4$ is unitarily equivalent to the operator $T = \gamma T_z + \gamma T_z^* + T_z^* T_z$, where $T_z$ is the operator of multiplication by the variable $z$ on the Bargmann space $B^2$ of entire functions on $\mathbb{C}$ that belong to $L^2(\mu)$, with $d\mu(z) = \frac{1}{\pi} e^{-|z|^2} dm(z)$, where $m$ is the Lebesgue measure [13].

Then

$$Tf = \lambda f$$

is equivalent to

$$\gamma zf + \gamma f' + (zf)' = \lambda f$$

$$f'/f = (\lambda - 1 - \gamma z)(z + \gamma)^{-1}.$$

Hence

$$f'/f = -\gamma + (\lambda + \gamma^2 - 1)(z + \gamma)^{-1}$$

$$\ln(f) = \ln[C(z + \gamma)^{1/2 + \gamma - 1} e^{-\gamma z}]$$

for some constant $C \neq 0$.

It follows that

$$f = C(z + \gamma)^{1/2 + \gamma - 1} e^{-\gamma z}$$
and so \( f \in B^2 \setminus \{0\} \Leftrightarrow \lambda + \gamma^2 - 1 \in \{0, 1, 2, \ldots\} \). Therefore;

\[
\sigma_p(J_4) = \sigma_p(T) = \{k + 1 - \gamma^2 : k = 0, 1, 2, \ldots\}.
\]

4. Rozenbljum theorem

In this section we are going to find an asymptotic behaviour of eigenvalues for a few Jacobi operators applying the following abstract theorem of Rozenbljum.

**Theorem 4.1 (Rozenbljum [14]).** Assume that \( A \) and \( B \) are self-adjoint bounded below operators having purely discrete spectra and \( 0 \) is not an eigenvalue of any of them. Let \( z \in (-\infty, 1) \). Assume that there exist bounded and boundedly invertible operators \( K_1, K_2 \) and bounded operators \( C_1, C_2 \) such that

\[
K_1^{-1} AM_1 - B = C_1 |B|^z,
\]

\[
K_2^{-1} BM_2 - A = C_2 |A|^z.
\]

If \( \sigma(A) = \{\lambda_n(A) : n \geq 1\} \), \( \sigma(B) = \{\lambda_n(B) : n \geq 1\} \) and \( \lambda_n(A) \leq \lambda_{n+1}(A), \lambda_n(B) \leq \lambda_{n+1}(B) \), \( n \geq 1 \) then

\[
|\lambda_n(A) - \lambda_n(B)| = O(|\lambda_n(A)|^z), \quad n \to \infty. \quad \square
\]

Let us mention that in [14] one can find much general version of Theorem 4.1 that includes similar result also for operators with discrete spectra but not bounded below. We apply Theorem 4.1 to the following Jacobi operators.

At the beginning we will complete analysis of asymptotics of eigenvalues for the Jacobi matrices on \( l^2 = l^2(\mathbb{N}) \) with the entries considered in Section 3.2 with the parameter \( c > 1 \). We consider the operator \( J_3 = \delta A_0 + SW + W S^* \) (like the one given by 3.12), where \( W = A_0 + cI \) for \( c > 1 \). In this case \( \tilde{\alpha} = \tilde{\alpha} + \tilde{c} \) for some \( \tilde{\alpha} \in \{1, 2, \ldots\} \) and \( \tilde{c} \in [0, 1] \). Take the Jacobi operator

\[
\tilde{J} - \tilde{\alpha} \delta I = (\delta A_0 - \tilde{\alpha} \delta I) + S(A_0 + \tilde{c} I) + (A_0 + \tilde{c} I) S^*
\]

\[
= \begin{pmatrix}
(1 - \tilde{\alpha}) \delta & 1 + \tilde{c} & & \\
1 + \tilde{c} & (2 - \tilde{\alpha}) \delta & & \\
& \ddots & \ddots & \ddots \\
& & \tilde{n} - 1 + \tilde{c} & 0 \delta & \tilde{n} + \tilde{c} \\
& & & \tilde{n} + \tilde{c} & 1 + \tilde{n} + \tilde{c} \\
& & & & \ddots \\
& & & & & 1 + \tilde{n} + \tilde{c} & 2 \delta & 2 + \tilde{n} + \tilde{c}
\end{pmatrix}.
\]

Notice that

\[
\tilde{J} - \tilde{\alpha} \delta I = \begin{pmatrix}
M_{\tilde{\alpha}} & S_c \\
S_c^* & J_3
\end{pmatrix},
\]

and so

\[
(\tilde{J} - \tilde{\alpha} \delta I) - \begin{pmatrix}
M_{\tilde{\alpha}} & 0 \\
0 & J_3
\end{pmatrix} = \begin{pmatrix}
0 & S_c \\
S_c^* & 0
\end{pmatrix} = C(\alpha)A_0^{-z},
\]

for some finite dimensional operator \( C(\alpha) \) and for any number \( \alpha \geq 1 \). Observe that

\[
\sigma_p(\tilde{J} - \tilde{\alpha} \delta I) = \{\lambda_n(\tilde{J}) - \tilde{\alpha} \delta : n = 0, 1, 2, \ldots\}
\]

