Ground-state positivity, negativity, and compactness
for a Schrödinger operator in $\mathbb{R}^N$ ★

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Abstract

We treat the Schrödinger operator $\mathcal{A} = -\Delta + q(x) \cdot $ on $L^2(\mathbb{R}^N)$ with the potential $q : \mathbb{R}^N \rightarrow [q_0, \infty)$ bounded below and satisfying some reasonable hypotheses on the growth at infinity (faster than $|x|^2$ as $|x| \rightarrow \infty$). We are concerned primarily with the compactness of the resolvent $(\mathcal{A} - \lambda I)^{-1}$ of $\mathcal{A}$ as an operator on the Banach space $X$,

$$X = \{ f \in L^2(\mathbb{R}^N) : f/\varphi \in L^\infty(\mathbb{R}^N) \}, \quad \|f\|_X = \text{ess sup}_{\mathbb{R}^N} (|f|/\varphi),$$

where $\varphi$ denotes the ground state for $\mathcal{A}$. If $\Lambda$ is the ground state energy for $\mathcal{A}$, we show that the restricted operator $(\mathcal{A} - \lambda I)^{-1} : X \rightarrow X$ is not only bounded, but also compact for $\lambda \in (-\infty, \Lambda)$. In particular, the spectra of $\mathcal{A}$ in $L^2(\mathbb{R}^N)$ and $X$ coincide; each eigenfunction belongs to $X$. As another consequence, we obtain a maximum and an anti-maximum principles.

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1. Introduction

This work is concerned with the positivity, negativity, and the asymptotic behavior at infinity of a weak solution \( u : \mathbb{R}^N \rightarrow \mathbb{R} \) to the (inhomogeneous) stationary Schrödinger equation

\[
-\Delta u + q(x)u = \lambda u + f(x) \quad \text{in} \quad L^2(\mathbb{R}^N),
\]

Here, \( q : \mathbb{R}^N \rightarrow \mathbb{R} \) is a given (electric) potential, \( \lambda \in \mathbb{R} \) is a real (or complex) spectral parameter, and \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) is a given function. Problem (1) is interpreted either in the Friedrichs representation setting in the Hilbert space \( L^2(\mathbb{R}^N) \), or in an operator-theoretical setting in a suitable Banach space \( X \) continuously embedded in \( L^2(\mathbb{R}^N) \). We assume that the potential \( q : \mathbb{R}^N \rightarrow \mathbb{R} \) is a continuous function that satisfies the following standard hypothesis:

\[
q_0 \overset{\text{def}}{=} \inf_{\mathbb{R}^N} q > 0 \quad \text{and} \quad q(x) \rightarrow +\infty \quad \text{as} \ |x| \rightarrow \infty.
\]

It is well known (Davies [6], Edmunds and Evans [8], or Reed and Simon [19]) that, under this hypothesis, the Schrödinger operator

\[
\mathcal{A} \equiv \mathcal{A}_q \overset{\text{def}}{=} -\Delta + q(x) \quad \text{on} \quad L^2(\mathbb{R}^N),
\]

defined to be the Friedrichs extension of \( \mathcal{A}|_{C^2(\mathbb{R}^N)} \), is selfadjoint and positive definite, and its inverse \( \mathcal{A}^{-1} \) on \( L^2(\mathbb{R}^N) \) is compact.

The principal eigenvalue \( \Lambda \equiv \Lambda_q \) of the operator \( \mathcal{A} \) is simple with the associated eigenfunction \( \varphi \equiv \varphi_q \) normalized by \( \varphi > 0 \) throughout \( \mathbb{R}^N \) and \( \|\varphi\|_{L^2(\mathbb{R}^N)} = 1 \). In the physics literature, \( \Lambda \) and \( \varphi \), respectively, are called the ground state energy and the ground state of the Schrödinger operator \( \mathcal{A} \).

The main goal of this article is to compare any solution \( u \) of problem (1) (in the sense of distributions) to the ground state \( \varphi \) under some suitable hypotheses on \( q \), \( \lambda \), and \( f \). More precisely, we investigate if any of the following statements holds:

(i) \( u/\varphi \in L^\infty(\mathbb{R}^N) \), and either
(ii+ \( u \geq c\varphi \) in \( \mathbb{R}^N \) (\( \varphi \)-positivity) ) or else
(ii− \( u \leq -c\varphi \) in \( \mathbb{R}^N \) (\( \varphi \)-negativity),

for some constant \( c > 0 \). Of course, answers depend on \( q \), \( \lambda \), and \( f \).

To begin with, let us focus on the radially symmetric eigenvalue problem

\[
\mathcal{A}v \equiv -\Delta v + q(|x|)v = \lambda v \quad \text{in} \quad L^2(\mathbb{R}^N), \quad 0 \neq v \in L^2(\mathbb{R}^N),
\]

i.e., let \( q(x) \equiv q(r) \) be radially symmetric, \( r = |x| \geq 0 \), and \( f \equiv 0 \) in \( \mathbb{R}^N \). First, consider the harmonic oscillator, that is, \( q(r) = r^2 \) for \( r \geq 0 \). One finds immediately that, except for the ground state \( \varphi \) itself, no other eigenfunction \( v \) of \( \mathcal{A} \) (associated with an eigenvalue \( \lambda \neq \Lambda \)) can satisfy \( v/\varphi \in L^\infty(\mathbb{R}^N) \). We refer to Davies [6, Section 4.3, pp. 113–117] for greater details when \( N = 1 \). On the other hand, if \( q(r) = r^{2+\varepsilon} \) for \( r \geq 0 \) (\( \varepsilon > 0 \) — a constant), then \( v/\varphi \in L^\infty(\mathbb{R}^N) \) holds for every eigenfunction \( v \) of \( \mathcal{A} \), again by results from Davies [6], Corollary 4.5.5 (p. 122) combined with Lemma 4.2.2 (p. 110) and Theorem 4.2.3 (p. 111). We refer to Davies...
and Simon [7, Theorem 6.3, p. 359] and M. Hoffmann-Ostenhof [14, Theorem 1.4(i), p. 67] for the same result under much weaker restrictions on \( q(x) \). In our present article we impose similar restrictions.

From these simple examples it is clear that, if (i) is to be satisfied, then the potential \( q(x) \) has to grow fast enough as \( |x| \to \infty \). We will see in this article that a natural sufficient condition, which implies the validity of (i), should look like

\[
\int_{r_0}^{\infty} \left( \inf_{|x| = r} q(x) \right)^{-1/2} dr < \infty \quad \text{for some } 0 < r_0 < \infty.
\]  

Moreover, a closely related condition on \( q \) is imposed in Alziary and Takáč [1, Theorem 2.1, p. 284] in order to obtain (ii+), whereas condition (5) itself is imposed in Alziary, Fleckinger, and Takáč [2, Theorem 2.1, p. 128] (for \( N = 2 \)) and [3, Theorem 2.1, p. 365] (for \( N \geq 2 \)) to establish (ii−). Again, the harmonic oscillator \( q(x) \equiv |x|^2 \) and a suitably chosen positive function \( f \) provide easy counterexamples to both, (ii+) and (ii−).

In the present work we treat potentials \( q(x) \) that are not necessarily radially symmetric. We impose quite general hypotheses on \( q \), \( \lambda \), and \( f \) that guarantee the validity of each of the statements (i), (ii+), and (ii−). Our method is based on rather precise estimates of the asymptotic behavior at infinity of the (unique) weak solution \( u \) to the Schrödinger equation (1), provided \( \lambda < \Lambda \) and the function \( f \) belongs to one of the following Banach spaces: \( L^2(\mathbb{R}^N) \),

\[ X \equiv X_q \overset{\text{def}}{=} \{ f \in L^2(\mathbb{R}^N) : f/\varphi \in L^\infty(\mathbb{R}^N) \}, \]  

or to its predual space \( X^\circ = L^1(\mathbb{R}^N; \varphi \, dx) \). Their respective norms are denoted by

\[
\| f \|_{L^2(\mathbb{R}^N)} \overset{\text{def}}{=} \left( \int_{\mathbb{R}^N} |f|^2 \, dx \right)^{1/2},
\]

\[
\| f \|_X \overset{\text{def}}{=} \text{ess sup}_{\mathbb{R}^N} (|f|/\varphi), \quad \text{and} \quad \| f \|_{X^\circ} \overset{\text{def}}{=} \int_{\mathbb{R}^N} |f| \varphi \, dx.
\]

Conditions (2) guarantee that, whenever \( -\infty < \lambda < \Lambda \), the resolvent

\[ (A - \lambda I)^{-1} = (\Delta + q(x) \bullet - \lambda I)^{-1} \]

of \( A \) on \( L^2(\mathbb{R}^N) \) is compact from \( L^2(\mathbb{R}^N) \) into itself. Under some additional conditions on \( q(x) \), (5) among them, we will show that also the restriction \( (A - \lambda I)^{-1}|_X \) of \( (A - \lambda I)^{-1} \) to \( X \) is compact from \( X \) into itself. (The restriction exists as a bounded linear operator on \( X \), by the weak maximum principle.) Moreover, \( (A - \lambda I)^{-1} \) extends to a compact linear operator \( (A - \lambda I)^{-1}|_{X^\circ} \) from \( X^\circ \) into itself, with a help from Schauder’s theorem (Edwards [9, Corollary 9.2.3, p. 621] or Yosida [23, Chapter X, Section 4, p. 282]). This compactness is the main new result of our paper stated in Theorem 3.2, together with a few important consequences. In particular, \( (A - \lambda I)^{-1}|_X \) compact implies that every eigenfunction \( v \) of \( A \) in \( L^2(\mathbb{R}^N) \) must belong to \( X \) as well, i.e., \( v/\varphi \in L^\infty(\mathbb{R}^N) \), which means that (i) holds. This is a new approach to problem (i).
From the proof of Theorem 3.2, part (a), we will derive (ii+) whenever \( f \in X, \ 0 \leq f \neq 0 \) in \( \mathbb{R}^N \), and \( \lambda < \Lambda \). This result is stated as Theorem 3.1. Directly from Theorem 3.2, part (c), we will derive also (ii−) whenever \( f \in X, \int_{\mathbb{R}^N} f \varphi \, dx > 0 \), and \( \Lambda < \lambda < \Lambda + \delta \ (\delta > 0 — \text{small enough}) \). This is the anti-maximum principle in Theorem 3.4.

The proof of Theorem 3.2 is based on estimates of \( u(x) \) and \( v(x) \) as \( |x| \to \infty \) (satisfying (1) and (4), respectively) which we establish gradually, first for \( q(x) = Q(|x|) \) radially symmetric from a special class (Q) of “auxiliary” potentials (defined in Section 2), and then for any potential \( q(x) \) satisfying

\[
Q_1(|x|) \leq q(x) \leq Q_2(|x|), \quad x \in \mathbb{R}^N,
\]

where \( Q_1 \) and \( Q_2 \) are some potentials of class (Q), such that \( Q_2/Q_1 \) is bounded on \( \mathbb{R}^N \) and

\[
\int_{r_0}^{\infty} \left( Q_2(r)^{1/2} - Q_1(r)^{1/2} \right) \, dr < \infty \quad \text{for some } 0 < r_0 < \infty.
\]

We note that such a potential \( q(x) \) obeys condition (5).

A key tool in obtaining precise asymptotic estimates of \( u(x) = u(|x|) \) and \( \varphi(x) = \varphi(|x|) \) as \( |x| \to \infty \), for \( q(x) = Q(|x|) \) radially symmetric of class (Q), is a WKB-type formula for the asymptotic behavior at infinity of a positive solution \( \psi : (R, \infty) \to \mathbb{R} \) to the radial Schrödinger equation

\[
-\psi''(r) - \frac{N-1}{r} \psi'(r) + Q(r) \psi(r) = \lambda \psi(r), \quad r > R,
\]

for some \( 0 < R < \infty \) (Lemma 4.1). This asymptotic formula is due to Hartman and Wintner [13, Eq. (xxv), p. 49]. It replaces Titchmarsh’s lemma [22, Section 8.2, p. 165] applied in Alziary and Takáč [1, Lemma 3.2, p. 286], and in Alziary, Fleckinger, and Takáč [2, p. 132] and [3, p. 366], with a different class of “auxiliary” potentials \( Q(r) \).

Asymptotic estimates for radially symmetric solutions of the Schrödinger equation with \( q(x) = Q_j(|x|) \) \( (j = 1, 2) \) are combined with standard comparison results for solutions with different, but pointwise ordered (nonradial) potentials in order to control the asymptotic behavior of these solutions at infinity, and thus retain the compactness of the resolvent from the radially symmetric case (Proposition 8.1). In our approach it is crucial that the ground states \( \varphi_j(x) \equiv \varphi_{Q_j}(|x|) \) corresponding to the potentials \( Q_j(|x|) \) \( (j = 1, 2) \) are comparable, that is, \( \varphi_1/\varphi_2 \in L^\infty(\mathbb{R}^N) \) (by Proposition 5.1) and \( \varphi_2/\varphi_1 \in L^\infty(\mathbb{R}^N) \) (by Corollary 8.2).

This article is organized as follows. In the next section (Section 2) we describe the type of potentials \( q(x) \) we are concerned with, together with some basic notations. Section 3 contains our main results and a few examples of potentials to which they apply. These results are proved in Sections 4–9.

2. Hypotheses and notations

We consider the Schrödinger equation (1), i.e.,

\[-\Delta u + q(x)u = \lambda u + f(x) \quad \text{in } L^2(\mathbb{R}^N).\]
Here, \( f \in L^2(\mathbb{R}^N) \) is a given function, \( \lambda \in \mathbb{C} \) is a complex parameter, and the potential \( q : \mathbb{R}^N \to \mathbb{R} \) is a continuous function; we always assume that \( q \) satisfies (2), i.e.,

\[
q_0 \overset{\text{def}}{=} \inf_{\mathbb{R}^N} q > 0 \quad \text{and} \quad q(x) \to +\infty \quad \text{as} \quad |x| \to \infty.
\]

We interpret Eq. (1) as the operator equation \( Au = \lambda u + f \) in \( L^2(\mathbb{R}^N) \), where the Schrödinger operator (3),

\[
A \equiv A_q \overset{\text{def}}{=} -\Delta + q(x) \quad \text{on} \quad L^2(\mathbb{R}^N),
\]

is defined formally as follows. We first define the quadratic (Hermitian) form

\[
Q_q(v, w) \overset{\text{def}}{=} \int_{\mathbb{R}^N} (\nabla v \cdot \nabla \bar{w} + q(x)v\bar{w}) \, dx
\]

for every pair \( v, w \in \mathcal{V}_q \) where

\[
\mathcal{V}_q \overset{\text{def}}{=} \{ f \in L^2(\mathbb{R}^N) : Q_q(f, f) < \infty \}.
\]

Then \( A \) is defined to be the Friedrichs representation of the quadratic form \( Q_q \) in \( L^2(\Omega) \); \( L^2(\Omega) \) is endowed with the natural inner product

\[
(v, w)_{L^2(\mathbb{R}^N)} \overset{\text{def}}{=} \int_{\mathbb{R}^N} v\bar{w} \, dx, \quad v, w \in L^2(\Omega).
\]

This means that \( A \) is a positive definite, selfadjoint linear operator on \( L^2(\Omega) \) with domain \( \text{dom}(A) \) dense in \( \mathcal{V}_q \) and

\[
\int_{\mathbb{R}^N} (Av)\bar{w} \, dx = Q_q(v, w) \quad \text{for all} \quad v, w \in \text{dom}(A);
\]

see Kato [15, Theorem VI.2.1, p. 322]. Notice that \( \mathcal{V}_q \) is a Hilbert space with the inner product \( (v, w)_q = Q_q(v, w) \) and the norm \( \|v\|_{\mathcal{V}_q} = ((v, v)_q)^{1/2} \). The embedding \( \mathcal{V}_q \hookrightarrow L^2(\mathbb{R}^N) \) is compact, by (2).

The principal eigenvalue \( \Lambda \equiv \Lambda_q \) of the operator \( A \equiv A_q \) can be obtained from the Rayleigh quotient

\[
\Lambda \equiv \Lambda_q = \inf \{ Q_q(f, f) : f \in \mathcal{V}_q \text{ with } \|f\|_{L^2(\mathbb{R}^N)} = 1 \}, \quad \Lambda > 0.
\]

This eigenvalue is simple with the associated eigenfunction \( \varphi \equiv \varphi_q \) normalized by \( \varphi > 0 \) throughout \( \mathbb{R}^N \) and \( \|\varphi\|_{L^2(\mathbb{R}^N)} = 1; \varphi \) is a minimizer for the Rayleigh quotient above. The reader is referred to Edmunds and Evans [8] or Reed and Simon [19, Chapter XIII] for these and other basic facts about Schrödinger operators.
We set $r = |x|$ for $x \in \mathbb{R}^N$, so $r \in \mathbb{R}_+$, where $\mathbb{R}_+ \overset{\text{def}}{=} [0, \infty)$. If $q$ is a radially symmetric potential, $q(x) = q(r)$ for $x \in \mathbb{R}^N$, then also the eigenfunction $\varphi$ must be radially symmetric. This follows directly from $\Lambda$ being a simple eigenvalue.

Since our technique is based on a perturbation argument for a relatively small perturbation of a radially symmetric potential, which is assumed to satisfy certain differentiability and growth conditions in the radial variable $r = |x|$, $r \in \mathbb{R}_+$, we bound the potential $q : \mathbb{R}^N \to \mathbb{R}$ by such radially symmetric potentials from below and above.

In order to formulate our hypotheses on the potential $q(x)$, $x \in \mathbb{R}^N$, we first introduce the following class (Q) of auxiliary functions $Q(r)$ of $r = |x| \geq 0$:

(Q) $Q : \mathbb{R}_+ \to (0, \infty)$ is a locally absolutely continuous function that satisfies the following conditions, for some $0 < r_0 < \infty$:

\begin{equation}
\int_{r_0}^{\infty} Q(r)^{-1/2} \, dr < \infty, \tag{13}
\end{equation}

and there is a constant $\gamma$, $1 < \gamma \leq 2$, such that

\begin{equation}
\int_{r_0}^{\infty} \left| \frac{d}{dr}(Q(r)^{-1/2}) \right|^\gamma Q(r)^{1/2} \, dr < \infty. \tag{14}
\end{equation}

Condition (14) is taken from Hartman’s monograph [12, Exercise 17.5, part (b), p. 320]. Originally, it appeared in the work of Hartman and Wintner [13], on p. 49, Eq. (xxiv), and on p. 80, Eq. (157), in an equivalent form

\begin{equation}
\int_{r_0}^{\infty} \left| \frac{Q'(r)}{Q(r)} \right|^\gamma (Q(r)^{1/2})^{1-\gamma} \, dr < \infty,
\end{equation}

where we have corrected the exponent $\gamma - 1$ to $1 - \gamma$.

Condition (14) replaces another condition,

\begin{equation}
\frac{d}{dr}(Q(r)^{-1/2}) \to 0 \quad \text{as } r \to \infty, \tag{15}
\end{equation}

from [12, Exercise 17.5, part (a), p. 320]. Also this condition appeared originally in [13], on p. 49, as the last condition in Eq. (xxii), and on p. 79, Eq. (153).

The following remarks are essential for understanding potentials of class (Q).

**Remark 2.1.** We claim that $Q(r) \to \infty$ as $r \to \infty$, which can be verified as follows. Owing to $1 < \gamma \leq 2$, the conjugate exponent $\gamma' = \gamma/(\gamma - 1)$ satisfies $2 \leq \gamma' < \infty$. Hence, for $r_0 < r \leq s < \infty$ we have

\begin{equation}
Q(s)^{-1/\gamma'} - Q(r)^{-1/\gamma'} = -\frac{1}{\gamma'} \int_r^s Q'(t)Q(t)^{-((1/\gamma')-1)} \, dt
\end{equation}
\[ = \frac{2}{\gamma'} \int_r^s \frac{d}{dr} (Q(t)^{-1/2}) (Q(t)^{1/2})^{1/\gamma} (Q(t)^{-1/2})^{1/\gamma'} \, dr. \]

We apply Hölder’s inequality to estimate

\[ |Q(s)^{-1/\gamma'} - Q(r)^{-1/\gamma'}| \leq \frac{2}{\gamma'} \left( \int_r^s \left| \frac{d}{dr} (Q(t)^{-1/2}) \right|^{\gamma} (Q(t)^{1/2}) \, dr \right)^{1/\gamma} \left( \int_r^s (Q(t)^{-1/2}) \, dr \right)^{1/\gamma'}. \]

Conditions (13) and (14) guarantee first that the limit \( L = \lim_{r \to \infty} Q(r)^{-1/\gamma'} \) exists in \( \mathbb{R}_+ \), then also \( L = 0 \) by (13) again.

**Remark 2.2.** Notice that there is no potential \( Q(r) \) of class (Q) that would satisfy both conditions (13) and (14) with \( \gamma = 1 \). Namely, by arguments similar to those used in Remark 2.1, one can show that the limit \( L = \lim_{r \to \infty} \log Q(r) \) exists in \( \mathbb{R} \), that is, \( \lim_{r \to \infty} Q(r) = e^L \in (0, \infty) \) which contradicts (13).

**Remark 2.3.** With the Liouville substitution

\[ \varrho(r) = \int_r^{r_0} Q(r)^{1/2} dr \quad \text{for } r \geq r_0 \]  

(see Hartman [12, Eq. (2.36), p. 331]), we have \( \frac{d}{dr} Q(r) = Q(r)^{1/2} > 0 \) and, therefore, conditions (13) and (14), respectively, read

\[ \int_0^\infty Q(r(\varrho))^{-1} d\varrho < \infty, \]  

\[ \int_0^\infty \left| \frac{d}{d\varrho} \log Q(r(\varrho)) \right|^{\gamma} d\varrho < \infty. \]

Writing \( x = rx' \) \( (x \in \mathbb{R}^N \setminus \{0\}) \) with the radial and azimuthal variables \( r = |x| \) and \( x' = x/|x| \), respectively, we impose the following restrictions on the growth of \( q(x) \) in \( r \) and the variation of \( q(x) \) in \( x' \).

**Hypothesis.** We assume that

\( (H_q) \) there exist two functions \( Q_1, Q_2 : \mathbb{R}_+ \to (0, \infty) \) of class (Q) such that the inequalities

\[ Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_{12} Q_1(|x|) \quad \text{hold for all } x \in \mathbb{R}^N, \]
with a constant $0 < C_{12} < \infty$, and for some $0 < r_0 < \infty$,

$$\int_{r_0}^{\infty} \frac{Q_2(r) - Q_1(r)}{(Q_1(r) + Q_2(r))^{1/2}} \, dr < \infty. \quad (20)$$

Notice that, assuming (19), the latter condition, (20), is equivalent with (8).

In fact, it suffices to assume inequalities (19) only for all $|x| = r > r_0$ with $r_0 > 0$ large enough, provided $q : \mathbb{R}^N \to (0, \infty)$ is a continuous function. Indeed, then one can find some extensions $\tilde{Q}_1, \tilde{Q}_2 : \mathbb{R}_+ \to (0, \infty)$ of class $(Q)$ of (the restrictions of) functions $Q_1, Q_2 : [r_0 + 1, \infty) \to (0, \infty)$, respectively, from $[r_0 + 1, \infty)$ to $\mathbb{R}_+$ such that $\tilde{Q}_j(r) = Q_j(r)$ for $r \geq r_0 + 1$; $j = 1, 2$, and inequalities (19) hold for all $x \in \mathbb{R}^N$ with $\tilde{Q}_j$ in place of $Q_j$.