and

\[
\sigma_p \left( \begin{pmatrix}
M_{\tilde{\alpha}} & 0 \\
0 & J_3
\end{pmatrix} \right) = \{z_0, \ldots, z_{\tilde{\alpha} - 1}\} \cup \{\lambda_0(J_3), \lambda_1(J_3), \ldots\}.
\]
We apply the Rozenbljum theorem (Theorem 4.1) to operators $A = (\tilde{J} - \tilde{n}\delta I) + \gamma I$ and $B = (\frac{M_0}{0} J) + \gamma I$, where the parameter $\gamma \in \mathbb{R}$ is such that $\sigma(A), \sigma(B) \subset (0, +\infty)$. Therefore, by the Rozenbljum theorem we obtain
\[
\lambda_n(J_3) = \lambda_n + \tilde{n}\delta + \tilde{r}_n,
\]
where $\tilde{r}_n = O(n^{-\gamma})$ for large $n$ and any natural number $\gamma \geq 1$. Finally, using Theorem 3.4, calculate that
\[
\lambda_n(J_3) = (n + \tilde{n})\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \delta + \sqrt{\delta^2 - 4} \right) + \tilde{c} \left( \sqrt{\delta^2 - 4} - \delta \right) - r_n\sqrt{\delta^2 - 4} - \tilde{n}\delta + \tilde{r}_n
\]
\[
= n\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \delta + \sqrt{\delta^2 - 4} \right) + c \left( \sqrt{\delta^2 - 4} - \delta \right) - \tilde{r}_n,
\]
for $n = 0, 1, 2, \ldots$, where $\tilde{r}_n = O(n^{-\gamma})$ for large $n$ and for any $\gamma \geq 1$.

Now, let us come back to the operator with diagonals like these considered in [12,6]. Let $\lambda_n = \sqrt{n+c}^2 + \tilde{n}$. Take the diagonal operators $Q = \text{diag}(\lambda_n)$ and define the Jacobi operator $J_5 = SW + WS^* + Q$.

Let $B = J_5 + \gamma I$, $A = J_3 + \gamma I$, where $J_3$ is given by (3.12) with the parameter $\delta > 2$ and $\gamma \geq 0$ and $\gamma \in \mathbb{R}$ is such that $\sigma(A), \sigma(B) \subset (0, +\infty)$. Applying Theorem 4.1, take $K_1 = K_2 = I$, $x = -1$. Define
\[
R := \text{diag} \left( \frac{b}{\sqrt{(n+c)^2 + b - (n+c)}} \right), \frac{b}{\sqrt{(n+c)^2 + b - (n+c)}} = O \left( \frac{1}{n} \right).
\]
Then we have $B - A = J_5 - J_3 = SR - RS^* = C_1 A^{-1} = C_2 B^{-1}$ for some bounded operators $C_1$ and $C_2$. Again by the Rozenbljum theorem and (3.27),
\[
\lambda_n(J_5) = \lambda_n(J_3) + O \left( \frac{1}{n} \right) = n\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \delta + \sqrt{\delta^2 - 4} \right) + c \left( \sqrt{\delta^2 - 4} - \delta \right) + O \left( \frac{1}{n} \right)
\]
for large $n$.

Recall that Masson and Repka [12] and Edward [6] proved that if $J_k'$ with a diagonal and weights like $J_5$ is considered as an operator in $l^2(\mathbb{Z})$ then
\[
\sigma_p(J_k') = \left\{ n\sqrt{\delta^2 - 4} + \frac{1}{2} \left( \delta + \sqrt{\delta^2 - 4} \right) + c \left( \sqrt{\delta^2 - 4} - \delta \right) : n \in \mathbb{Z} \right\}.
\]
Moreover, they found exact formulae for eigenvalues of $J_k'$ and proved that the eigenvalues do not depend on the parameter $b \geq 0$. We see that the asymptotic behaviour in $+\infty$ of the point spectra of $J_k'$ and $J_5$ is obviously similar, but generally the eigenvalues of the operator $J_5$ acting in $l^2(\mathbb{N})$ do not equal to formulae that give their asymptotics, what can be observed by looking at (3.23) and (3.24) in Section 3.2.

The final (easy) application of Theorem 4.1 is given to a Jacobi operator which also does not fall into the class described in Theorem 2.1. Let us try to calculate asymptotic behaviour of the point spectrum of $J_6 = SW + WS^* + Q$,
\[
\lambda_n(J_6) = \sqrt{n} + O \left( \frac{1}{\sqrt{n}} \right),
\]
n $\geq 1$. 

However, using the Rozenbljum theorem we can obtain even better result. Notice that $J_6^2 = Q^2 + W^2 + SW^2S^* + (S + QSW)^2 + (SW)^2 + (WS^*)^2$, $Q^2 = A_0$ and $S + QSW = 2S + SD$, where $De_n = \frac{1}{\sqrt{n(n+1)}} \epsilon_n$ for $n = 1, 2, \ldots$. Then $J_6^2 - (A_0 + 2S + 2S^*) \in \Sigma_{1/1}^b$, so using the result of Section 3 and the Rozenbljum theorem we have the asymptotic for the eigenvalues of $J_6^2$ given by $\hat{\lambda}_n(J_6^2) = n + O(\frac{1}{n})$ and the asymptotics for $J_6$ is $\hat{\lambda}_n(J_6) = \sqrt{n} + O\left(\frac{1}{n^{1/2}}\right)$.

References