3. Main results and examples

For any complex number $\lambda \in \mathbb{C}$ that is not an eigenvalue of the operator $\mathcal{A} = -\Delta + q(x) \bullet$ on $L^2(\mathbb{R}^N)$, we denote by

$$(\mathcal{A} - \lambda I)^{-1} = (-\Delta + q(x) \bullet - \lambda I)^{-1}$$

the resolvent of $\mathcal{A}$ on $L^2(\mathbb{R}^N)$ given by

$$u(x) = \left[(\mathcal{A} - \lambda I)^{-1} f\right](x), \quad x \in \mathbb{R}^N.$$

Now let us fix any real number $\lambda < \Lambda$ and consider the resolvent $(\mathcal{A} - \lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$. By the weak maximum principle (see the proof of Proposition 5.1), the operator $(\mathcal{A} - \lambda I)^{-1}$ is positive, that is, for $f \in L^2(\mathbb{R}^N)$ and $u = (\mathcal{A} - \lambda I)^{-1} f$ we have

$$f \geq 0 \quad \text{a.e. in } \mathbb{R}^N \quad \Rightarrow \quad u \geq 0 \quad \text{a.e. in } \mathbb{R}^N. \quad (21)$$

Consequently, given any constant $C > 0$, we have also

$$|f| \leq C\varphi \quad \text{in } \mathbb{R}^N \quad \Rightarrow \quad |u| \leq C(\Lambda - \lambda)^{-1}\varphi \quad \text{in } \mathbb{R}^N, \quad (22)$$

by linearity. We denote by $\mathcal{K}|_X$ the restriction of $\mathcal{K} = (\mathcal{A} - \lambda I)^{-1}$ to the Banach space $X$ defined in (6). Hence, $\mathcal{K}|_X$ is a bounded linear operator on $X$ with the operator norm $\leq (\Lambda - \lambda)^{-1}$, by (22).

Clearly, $X$ is the dual space of the Lebesgue space $X^\odot = L^1(\mathbb{R}^N; \varphi \, dx)$ with respect to the duality induced by the natural inner product on $L^2(\mathbb{R}^N)$. The embeddings

$$X \hookrightarrow L^2(\mathbb{R}^N) \hookrightarrow X^\odot$$

are dense and continuous. Furthermore, $\mathcal{K}$ possesses a unique extension $\mathcal{K}|_{X^\odot}$ to a bounded linear operator on $X^\odot$ (by Lemma 4.3). Finally, it is obvious that $\mathcal{K}|_X : X \to X$ is the adjoint of $\mathcal{K}|_{X^\odot} : X^\odot \to X^\odot$. 
3.1. Main theorems

Throughout this subsection we assume that $q(x)$ is a potential that satisfies hypothesis $(H_q)$. Under this hypothesis we are able to show the following ground-state positivity of the weak solution to the Schrödinger equation (1) in $X^\odot$.

**Theorem 3.1.** Let hypothesis $(H_q)$ be satisfied and let $-\infty < \lambda < \Lambda$. Assume that $f \in X^\odot$ satisfies $f \geq 0$ almost everywhere and $f \neq 0$ in $\mathbb{R}^N$. Then the (unique) solution $u \in X^\odot$ to Eq. (1) (in the sense of distributions on $\mathbb{R}^N$) is given by $u = (A - \lambda I)^{-1}|X^\odot f$ and satisfies $u \geq c\varphi$ almost everywhere in $\mathbb{R}^N$, with some constant $c \equiv c(f) > 0$.

In the literature, the inequality $u \geq c\varphi$ is often called briefly $\varphi$-positivity. In Protter and Weinberger [17, Chapter 2, Theorem 10, p. 73], a similar result is referred to as the generalized maximum principle.

This result has been established in Alziary and Takáč [1, Theorem 2.1, p. 284] under somewhat different hypotheses on the potential $q(x)$ using a different class $(Q)$ where $Q(r)$ still satisfies a condition similar to (13), but is required to be monotone increasing (i.e., nondecreasing) on some interval $(r_0, \infty)$, $r_0 > 0$, instead of condition (14). Our proof of Theorem 3.1 follows a similar pattern as in [1]; Lemma 3.2 on p. 286 in [1] needs to be replaced by Lemma 4.1 in our present paper. Theorem 3.1 will be proved first for $q(x) = Q(|x|)$ of class $(Q)$, as Proposition 5.1 in Section 5, and then in its full generality in Section 9.2, after the proof of Theorem 3.2, a part of which will be needed (stated below as Corollary 3.3).

The following compactness result is the most important new result of our present paper. It opens new ways to approach several classical problems for Schrödinger operators, such as domination of any eigenfunction by the ground state, an anti-maximum principle for the Schrödinger equation, and independence of the spectrum from the choice of space among $L^2(\mathbb{R}^N)$, $X$, or $X^\odot$.

**Theorem 3.2.** Let hypothesis $(H_q)$ be satisfied. Then we have the following three statements for the resolvent $K = (A - \lambda I)^{-1}$ of $A$ on $L^2(\mathbb{R}^N)$.

(a) If $-\infty < \lambda < \Lambda$ then both operators $K|_X : X \to X$ and $K|_{X^\odot} : X^\odot \to X^\odot$ are compact (and positive, see (21)).

(b) If $\lambda \in \mathbb{C}$ is an eigenvalue of $A$, that is, $Av = \lambda v$ for some $v \in L^2(\mathbb{R}^N)$, $v \neq 0$, then $v \in X (\subset L^2(\mathbb{R}^N) \subset X^\odot)$ and $\lambda \in \mathbb{R}$, $\lambda \geq \Lambda$.

(c) If $\lambda \in \mathbb{C}$ is not an eigenvalue of $A$, then the restriction $K|_X$ of $K$ to $X$ is a bounded linear operator from $X$ into itself and, moreover, $K$ possesses a unique extension $K|_{X^\odot}$ to a bounded linear operator from $X^\odot$ into itself. Again, both operators $K|_X : X \to X$ and $K|_{X^\odot} : X^\odot \to X^\odot$ are compact.

Part (a) is the most difficult one to prove. Since $K|_X : X \to X$ is compact if and only if $K|_{X^\odot} : X^\odot \to X^\odot$ is compact, by Schauder’s theorem (Edwards [9, Corollary 9.2.3, p. 621] or Yosida [23, Chapter X, Section 4, p. 282]), it suffices to prove that either of them is compact. Thus, our proof of part (a) begins with the compactness of the restriction of $K|_X$ to (the corresponding subspace of) radially symmetric functions with $q(x) = Q(|x|)$ of class $(Q)$ and only for $\lambda < \Lambda$, see Lemma 7.2. So we may apply Schauder’s theorem to get the compactness of the restriction of $K|_{X^\odot}$ to radially symmetric functions with $q = Q$. Then we extend this result to $K|_{X^\odot}$ on $X^\odot$ with $q = Q$ again, see Proposition 7.1. Finally, from there we derive that $K|_{X^\odot}$ is...
compact for any \( q(x) \) satifying hypothesis (H_\( q \)), first only for \( \lambda < \Lambda \) and then for any \( \lambda \in \mathbb{C} \) that is not an eigenvalue of \( \mathcal{A} \), see Section 9.1. Parts (b) and (c) are proved immediately thereafter; they will be derived from part (a) by standard arguments based on the Riesz–Schauder theory of compact linear operators (Edwards [9, Section 9.10, pp. 677–682] or Yosida [23, Chapter X, Section 5, pp. 283–286]).

An important by-product of our proof of Theorem 3.2, part (a), is the following comparison result for the ground states.

**Corollary 3.3.** Let hypothesis (H_\( q \)) be satisfied. Then the ground states \( \varphi_\lambda, \varphi_{Q_1}, \) and \( \varphi_{Q_2} \) corresponding to the potentials \( q, Q_1, \) and \( Q_2 \), respectively, are comparable, that is, there exist some constants \( 0 < \gamma_1 \leq \gamma_2 < \infty \) such that \( \gamma_1 \varphi_\lambda \leq \varphi_{Q_j} \leq \gamma_2 \varphi_\lambda \) in \( \mathbb{R}^N; \ j = 1, 2. \) Equivalently, we have \( X_q = X_{Q_1} = X_{Q_2}. \)

Another interesting consequence of Theorem 3.2, part (c), is the anti-maximum principle for the Schrödinger operator \( \mathcal{A} = -\Delta + q(x) \bullet \) which complements the ground-state positivity of Theorem 3.1.

**Theorem 3.4.** Let hypothesis (H_\( q \)) be satisfied and let \( f \in X \) satisfy \( \int_{\mathbb{R}^N} f \varphi \, dx > 0. \) Then there exists a number \( \delta \equiv \delta(f) > 0 \) such that, for every \( \lambda \in (\Lambda, \Lambda + \delta) \), the inequality \( u \leq -c \varphi \) is valid a.e. in \( \mathbb{R}^N \) with some constant \( c \equiv c(f) > 0. \)

This theorem has been obtained in Alziary, Fleckinger, and Takáč [2, Theorem 2.1, p. 128] (for \( N = 2 \)) and [3, Theorem 2.1, p. 365] (for \( N \geq 2 \)) under different hypotheses on the potential \( q(x) \) assuming that \( q(x) = q(|x|) \) is radially symmetric and bounded below by \( Q(|x|) \) using a different class \( \mathcal{Q} \). In particular, in addition to (13), \( Q(r) \) is required to be monotone increasing on some interval \( (r_0, \infty), \ r_0 > 0, \) instead of condition (14). Furthermore, in [2,3] the function \( f \) is assumed to be a “sufficiently smooth” perturbation of a radially symmetric function from \( X \).

Theorem 3.4 is an immediate consequence of the spectral decomposition of the resolvent of \( \mathcal{A} \) as

\[(\lambda I - \mathcal{A})^{-1} = (\lambda - \Lambda)^{-1} \mathcal{P} + \mathcal{H}(\lambda) \quad \text{for} \ 0 < |\lambda - \Lambda| < \eta, \] (23)

see, e.g., Sweers [20, Theorem 3.2(ii), p. 259] or Takáč [21, Eq. (6), p. 67]. Here, \( \lambda \in \mathbb{C} \), \( \eta > 0 \) is small enough, \( \mathcal{P} \) denotes the spectral projection onto the eigenspace spanned by \( \varphi \), and \( \mathcal{H}(\lambda) : L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N) \) is a holomorphic family of compact linear operators parameterized by \( \lambda \) with \( |\lambda - \Lambda| < \eta \). Moreover, \( \mathcal{P} \) is selfadjoint and \( \mathcal{P} \mathcal{H}(\lambda) = \mathcal{H}(\lambda) \mathcal{P} = 0 \) on \( L^2(\mathbb{R}^N) \). Formula (23) is used to prove the anti-maximum principle also in Alziary, Fleckinger, and Takáč [2, Eq. (6), p. 124] and [3, Eq. (6), p. 361]. The main idea of the proof of Theorem 3.4 is to show that each of the linear operators \( \{\mathcal{H}(\lambda) : |\lambda - \Lambda| < \eta\} \) is bounded not only on \( L^2(\mathbb{R}^N) \) but also on \( X \). Clearly, given the Neumann series expansion of \( \mathcal{H}(\lambda) \) for \( |\lambda - \Lambda| < \eta \), it suffices to show that the restriction \( \mathcal{H}(\lambda)|_X \) of \( \mathcal{H}(\lambda) \) to \( X \) is a bounded linear operator on \( X \). But this clearly follows from Theorem 3.2, part (c), with a help from formula (6.32) in Kato [15, Chapter III, Section 6.5, p. 180] or formula (1) in Yosida [23, Chapter VIII, Section 8, p. 228].

In various common versions of the anti-maximum principle in a bounded domain \( \Omega \subset \mathbb{R}^N, \ N \geq 1, \) besides the assumption \( 0 \leq f \neq 0 \) in \( \Omega \), it is only assumed that \( f \in L^p(\Omega) \) for some \( p > N \) (cf. Clément and Peletier [5, Theorem 1, p. 222], Sweers [20] or Takáč [21]). For \( \Omega = \mathbb{R}^N \) the authors [3, Example 4.1, pp. 377–379] have constructed an example of a simple potential \( q(r) \).
and a function \( f(r) \), both radially symmetric, \( f \in L^2(\mathbb{R}^N) \setminus X \), and \( 0 \leq f \neq 0 \) in \( \mathbb{R}^N \), in which even the inequality \( u \leq 0 \) a.e. in \( \mathbb{R}^N \) (weaker than the anti-maximum principle of Theorem 3.4) is violated. More precisely, if \( |\lambda - \Lambda| > 0 \) is small enough, then even \( u(r) > 0 \) for every \( r > 0 \) large enough.

3.2. Some examples of potentials

Here we give a few examples of radially symmetric potentials \( q(x) = q(r) \) that do or do not belong to class \((Q)\). These examples illustrate how “large” class \((Q)\) actually is.

First, we give a typical example of two nonmonotone potentials \( q(r) \), with rather rapidly oscillating derivative \( q'(r) \), which (under a simple condition) do or do not belong to class \((Q)\).

**Example 3.5.** Define the “saw tooth” function \( \theta : \mathbb{R} \to [0, 1] \) by \( \theta(t) \overset{\text{def}}{=} \min_{k \in \mathbb{Z}} |t - 2k| \) for \( t \in \mathbb{R} \), where \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \). Hence, \( \theta \) is a continuous periodic function on \( \mathbb{R} \) with period 2. In particular, \( \theta'(t) = \pm 1 \) whenever \( t \in \mathbb{R} \setminus \mathbb{Z} \).

(a) Take \( q : \mathbb{R}_+ \to (0, \infty) \) with

\[
q(r)^{1/2} = \begin{cases} 
1 + \theta(1) & \text{for } 0 \leq r < 1; \\
 r^{\alpha}(1 + \theta(r^\beta)) & \text{for } r \geq 1,
\end{cases}
\]

where \( \alpha, \beta \geq 0 \) are some constants to be determined. Clearly, \( Q = q \) satisfies condition (13) if and only if \( \alpha > 1 \). Now we compute

\[
\frac{d}{dr} \left( q(r)^{-1/2} \right) = -\left( \frac{\alpha}{r^\beta} + \frac{\beta \theta'(r^\beta)}{1 + \theta(r^\beta)} \right) r^{\beta-1} q(r)^{-1/2}, \quad r > 1,
\]

which yields, for \( \alpha > 1 \) and \( \beta \geq 0 \),

\[
c_1 r^{\beta-\alpha-1} \leq \left| \frac{d}{dr} \left( q(r)^{-1/2} \right) \right| \leq c_2 r^{\beta-\alpha-1}, \quad r > r_0,
\]

where \( 0 < c_1 < c_2 < \infty \) are some constants and \( r_0 = \max\{r_1, 1\} \),

\[
r_1 = \begin{cases} 
1 & \text{if } \beta = 0; \\
(4\alpha/\beta)^{1/\beta} & \text{if } \beta > 0.
\end{cases}
\]

It follows that, for \( r > r_0 \),

\[
c_1 r^{(\beta-1)\gamma-\alpha(\gamma-1)} \leq \left| \frac{d}{dr} \left( q(r)^{-1/2} \right) \right|^\gamma q(r)^{1/2} \leq 2 c_2 r^{(\beta-1)\gamma-\alpha(\gamma-1)}.
\]

This shows that condition (14) holds if and only if \( \beta \in \mathbb{R}_+ \) satisfies \((\beta - 1)\gamma - \alpha(\gamma - 1) < -1\), i.e.,

\[
0 \leq \beta < (\alpha + 1) \left( 1 - \frac{1}{\gamma} \right) = (\alpha + 1)/\gamma'.
\]
Consequently, given any $\beta$ with $0 \leq \beta \leq (\alpha + 1)/2$, we may take $\gamma = 2$ to satisfy condition (14) with $Q = q$. On the other hand, if $\beta > (\alpha + 1)/2$ then (14) cannot be satisfied for any $1 < \gamma \leq 2$.

(b) Now let $q(r)^{1/2} = e^{\alpha r}(1 + \theta(e^{\beta r}))$ for $r \geq 0$, where $\alpha, \beta \geq 0$ are some constants to be determined. Condition (13) for $Q = q$ holds if and only if $\alpha > 0$. Now we compute

$$\frac{d}{dr}(q(r)^{-1/2}) = -\left(\frac{\alpha}{e^{\beta r}} + \frac{\beta \theta'(e^{\beta r})}{1 + \theta(e^{\beta r})}\right)e^{\beta r}q(r)^{-1/2}, \quad r \geq 0,$$

which yields, for $\alpha > 0$ and $\beta \geq 0$,

$$c_1e^{(\beta - \alpha)r} \leq \left|\frac{d}{dr}(q(r)^{-1/2})\right| \leq c_2e^{(\beta - \alpha)r}, \quad r \geq r_0,$$

where $0 < c_1 < c_2 < \infty$ are some constants and

$$r_0 = \begin{cases} 0 & \text{if } \beta = 0; \\ \frac{1}{\beta} \log + \frac{4\alpha}{\beta} & \text{if } \beta > 0, \end{cases}$$

with $t^+ \overset{\text{def}}{=} \max\{t, 0\}$ for $t \in \mathbb{R}$. It follows that, for $r \geq r_0$,

$$c_1e^{(\beta \gamma - \alpha(\gamma - 1))r} \leq \left|\frac{d}{dr}(q(r)^{-1/2})\right|^\gamma q(r)^{1/2} \leq 2c_2e^{(\beta \gamma - \alpha(\gamma - 1))r}.$$ 

Thus, condition (14) holds if and only if $\beta \in \mathbb{R}_+$ satisfies $\beta \gamma - \alpha(\gamma - 1) < 0$, i.e.,

$$0 \leq \beta < \alpha\left(1 - \frac{1}{\gamma}\right) = \alpha/\gamma'. \quad (25)$$

Consequently, given any $\beta$ with $0 \leq \beta \leq \alpha/2$, we may take $\gamma = 2$ to satisfy condition (14) with $Q = q$. If $\beta > \alpha/2$ then (14) cannot be satisfied for any $1 < \gamma \leq 2$.

Now we give an example of a monotone increasing potential $q(r)$ which does not belong to class $(Q)$; it fails to satisfy condition (14) for any $\gamma > 0$. In this example, for $r \in \mathbb{R}_+$ we set either $q'(r) = 0$ or else $q'(r) = 2q(r)^{3/2}$, which yields a “very fast” growth of $q(r)$ on a sequence of pairwise disjoint, nonempty intervals $(n - \varrho_n, n + \varrho_n)$; $n = 1, 2, 3, \ldots$, of total length $2 \sum_{n=1}^\infty \varrho_n = 1$, where $\varrho_n \to 0$ sufficiently fast as $n \to \infty$, say, $\varrho_n = O(1/n^3)$.

We remark that this potential $q(r)$ still belongs to a different class $(Q)$ used in Alziary, Fleckinger, and Takáč [2, Eq. (9), p. 127] (for $N = 2$) and [3, Eq. (10), p. 363] (for $N \geq 2$).

**Example 3.6.** We define $q : \mathbb{R}_+ \to (0, \infty)$ by $q(r) = \theta(r)^{-2}$ for $r \in \mathbb{R}_+$, where $\theta : \mathbb{R}_+ \to (0, 1]$ is a monotone decreasing, piecewise linear, continuous function defined as follows. Let $\{\varrho_n\}_{n=1}^\infty \subset (0, 1/2)$ be a sequence of numbers satisfying

$$\sum_{n=1}^\infty \varrho_n = 1/2. \quad (26)$$
Given $r \geq 0$, we set $\theta(0) = 1$ and

$$ \frac{d\theta}{dr}(r) = \begin{cases} -1 & \text{if } |r - n| < \varrho_n \text{ for some } n \in \mathbb{N}; \\ 0 & \text{otherwise}, \end{cases}$$

where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Setting $R_0 = 1$ and abbreviating

$$ R_n = 1 - 2 \sum_{k=1}^{n} \varrho_k > 0 \quad \text{for } n = 1, 2, \ldots,$$

we compute for $r \geq 0$:

$$ \theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1 - \varrho_1; \\ R_{n-1} - ((r - n) + \varrho_n) & \text{if } |r - n| < \varrho_n \text{ for some } n \in \mathbb{N}; \\ R_n & \text{if } \varrho_n \leq r - n \leq 1 - \varrho_n + 1 \text{ for some } n \in \mathbb{N}. \end{cases}$$

Clearly, $\theta : \mathbb{R}_+ \to (0, 1]$ is monotone decreasing, piecewise linear, and continuous. It satisfies $\theta(r) \to 0$ as $r \to 0+$, by (26). Next, we compute

$$ \int_{1-\varrho_1}^{\infty} \theta(r) \, dr = \sum_{n=1}^{\infty} (R_{n-1} - \varrho_n) \cdot 2\varrho_n + \sum_{n=1}^{\infty} R_n (1 - \varrho_{n+1} - \varrho_n) < 1 + 2 \sum_{n=1}^{\infty} R_n.$$

Furthermore, for any $\gamma > 0$ we get

$$ \int_{0}^{\infty} \left| \frac{d}{dr} (q(r)^{-1/2}) \right|^{\gamma} q(r)^{1/2} \, dr = \sum_{n=1}^{\infty} \int_{|r-n| < \varrho_n} \left[ R_{n-1} - ((r - n) + \varrho_n) \right]^{-1} \, dr > 2 \sum_{n=1}^{\infty} \varrho_n R_{n-1}^{-1}. $$

We will have an example of a potential $q(r)$ with the desired properties as soon as we find a sequence $\{\varrho_n\}_{n=1}^{\infty} \subset (0, 1/2)$ that satisfies all conditions (26).

$$ \sum_{n=1}^{\infty} R_n < \infty, \quad \text{(27)} $$

and

$$ \sum_{n=1}^{\infty} \varrho_n R_{n-1}^{-1} = \infty. \quad \text{(28)} $$
A simple choice of such $\varrho_n$’s is, for instance,

$$\varrho_n = \frac{1}{2} \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) = \frac{1 + \frac{1}{2n}}{n(n+1)^2} \quad \text{for } n = 1, 2, \ldots, \quad (29)$$

which renders

$$R_n = \frac{1 + \frac{1}{2n}}{n(n+1)^2} \quad \text{and} \quad \varrho_n R_{n-1}^{-1} = \frac{1 + \frac{1}{2n}}{n(1 + \frac{1}{n})^2} \quad \text{for } n = 1, 2, \ldots. \quad (29)$$

It is easy to see that these $\varrho_n$’s satisfy all conditions (26)–(28).

4. Preliminary results

In this section we first state an asymptotic formula (in Lemma 4.1) for the ground state $\psi \equiv \psi_Q$ associated with a potential $Q(r)$ of class $(Q)$. Then we prove a few obvious, but necessary facts (Lemma 4.3) about extensions of bounded symmetric operators defined on $X$ to $L^2(\mathbb{R}^N)$ and $X^\ominus$.

To state the Hartman–Wintner asymptotic formula [13], let us consider a more general setting for the eigenvalue problem $A\psi = \Lambda \psi$ for the ground state $\psi$ corresponding to a potential $q(x) = Q(|x|)$ $(x \in \mathbb{R}^N)$ of class $(Q)$, namely,

$$-\Delta u + Q(|x|)u = \lambda u \quad \text{in } \Omega_R = \left\{ x \in \mathbb{R}^N : |x| > R \right\}, \quad (30)$$

for some $0 < R < \infty$. Here, $\lambda \in \mathbb{R}$ is arbitrary and a weak solution $u$ is any function $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^N)$ satisfying Eq. (30) in the sense of distributions on $\Omega_R$. If $u(x) \equiv \psi(|x|)$ is radially symmetric, then $\psi : (R, \infty) \to \mathbb{R}$ satisfies the radial Schrödinger equation (9). Consequently, $\psi$ is a $C^2$ function on $(R, \infty)$.

The following asymptotic formula for a positive solution $\psi$ of (9), with $\psi(r) \to 0$ as $r \to \infty$, plays an essential role in our present work.

**Lemma 4.1.** Let $Q(r)$ be of class $(Q)$ and $\lambda \in \mathbb{R}$. Assume that, for some $0 < R < \infty$, $\psi : (R, \infty) \to (0, \infty)$ is a $C^2$ function that satisfies the radial Schrödinger equation (9), such that $\psi(r) \to 0$ as $r \to \infty$. Denote

$$V(r) \overset{\text{def}}{=} Q(r) - \lambda + \frac{(N - 1)(N - 3)}{4r^2} \quad \text{for } r > r_0, \quad (31)$$

with $r_0 \geq R$ large enough, so that $V(r) > 0$ for all $r > r_0$. Then we have

$$r^{(N-1)/2} \psi(r) = c V(r)^{-1/4} \exp \left( \eta(r) - \int_{r_0}^{r} V(t)^{1/2} \, dt \right), \quad r > r_0, \quad (32)$$

where $c > 0$ is a constant and $\eta(r) \to 0$ as $r \to \infty$. 

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*Note: The content is a continuation of the text from the previous image.*
This lemma is proved in Hartman and Wintner [13, pp. 81, 82] for the unknown function \( v(r) = r^{(N-1)/2} \psi'(r) > 0 \) satisfying the equation

\[
-v''(r) + V(r)v(r) = 0, \quad r > R,
\]

and the “boundary condition” \( r^{-(N-1)/2}v(r) = \psi'(r) \to 0 \) as \( r \to \infty \). It suffices to realize that this equation is equivalent with the radial Schrödinger equation (9) for \( \psi \) above. Formula (32) above corresponds to Eq. (xxv) on p. 49 and to Eq. (158) on p. 80 in [13].

**Remark 4.2.** Notice that also the potential \( V(r) \) defined in (31) belongs to class (Q) provided \( r_0 > 0 \) is chosen large enough, so that \( V(r) > 0 \), by \( Q(r) \to \infty \) as \( r \to \infty \). Formula (32) still remains valid if the potential \( V \) is replaced by \( Q \). Here, the term \(-\lambda + (N-1)(N-3)/4r^2\) has been added for convenience only (easy comparison with the setting in [13]); it may be left out by taking \( r_0 > 0 \) large enough.

The following lemma on extensions of symmetric operators is an easy consequence of the Riesz–Thorin interpolation theorem (Reed and Simon [18, Section IX.4, Theorem IX.17, p. 27]). We will apply it to the resolvent \( \mathcal{K} = (\mathcal{A} - \lambda I)^{-1} \) on \( L^2(\mathbb{R}^N) \), for \( \lambda < A \), which is bounded on \( X \) by inequality (22), and to similar operators as well.

**Lemma 4.3.** Let \( q, \phi, X \), and \( X^\odot \) be as in Section 3. Assume that \( T : X \to X \) is a bounded linear operator that satisfies the symmetry condition

\[
\int_{\mathbb{R}^N} (Tf)\overline{g} \, dx = \int_{\mathbb{R}^N} f(T\overline{g}) \, dx \quad \text{for all } f, g \in X. \tag{33}
\]

Then \( T \) possesses a unique extension \( T|_{X^\odot} \) to a bounded linear operator on \( X^\odot \). \( T \) is the adjoint of \( T|_{X^\odot} \), and \( T|_{X^\odot} \) restricts to a bounded selfadjoint operator \( T|_{L^2(\mathbb{R}^N)} \) on \( L^2(\mathbb{R}^N) \). Moreover, the operator norms of \( T|_{L^2(\mathbb{R}^N)} \), \( T|_{X^\odot} \), and \( T \), respectively, satisfy

\[
\|T\|_{L^2(\mathbb{R}^N)} \|L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)} \leq \|T\|_{X^\odot} = \|T\|_{X^\odot} \leq \|T\|_{X^\odot} < \infty. \tag{34}
\]

The spectrum of \( T|_{L^2(\mathbb{R}^N)} \) is contained in the spectrum of \( T \). Finally, if \( T \) is compact, then so is \( T|_{L^2(\mathbb{R}^N)} \) and their spectra coincide; in particular, if \( T|_{L^2(\mathbb{R}^N)} v = \lambda v \) for some \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( v \in L^2(\mathbb{R}^N) \setminus \{0\} \), then \( \lambda \in \mathbb{R} \) and \( v \in X \).

**Proof.** Let \( f \in X \) be arbitrary and take \( g \in X \). We apply the symmetry condition (33) to estimate

\[
\|Tf\|_{X^\odot} = \sup_{\|g\|_{X^\odot}} \|Tf, g\|_{L^2(\mathbb{R}^N)} = \sup_{\|g\|_{X^\odot}} \|Tg\|_{L^2(\mathbb{R}^N)} \leq \|T\|_{X^\odot} \sup_{\|g\|_{X^\odot}} \|Tg\|_X \leq \|T\|_{X^\odot} \sup_{\|g\|_{X^\odot}} \|Tg\|_X \|f\|_{X^\odot}. \tag{35}
\]

Hence, \( T \) is densely defined and bounded on \( X^\odot \) and, consequently, it possesses a unique extension \( T|_{X^\odot} \) to a bounded linear operator on \( X^\odot \). Again, the symmetry condition (33) yields that \( T \) is the adjoint of \( T|_{X^\odot} \) and, therefore,

\[
\|T\|_{X^\odot} \|_{X^\odot} \leq \|T\|_{X^\odot} \leq \|T\|_{X^\odot} \leq \|T\|_{X^\odot} \leq \|T\|_{X^\odot} \leq \|T\|_{X^\odot}. \]
Next, the measure $d\mu(x) = \varphi(x)^2 \, dx$ is a probability measure on $\mathbb{R}^N$. Let us define the linear operator $S : L^\infty(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)$ by

$$S f \overset{\text{def}}{=} \varphi^{-1} \cdot T(f \varphi) \quad \text{for } f \in L^\infty(\mathbb{R}^N).$$

So $S$ is bounded on $L^\infty(\mathbb{R}^N)$ and densely defined and bounded on $L^1(\mathbb{R}^N; d\mu)$, by what we have proved above, with the operator norms

$$\|S\|_{L^1(\mathbb{R}^N; d\mu) \to L^1(\mathbb{R}^N; d\mu)} = \|T|_{X^\otimes} \|_{X^\otimes \to X^\otimes} = \|S\|_{L^\infty(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)} = \|T\|_{X \to X}.$$

Now we can apply the Riesz–Thorin interpolation theorem (Reed and Simon [18, Section IX.4, Theorem IX.17, p. 27]) to conclude that $S$ is densely defined and bounded on $L^2(\mathbb{R}^N; d\mu)$ with the operator norm

$$\|S\|_{L^2(\mathbb{R}^N; d\mu) \to L^2(\mathbb{R}^N; d\mu)} \leq \|S\|_{L^1(\mathbb{R}^N; d\mu) \to L^1(\mathbb{R}^N; d\mu)}^{1/2} \|S\|_{L^\infty(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)}^{1/2} = \|S\|_{L^2(\mathbb{R}^N; d\mu) \to L^1(\mathbb{R}^N; d\mu)} = \|S\|_{L^2(\mathbb{R}^N) \to L^\infty(\mathbb{R}^N)}.$$

Inequality (34) follows immediately. Moreover, by (33), $T|_{L^2(\mathbb{R}^N)}$ is selfadjoint on $L^2(\mathbb{R}^N)$.

We apply (34) to the resolvents of $T|_{L^2(\mathbb{R}^N)}$ and $T$ to deduce that the spectrum of $T|_{L^2(\mathbb{R}^N)}$ is contained in the spectrum of $T$.

Finally, assume that $T$ is compact on $X$. Then also $T|_{L^2(\mathbb{R}^N)}$ is compact on $L^2(\mathbb{R}^N)$, by Davies [6, Theorem 1.6.1, p. 35]. The spectra of $T|_{L^2(\mathbb{R}^N)}$ and $T$ coincide, and so do the spectral projections on each (finite-dimensional) eigenspace associated with a nonzero eigenvalue, by [6, Corollary 1.6.2, p. 35].

\section{5. Positivity for potentials of class (Q)}

Throughout this section we consider only a radially symmetric potential $q$ of class (Q), $q(x) = Q(|x|)$ for all $x \in \mathbb{R}^N$. Therefore, all symbols $A$, $\Lambda$, $\varphi$, $X$, $X^\otimes$, etc. are considered only for this special type of potential. Here we prove Theorem 3.1 in this special case, that is, for the Schrödinger equation

$$-\Delta u + Q(|x|) u = \lambda u + f(x) \quad \text{in } X^\otimes. \tag{36}$$

\textbf{Proposition 5.1.} Let $Q(r)$ be of class (Q) and $-\infty < \lambda < \Lambda$. Assume that $f \in X^\otimes$ satisfies $f \geq 0$ almost everywhere and $f \not\equiv 0$ in $\mathbb{R}^N$. Then the (unique) solution $u \in X^\otimes$ to the Schrödinger equation (36) (in the sense of distributions on $\mathbb{R}^N$) is given by $u = (A - \lambda I)^{-1}|X^\otimes f$ and satisfies $u \geq c\varphi$ almost everywhere in $\mathbb{R}^N$, with some constant $c \equiv c(f) > 0$.

\textbf{Proof.} We begin with a standard application of the weak maximum principle to Eq. (36) which states that $u = (A - \lambda I)^{-1}|X^\otimes f$ satisfies $u \geq 0$ a.e. in $\mathbb{R}^N$. Indeed, since $(A - \lambda I)^{-1}|X^\otimes$ is the unique extension of the resolvent $(A - \lambda I)^{-1}$ to a bounded linear operator on $X^\otimes$ (by Lemma 4.3), it suffices to verify that $0 \leq f \in L^2(\Omega)$ implies $0 \leq u \in L^2(\Omega)$. As we have
$u \in \mathcal{V}_Q$, we may multiply Eq. (36) by the negative part $u^- \overset{\text{def}}{=} \max\{-u, 0\}$ of $u$, $u^- \in \mathcal{V}_Q$, and then integrate the product over $\mathbb{R}^N$, thus arriving at

$$-\int_{\mathbb{R}^N} |\nabla u^-|^2 \, dx - \int_{\mathbb{R}^N} Q(|x|)|u^-|^2 \, dx = -\lambda \int_{\mathbb{R}^N} |u^-|^2 \, dx + \int_{\mathbb{R}^N} f(x)u^- \, dx$$

which implies

$$(A - \lambda) \int_{\mathbb{R}^N} |u^-|^2 \, dx \leq Q_{Q}(u^-, u^-) - \lambda \int_{\mathbb{R}^N} |u^-|^2 \, dx = -\int_{\mathbb{R}^N} f(x)u^- \, dx \leq 0,$$

by the Rayleigh quotient (12) and $f \geq 0$ a.e. in $\mathbb{R}^N$. This is possible only if $u^- = 0$ a.e. in $\mathbb{R}^N$, because of $A - \lambda > 0$.

Next, we replace $f \in X^0 = L^1(\mathbb{R}^N; \varphi \, dx)$, $f \geq 0$ a.e. and $f \not= 0$ in $\mathbb{R}^N$, by the truncated function $\tilde{f} = \min\{f, \varphi\}$. Clearly, we have $0 \leq \tilde{f} \leq f$ in $\mathbb{R}^N$ and $0 < \|\tilde{f}\|_X \leq 1$. Again, the weak maximum principle guarantees that $u = (A - \lambda I)^{-1}f$ and $\tilde{u} = (A - \lambda I)^{-1}\tilde{f}$ satisfy $0 \leq \tilde{u} \leq u$ in $\mathbb{R}^N$ and $\|\tilde{u}\|_X \leq (A - \lambda)^{-1}$. Consequently, it suffices to replace $f$ and $u$, respectively, by $\tilde{f}$ and $\tilde{u}$. Thus, from now on we may assume $0 \leq f \leq \varphi$ in $\mathbb{R}^N$, together with $f \not= 0$ and, consequently, also $u \not= 0$.

Furthermore, standard local $L^p$-regularity theory applied to Eq. (36) with $f \in L^p_{\text{loc}}(\mathbb{R}^N)$ for some $p$ with $2 \leq p < \infty$, guarantees $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$; see Gilbarg and Trudinger [11, Theorem 9.15, p. 241]. In particular, if $p > N$ then $u \in C^1(\mathbb{R}^N)$, by the Sobolev imbedding theorem [11, Theorem 7.10, p. 155]. Now it follows that $\varphi \in C^1(\mathbb{R}^N)$, so indeed $0 \leq f \leq \varphi$ a.e. in $\mathbb{R}^N$ entails $f \in L^p_{\text{loc}}(\mathbb{R}^N)$ for any $p > N$; consequently, $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N) \subset C^1(\mathbb{R}^N)$. Finally, we apply the strong maximum and boundary point principles, which are due to Bony [4] for weak solutions (see also P.-L. Lions [16]), in order to conclude that $u > 0$ everywhere in $\mathbb{R}^N$.

Fix any $\lambda' \in (-\infty, \lambda \in (A)$ and take $0 < r_0 < \infty$ large enough, so that

$$W(r) \overset{\text{def}}{=} Q(r) - \lambda' + \frac{(N - 1)(N - 3)}{4r^2} > 0 \quad \text{for all } r > r_0. \quad (37)$$

Equation (36) may be rewritten as

$$-\Delta u + Q(|x|)u - \lambda' u = g(x) \overset{\text{def}}{=} (\lambda - \lambda')u(x) + f(x) \quad \text{in } X. \quad (38)$$

Now define

$$h(r) = \chi_{[0,r_0]}(r) \cdot \min\left\{ (\lambda - \lambda') \cdot \min_{|x|=r} u(x), \varphi(r) \right\} \quad \text{for every } r \geq 0, \quad (39)$$

where $x \in \mathbb{R}^N$ and $\chi_{[0,r_0]}$ is the characteristic function of the compact interval $[0, r_0]$. We observe that the function $h(x) \overset{\text{def}}{=} h(|x|)$ satisfies $0 \leq h(|x|) \leq g(x)$ for a.e. $x \in \mathbb{R}^N$, together with $0 < h(r) \leq \varphi(r)$ for all $0 \leq r \leq r_0$, and $h(r) = 0$ for all $r > r_0$. The weak maximum principle guarantees that $u = (A - \lambda'I)^{-1}g$ and $w = (A - \lambda'I)^{-1}h$ satisfy $0 \leq w \leq u$ and
$w \leq (\Lambda - \lambda')^{-1} \varphi$ in $\mathbb{R}^N$. Moreover, $w(x) \equiv w(|x|)$ is radially symmetric, $w \in C^1(\mathbb{R}^N)$, $w > 0$ everywhere in $\mathbb{R}^N$, and $w(r)$ satisfies the radial Schrödinger equation

$$-w''(r) - \frac{N - 1}{r} w'(r) + Q(r)w(r) = \lambda' w(r), \quad r_0 < r < \infty,$$

thanks to $h(r) = 0$ for all $r > r_0$. We apply Lemma 4.1 with $Q(r) + \Lambda - \lambda'$ in place of $Q(r)$, i.e., with $W(r)$ (defined by (37)) in place of $V(r)$, to conclude that

$$r^{(N-1)/2}w(r) = \hat{c}W(r)^{-1/4} \exp \left( \tilde{\eta}(r) - \int_{r_0}^{r} W(t)^{1/2} \, dt \right), \quad r > r_0,$$

where $\hat{c} > 0$ is a constant and $\tilde{\eta}(r) \to 0$ as $r \to \infty$. We combine formulas (32) and (40) to obtain

$$\frac{w(r)}{\varphi(r)} = \hat{c} \left( \frac{V(r)}{W(r)} \right)^{1/4} \exp \left( \tilde{\eta}(r) - \int_{r_0}^{r} \left[ W(t)^{1/2} - V(t)^{1/2} \right] \, dt \right)$$

for all $r > r_0$, where $\hat{c} = \hat{c}/c > 0$ is a constant and $\tilde{\eta}(r) = \hat{\eta}(r) - \eta(r) \to 0$ as $r \to \infty$. With regard to Remark 4.2, this formula yields $c_0 = \inf_{r \geq R_0} (w(r)/\varphi(r)) > 0$ for some $R_0 > r_0$, owing to conditions (13) and (14). Here, we have used the fact that $W - V = \Lambda - \lambda'$ implies the identity

$$W(t)^{1/2} - V(t)^{1/2} = (\Lambda - \lambda')[W(t)^{1/2} + V(t)^{1/2}]^{-1}$$

with both $V(r)$ and $W(r)$ of class (Q), by Remark 4.2 again.

Finally, we combine $c_0 > 0$ with $w, \varphi \in C^1(\mathbb{R}^N)$ and $w, \varphi > 0$ everywhere in $\mathbb{R}^N$, thus arriving at $\gamma = \inf_{\mathbb{R}^N}(w/\varphi) > 0$. Owing to $w \leq u$ in $\mathbb{R}^N$, this entails the conclusion of the proposition, that is, $u \geq \gamma \varphi$ in $\mathbb{R}^N$. \hfill \Box

6. A local compactness result

We denote by $(r, x')$ the spherical coordinates in $\mathbb{R}^N$, that is, $x = rx' \in \mathbb{R}^N$ where $r = |x|$ and $x' = r^{-1}x \in \mathbb{S}^{N-1}$ if $x \neq 0$; we set $r = 0$ and leave $x' \in \mathbb{S}^{N-1}$ arbitrary if $x = 0$. As usual, $\mathbb{S}^{N-1}$ denotes the unit sphere in $\mathbb{R}^N$ centered at the origin. We refer to $r$ and $x'$ as the radial and azimuthal variables, respectively. The surface measure on $\mathbb{S}^{N-1}$ is denoted by $\sigma$; we let $\sigma_{N-1} = \sigma(\mathbb{S}^{N-1})$ be the surface area of $\mathbb{S}^{N-1}$.

The potential $q$ is assumed to satisfy only conditions (2) in this section. In the Banach lattice $X^\circ = L^1(\mathbb{R}^N; \varphi \, dx)$ we denote by

$$\overline{B}_{X^\circ} = \{ f \in X^\circ : \|f\|_{X^\circ} \leq 1 \}$$

the closed unit ball centered at the origin, and

$$\overline{B}_{X^\circ}^+ = \{ f \in \overline{B}_{X^\circ} : f \geq 0 \text{ in } \mathbb{R}^N \}.$$

If $B_R(0)$ is an open ball of radius $R$ ($0 < R < \infty$) in $\mathbb{R}^N$ centered at the origin, let $u|_{B_R(0)}$ denote the restriction of a function $u : \mathbb{R}^N \to \mathbb{R}$ to $B_R(0)$. 

Proposition 6.1. Assume that \( q : \mathbb{R}^N \to \mathbb{R} \) is a continuous function satisfying (2), and let \( \lambda < \Lambda \). Then, given any \( 0 < R < \infty \), the restricted resolvent

\[
\mathcal{R}_R : X^\oplus \to L^1(B_R(0)) : f \mapsto u|_{B_R(0)}
\]

is compact, where \( u = (A - \lambda I)^{-1}|_{X^\oplus} f \).

Proof. Equivalently, we need to show that the image \( \mathcal{R}_R(B_{X^\oplus}) \) of the closed unit ball \( B_{X^\oplus} \) in \( X^\oplus \) under the operator \( \mathcal{R}_R \) has compact closure in \( L^1(B_R(0)) \). Since \( X^\oplus \) is a Banach lattice, it suffices to show that \( \mathcal{R}_R(B_{X^\oplus}^+) \) has compact closure in \( L^1(B_R(0)) \).

Thus, let us assume \( f \in B_{X^\oplus}^+ \); hence \( u \geq 0 \) a.e. in \( \mathbb{R}^N \) as well. Moreover, by (35), we have

\[
\int_{\mathbb{R}^N} u(x)\varphi(x) \, dx \leq (\Lambda - \lambda)^{-1} \int_{\mathbb{R}^N} f(x)\varphi(x) \, dx \leq (\Lambda - \lambda)^{-1}.
\]

In particular, for every \( 0 < s < \infty \) we get

\[
\int_{|x| \leq s} f(x) \, dx \leq C_s, \quad (42)
\]

\[
\int_{|x| \leq s} u(x) \, dx \leq C_s(\Lambda - \lambda)^{-1}, \quad (43)
\]

where

\[
C_s \equiv \left( \inf_{|x| \leq s} \varphi(x) \right)^{-1} < \infty.
\]

We rewrite the Schrödinger equation (1) as

\[
-\Delta u = f^z(x) \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N), \quad (44)
\]

where

\[
f^z(x) \equiv (\lambda - q(x))u(x) + f(x), \quad x \in \mathbb{R}^N. \quad (45)
\]

This equation holds in the sense of distributions on \( \mathbb{R}^N \). The function \( f^z \) satisfies

\[
-\left(|\lambda| + \sup_{0 \leq |x| \leq s} q(x)\right)u(x) \leq f^z(x) \leq |\lambda|u(x) + f(x) \quad \text{whenever } |x| \leq s,
\]

for every \( 0 < s < \infty \), which implies

\[
|f^z(x)| \leq c_s u(x) + f(x), \quad |x| \leq s,
\]
where
\[ c_s \overset{\text{def}}{=} |\lambda| + \sup_{0 \leq |x| \leq s} q(x) < \infty. \]

Applying (42) and (43) we thus get
\[
\int_{|x| \leq s} |f^\sharp(x)| \, dx \leq \int_{|x| \leq s} \left( c_s u(x) + f(x) \right) \, dx \leq c_s C_s (A - \lambda)^{-1} + C_s
\]
\[
= C_s (c_s (A - \lambda)^{-1} + 1) \equiv C'_s < \infty
\]
for every \( 0 < s < \infty \).

Next, we split the function \( u \) in Eq. (44) as
\[ u = N_s f^\sharp + u^\sharp \quad \text{a.e. in } B_s(0), \]
where \( N_s f^\sharp \) denotes the Newton potential of \( f^\sharp \) on \( B_s(0) \) and \( u^\sharp : B_s(0) \to \mathbb{R} \) is a harmonic function, \( \Delta u^\sharp = 0 \) in \( B_s(0) \). More precisely,
\[
(N_s f^\sharp)(x) \overset{\text{def}}{=} \int_{|y| \leq s} \Phi(|x - y|) f^\sharp(y) \, dy, \quad x \in \mathbb{R}^N,
\]
where
\[
\Phi(r) = \begin{cases} -1 \frac{\log r}{2\pi} & \text{if } N = 2; \\ \frac{1}{(N-2)\sigma_{N-1}} r^{-(N-2)} & \text{if } N \geq 3,
\end{cases}
\]
for \( r > 0 \). The operator \( N_s : L^1(B_s(0)) \to L^1(B_s(0)) \) is compact, by Lemma 6.2. From now on we take \( s = R + 2 \). Consequently, also the operator
\[
N_{R+2}|_{B_R(0)} : f^\sharp \mapsto \left( N_{R+2} f^\sharp \right)|_{B_R(0)} : L^1(B_{R+2}(0)) \to L^1(B_R(0))
\]
of restrictions of \( N_{R+2} f^\sharp \) to \( B_R(0) \) is compact. Furthermore, by another auxiliary result below, Lemma 6.3, also the operator \( f \mapsto u^\sharp|_{B_R(0)} : X^\ominus \to L^1(B_R(0)) \) is compact. These two compactness results combined with (47) imply that, indeed, the restricted resolvent \( R_R : X^\ominus \to L^1(B_R(0)) : f \mapsto u|_{B_R(0)} \) is compact as claimed. \( \square \)

**Lemma 6.2.** Given any \( 0 < s < \infty \), the operator \( N_s : L^1(B_s(0)) \to L^1(B_s(0)) \) defined in (48) is compact.

**Proof.** One shows easily that, for \( 0 \leq s_1 < s_2 < \infty \),

\[
\Phi(r) = \begin{cases} -1 \frac{\log r}{2\pi} & \text{if } N = 2; \\ \frac{1}{(N-2)\sigma_{N-1}} r^{-(N-2)} & \text{if } N \geq 3,
\end{cases}
\]
for \( r > 0 \). The operator \( N_s : L^1(B_s(0)) \to L^1(B_s(0)) \) is compact, by Lemma 6.2. From now on we take \( s = R + 2 \). Consequently, also the operator
\[
N_{R+2}|_{B_R(0)} : f^\sharp \mapsto \left( N_{R+2} f^\sharp \right)|_{B_R(0)} : L^1(B_{R+2}(0)) \to L^1(B_R(0))
\]
of restrictions of \( N_{R+2} f^\sharp \) to \( B_R(0) \) is compact. Furthermore, by another auxiliary result below, Lemma 6.3, also the operator \( f \mapsto u^\sharp|_{B_R(0)} : X^\ominus \to L^1(B_R(0)) \) is compact. These two compactness results combined with (47) imply that, indeed, the restricted resolvent \( R_R : X^\ominus \to L^1(B_R(0)) : f \mapsto u|_{B_R(0)} \) is compact as claimed. \( \square \)

**Lemma 6.2.** Given any \( 0 < s < \infty \), the operator \( N_s : L^1(B_s(0)) \to L^1(B_s(0)) \) defined in (48) is compact.

**Proof.** One shows easily that, for \( 0 \leq s_1 < s_2 < \infty \),
\[
\int_{s_1 \leq |x| \leq s_2} \Phi(|x|) \, dx = \sigma_{N-1} \int_{s_1}^{s_2} \Phi(r)r^{N-1} \, dr
\]

\[
= \begin{cases} 
\frac{1}{2} \left[ s_2^2 \left( \frac{1}{2} - \log s_2 \right) - s_1^2 \left( \frac{1}{2} - \log s_1 \right) \right] & \text{if } N = 2; \\
\frac{1}{2} (s_2^2 - s_1^2) & \text{if } N \geq 3,
\end{cases}
\]

which implies

\[
\int_{|x| \leq s} |\Phi(|x|)| \, dx \leq \begin{cases} 
\frac{1}{2} [1 + s^2 (\frac{1}{2} + |\log s|)] & \text{if } N = 2; \\
\frac{1}{2} s^2 & \text{if } N \geq 3,
\end{cases}
\]

for every \(0 < s < \infty\). Now assume that \(f^\sharp \in L^1(B_s(0))\) satisfies (46) where \(C'_s > 0\) may be an arbitrary constant. It follows that

\[
\int_{|x| \leq s} |(N_s f^\sharp)(x)| \, dx \leq \left( \sup_{|y| \leq s} \int_{|x| \leq s} |\Phi(|x|) - \Phi(|x-y|)| \, dx \right) \int_{|y| \leq s} |f^\sharp(y)| \, dy
\]

\[
\leq \left( \int_{|x| \leq 2s} |\Phi(|x|)| \, dx \right) \int_{|y| \leq s} |f^\sharp(y)| \, dy
\]

\[
\leq \frac{1}{2} \left[ 1 + (2s)^2 \left( 1 + |\log (2s)| \right) \right] C'_s = C''_s < \infty
\]

(49)

for each \(N \geq 2\), by (46). Furthermore, owing to

\[
\nabla \Phi(x) = -\frac{1}{\sigma_{N-1}} \cdot \frac{x}{|x|^N} \quad \text{for } 0 \neq x \in \mathbb{R}^N \text{ and } N \geq 2,
\]

we have

\[
\nabla (N_s f^\sharp)(x) = -\frac{1}{\sigma_{N-1}} \int_{|y| \leq s} \frac{x - y}{|x - y|^N} f^\sharp(y) \, dy, \quad x \in \mathbb{R}^N,
\]

which gives the estimate

\[
\int_{|x| \leq s} |\nabla (N_s f^\sharp)(x)| \, dx \leq \frac{1}{\sigma_{N-1}} \left( \sup_{|y| \leq s} \int_{|x| \leq s} |x - y|^{-(N-1)} \, dx \right) \int_{|y| \leq s} |f^\sharp(y)| \, dy
\]

\[
\leq \frac{1}{\sigma_{N-1}} \left( \int_{|x| \leq 2s} |x|^{-(N-1)} \, dx \right) \int_{|y| \leq s} |f^\sharp(y)| \, dy
\]

\[
= \left( \int_{0}^{2s} dr \right) \int_{|y| \leq s} |f^\sharp(y)| \, dy = 2s \int_{|y| \leq s} |f^\sharp(y)| \, dy \leq 2s C'_s
\]

(50)
for every $0 < s < \infty$. We combine (49) and (50) to get the Sobolev norm
\[
\| N_s f^\sharp \|_{W^{1,1}(B_s(0))} = \int_{|x| \leq s} |\nabla (N_s f^\sharp)| \, dx + \int_{|x| \leq s} |(N_s f^\sharp)| \, dx 
\leq 2sC_s' + C_s'' \quad \text{for every } 0 < s < \infty.
\]

From this estimate combined with Rellich’s theorem in $W^{1,1}(B_s(0))$ we deduce that the set
\[
\{ N_s f^\sharp : \| f^\sharp \|_{L^1(B_s(0))} \leq C_s' \}
\]
has compact closure in $L^1(B_s(0))$.  

**Lemma 6.3.** Given any $0 < R < \infty$, the mapping $f \mapsto u^\sharp|_{B_R(0)} : X^\circ \to L^1(B_R(0))$, defined as follows, is a compact linear operator: for $f \in X^\circ$ take first $u = (A - \lambda I)^{-1}|_{X^\circ} f$, then define $f^\sharp$ by (45), and finally set $u^\sharp = N_{R+2} f^\sharp - u$ in $\mathbb{R}^N$.

**Proof.** Let $s = R + 2$. We take $f \in \overline{B}_{X^\circ}$ arbitrary. Applying estimates (43) and (49) to $u^\sharp = N_s f^\sharp - u$ we get
\[
\int_{|x| \leq s} |u^\sharp(x)| \, dx \leq \int_{|x| \leq s} u(x) \, dx + \int_{|x| \leq s} |(N_s f^\sharp)(x)| \, dx 
\leq C_s (A - \lambda)^{-1} + C_s'' \equiv C_s'' < \infty
\] (51)
for every $0 < s < \infty$. The function $u^\sharp$ being harmonic in $B_s(0)$, it is continuous and has the mean value property (see Evans [10, Theorem 2, p. 25])
\[
u^\sharp(x) = \frac{1}{\sigma_{N-1}r^{N-1}} \int_{|y|=r} u^\sharp(x + y) \, d\sigma(y) = \frac{N}{\sigma_{N-1}r^N} \int_{|y| \leq r} u^\sharp(x + y) \, dy
\]
which holds for any closed ball
\[
\overline{B}_r(x) = \{ z \in \mathbb{R}^N : |z - x| \leq r \} = x + \overline{B}_r(0)
\]
contained in $B_s(0)$. Hence,
\[
|u^\sharp(x)| \leq \frac{N}{\sigma_{N-1}r^N} \int_{|y| \leq r} |u^\sharp(x + y)| \, dy \leq \frac{N}{\sigma_{N-1}r^N} \int_{|y| \leq s} |u^\sharp(y)| \, dy 
\leq \frac{N}{\sigma_{N-1}r^N} C_s^{'''},
\] (52)
by (51). Using Green’s theorem we compute (following Evans [10, proof of Theorem 7, p. 29])
\[ \nabla u^\#(x) = \frac{N}{\sigma_{N-1} r^N} \int_{|y| \leq r} \nabla u^\#(x + y) \, dy \]

\[ = \frac{N}{\sigma_{N-1} r^N} \int_{|y| = r} u^\#(x + y) \frac{y}{|y|} \, d\sigma(y) \]

\[ = \frac{N}{\sigma_{N-1} r} \int_{|y'| = 1} u^\#(x + ry') y' \, d\sigma(y') \]

whenever \( \overline{B}_r(x) \subset B_s(0) \). Integration with respect to \( r \) now yields

\[ (N + 1)^{-1} r^{N+1} |\nabla u^\#(x)| = \left( \int_{0}^{r} t^{N} \, dt \right) |\nabla u^\#(x)| \]

\[ \leq \frac{N}{\sigma_{N-1}} \int_{0}^{r} \int_{|y'| = 1} |u^\#(x + ty')| \, d\sigma(y') t^{N-1} \, dt \]

\[ = \frac{N}{\sigma_{N-1}} \int_{|y| \leq r} |u^\#(x + y)| \, dy, \]

that is,

\[ |\nabla u^\#(x)| \leq \frac{N(N + 1)}{\sigma_{N-1} r^{N+1}} \int_{|y| \leq r} |u^\#(x + y)| \, dy \leq \frac{N(N + 1)}{\sigma_{N-1} r^{N+1}} \int_{|y| \leq s} |u^\#(y)| \, dy \]

\[ \leq \frac{N(N + 1)}{\sigma_{N-1} r^{N+1}} C''', \tag{53} \]

by (51), whenever \( \overline{B}_r(x) \subset B_s(0) \). Finally, we take \( |x| \leq R \) and \( r = 1 \) in (52) and (53), and recall \( s = R + 2 \), thus obtaining the Hölder norm

\[ \|u^\#\|_{C^1(\overline{B}_R(0))} \equiv \sup_{|x| \leq R} |\nabla u^\#(x)| + \sup_{|x| \leq R} |u^\#(x)| \]

\[ \leq \frac{N(N + 1)}{\sigma_{N-1}} C'' + \frac{N}{\sigma_{N-1}} C''' = \frac{N(N + 2)}{\sigma_{N-1}} C'''. \]

Having established this estimate, we may apply Arzelà–Ascoli’s compactness criterion to conclude that the set

\[ \left\{ u^\# : \|u^\#\|_{C^1(\overline{B}_R(0))} \leq \frac{N(N + 2)}{\sigma_{N-1}} C''' \right\} \]

has compact closure in \( C(\overline{B}_R(0)) \) and, hence, in \( L^1(B_R(0)) \) as well. It follows that the operator \( f \mapsto u^\#|_{B_R(0)} : X^\ominus \to L^1(B_R(0)) \) is compact as claimed. \( \Box \)
7. Compactness for potentials of class \((Q)\)

Throughout this section we consider only a radially symmetric potential \(q\) of class \((Q)\), \(q(x) = Q(|x|)\) for all \(x \in \mathbb{R}^N\). All symbols \(A, \Lambda, \varphi, X, X^\odot\), etc. are considered only for this special type of potential. Under the hypotheses in \((Q)\), we are able to show the following special case of Theorem 3.2, part (a).

**Proposition 7.1.** Both operators \(((A - \lambda I)^{-1}|_X : X \to X)\) and \(((A - \lambda I)^{-1}|_{X^\odot} : X^\odot \to X^\odot)\) are compact.

Also the compactness of \(((A - \lambda I)^{-1}|_X : X \to X)\) is equivalent to that of \(((A - \lambda I)^{-1}|_{X^\odot} : X^\odot \to X^\odot)\), by Schauder’s theorem (Edwards [9, Corollary 9.2.3, p. 621] or Yosida [23, Chapter X, Section 4, p. 282]); we will prove the latter one.

We split the proof of Proposition 7.1 into Sections 7.1 and 7.2. We set \(K = (A - \lambda I)^{-1}\) on \(L^2(\mathbb{R}^N)\). In Section 7.1, we restrict the operators \(K|_X\) and \(K|_{X^\odot}\) to the corresponding subspaces of radially symmetric functions and show that Proposition 7.1 is valid in these subspaces; see Lemmas 7.2 and 7.3. In Section 7.2, we take advantage of Lemma 7.3 to prove the compactness of \(K|_{X^\odot}\) in Proposition 7.1.

7.1. Compactness on the space of radial functions

Throughout this subsection, we denote by \(X_{\text{rad}}, L^2_{\text{rad}}(\mathbb{R}^N),\) and \(X^\odot_{\text{rad}}\), respectively, the subspaces of \(X, L^2(\mathbb{R}^N),\) and \(X^\odot\) that consist of all radially symmetric functions from these spaces. All these subspaces are closed. Moreover, since the potential \(Q\) is radially symmetric, all subspaces above are invariant under the operator \(K|_{X^\odot}\). We denote by \(K|_{X_{\text{rad}}}, K|_{L^2_{\text{rad}}(\mathbb{R}^N)},\) and \(K|_{X^\odot_{\text{rad}}}\), respectively, the restrictions of \(K|_{X^\odot}\) to the spaces \(X_{\text{rad}}, L^2_{\text{rad}}(\mathbb{R}^N),\) and \(X^\odot_{\text{rad}}\). These restrictions have similar properties as \(K|_X, K,\) and \(K|_{X^\odot}\), respectively, above.

**Lemma 7.2.** Under the hypotheses in \((Q)\) the operator \(K|_{X_{\text{rad}}} : X_{\text{rad}} \to X_{\text{rad}}\) is compact.

By Schauder’s theorem again (Edwards [9, Corollary 9.2.3, p. 621] or Yosida [23, Chapter X, Section 4, p. 282]), this lemma is equivalent to

**Lemma 7.3.** Under the hypotheses in \((Q)\) the operator \(K|_{X^\odot_{\text{rad}}} : X^\odot_{\text{rad}} \to X^\odot_{\text{rad}}\) is compact.

We prove Lemma 7.2 directly using Arzelà–Ascoli’s compactness criterion for continuous functions on the one point compactification \(\mathbb{R}^*_+ = \mathbb{R}_+ \cup \{\infty\}\) of \(\mathbb{R}_+\). The metric on \(\mathbb{R}^*_+\) is defined by

\[
d(x, y) \overset{\text{def}}{=} \begin{cases} \frac{|x - y|}{1 + |x - y|} & \text{for } x, y \in \mathbb{R}_+; \\ 1 & \text{for } 0 \leq x < y = \infty \text{ or } 0 \leq y < x = \infty; \\ 0 & \text{for } x = y = \infty. \end{cases}
\]

We denote by \(C(\mathbb{R}^*_+)\) the Banach space of all continuous functions on the compact metric space \(\mathbb{R}^*_+\) endowed with the supremum norm from \(L^\infty(\mathbb{R}_+)\).

**Proof of Lemma 7.2.** Given \(f, u \in X_{\text{rad}}\), \(u = Kf\) is equivalent with the ordinary differential equation
\[-u''(r) - \frac{N-1}{r}u'(r) + q(r)u(r) = \lambda u(r) + f(r) \quad \text{for } 0 < r < \infty\]

supplemented by the conditions

\[
\lim_{r \to 0^+} u'(r) = 0 \quad \text{and} \quad \sup_{0 < r < \infty} \left| \frac{u(r)}{\varphi(r)} \right| \leq (A - \lambda)^{-1} \cdot \sup_{0 < r < \infty} \left| \frac{f(r)}{\varphi(r)} \right|.
\]

Clearly, the former one is a boundary condition at zero that follows from the radial symmetry, whereas the latter one follows from the weak maximum principle.

Substituting \( g = f/\varphi \) and \( v = u/\varphi \), combined with

\[-\varphi''(r) - \frac{N-1}{r} \varphi'(r) + q(r)\varphi(r) = \Lambda \varphi(r) \quad \text{for } 0 < r < \infty, \]

we have equivalently

\[-v''(r) - \frac{N-1}{r} v'(r) - 2 \left( \log \varphi(r) \right)'v'(r) + (A - \lambda)v(r) = g(r) \quad \text{for } 0 < r < \infty \quad (54)\]

subject to the conditions

\[
\lim_{r \to 0^+} v'(r) = 0 \quad \text{and} \quad \sup_{0 < r < \infty} |v(r)| \leq (A - \lambda)^{-1} \cdot \sup_{0 < r < \infty} |g(r)|. \quad (55)
\]

Then \( K|_{X_{\text{rad}}} \) is compact on \( X_{\text{rad}} \) if and only if the linear operator \( K_\varphi : L^\infty(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+) \), defined by

\[ K_\varphi g \overset{\text{def}}{=} v = \varphi^{-1} \cdot K(g\varphi) \quad \text{for } g \in L^\infty(\mathbb{R}_+), \]

is compact.

We will apply Arzelà–Ascoli’s compactness criterion in the Banach space \( C(\mathbb{R}_+) \) in order to show that the image \( K_\varphi(\overline{B}_{L^\infty(\mathbb{R}_+)}) \) of the ball

\[ \overline{B}_{L^\infty(\mathbb{R}_+)} = \{ g \in L^\infty(\mathbb{R}_+) : \| g \|_{L^\infty(\mathbb{R}_+)} \leq 1 \} \]

has compact closure in \( C(\mathbb{R}_+) \). Since \( L^\infty(\mathbb{R}_+) \) is a Banach lattice, it suffices to show that \( K_\varphi(\overline{B}_{L^\infty(\mathbb{R}_+)})^+ \) has compact closure in \( C(\mathbb{R}_+) \), where

\[ \overline{B}_{L^\infty(\mathbb{R}_+)^+} = \{ g \in \overline{B}_{L^\infty(\mathbb{R}_+)}^+: g \geq 0 \text{ in } \mathbb{R}_+ \}. \]

Clearly, the function \( v \) from (54) and (55) above satisfies \( v \in C^1(\mathbb{R}_+) \); we will show also \( v \in C(\mathbb{R}_+) \). Therefore, we need to show that the linear operator

\[ K_\varphi : L^\infty(\mathbb{R}_+) \to C(\mathbb{R}_+) \subset L^\infty(\mathbb{R}_+) \]

is compact.
So let \( g \in L^\infty(\mathbb{R}_+) \) be arbitrary with \( 0 \leq g(r) \leq 1 \) for \( r \in \mathbb{R}_+ \). Hence, \( v = K\varphi g \) satisfies \( v \in C^1(\mathbb{R}_+) \) and also \( 0 \leq v(r) \leq (\Lambda - \lambda)^{-1} \), by (22). It follows that the function

\[
g^\# \overset{\text{def}}{=} g - (\Lambda - \lambda)v
\]
satisfies \(-1 \leq g^\# \leq 1\), and the derivative \( w \overset{\text{def}}{=} v' \) verifies the ordinary differential equation

\[
-w'(r) - \frac{N - 1}{r} w(r) - 2(\log \varphi(r))' w(r) = g^\#(r) \quad \text{for } 0 < r < \infty \tag{56}
\]

subject to the conditions

\[
\lim_{r \to 0^+} w(r) = 0 \quad \text{and} \quad \sup_{0 < r < \infty} \left| \int_0^r w(s) \, ds \right| \leq (\Lambda - \lambda)^{-1}.
\]

The latter condition has been obtained from

\[
\int_0^r w(s) \, ds = v(r) - v(0) \quad \text{with } 0 \leq v(r) \leq (\Lambda - \lambda)^{-1}
\]

for all \( r \geq 0 \). Since \( w \) is continuous, this condition implies that there exists a sequence \( \{r_n\}_{n=1}^\infty \subset \mathbb{R}_+ \) such that \( r_n \to \infty \) and \( w(r_n) \to 0 \) as \( n \to \infty \).

The differential equation (56) is equivalent to

\[
-\frac{d}{dr} \left( r^{N-1} \varphi(r)^2 w(r) \right) = r^{N-1} \varphi(r)^2 g^\#(r) \quad \text{for } 0 < r < \infty.
\]

After integration, we thus arrive at

\[
r^{N-1} \varphi(r)^2 w(r) - s^{N-1} \varphi(s)^2 w(s) = \int_r^s t^{N-1} \varphi(t)^2 g^\#(t) \, dt
\]

whenever \( 0 \leq r, s < \infty \). Applying \( \lim_{s \to 0^+} w(s) = 0 \) we obtain

\[
r^{N-1} \varphi(r)^2 w(r) = -\int_0^r t^{N-1} \varphi(t)^2 g^\#(t) \, dt \quad \text{for all } r \geq 0. \tag{57}
\]

Taking \( s = r_n \) and letting \( n \to \infty \) we obtain also

\[
r^{N-1} \varphi(r)^2 w(r) = \int_r^\infty t^{N-1} \varphi(t)^2 g^\#(t) \, dt \quad \text{for all } r \geq 0. \tag{58}
\]
Here we have used the facts that $s^{N-1} \phi(s)^2 \to 0$ as $s \to \infty$ together with $r_n \to \infty$ and $w(r_n) \to 0$ as $n \to \infty$. Recall the normalization $\int_0^\infty \phi(t)^2 \, \sigma(\cdot) \, dt = \sigma(\cdot)$ Below we will take advantage of formulas (57) and (58) to estimate $|w(r)|$ as $r \to 0+$ and $r \to \infty$, respectively.

Because of $|g^\sharp| \leq 1$, Eq. (57) yields $|w| \leq w_0^\sharp$ where $w_0^\sharp : \mathbb{R}_+ \to \mathbb{R}_+$ is the function defined by $w_0^\sharp(0) = 0$ and

$$r^{N-1} \phi(r)^2 w_0^\sharp(r) = \int_0^r t^{N-1} \phi(t)^2 \, dt \quad \text{for all } r > 0.$$

Using $\lim_{r \to 0+} \phi(r) = \phi(0) > 0$ we conclude that

$$\lim_{r \to 0+} \frac{w_0^\sharp(r)}{r} = \lim_{r \to 0+} \frac{1}{r} \int_0^r \left( \frac{t}{r} \right)^{N-1} \phi(t)^2 \, dt = \lim_{r \to 0+} \frac{1}{r} \int_0^r \left( \frac{t}{r} \right)^{N-1} \, dt = \frac{1}{N}. $$

Because of $|g^\sharp| \leq 1$, Eq. (58) yields $|w| \leq w_\infty^\sharp$ where $w_\infty^\sharp : (0, \infty) \to \mathbb{R}_+$ is the function defined by

$$w_\infty^\sharp(r) = r^{-(N-1)} \phi(r)^{2} \int_r^\infty t^{N-1} \phi(t)^2 \, dt \quad \text{for all } r > 0. \quad (59)$$

Next, we wish to show

$$\int_{r_0}^\infty w_\infty^\sharp(r) \, dr \leq \int_{r_0}^\infty V(r)^{-1/2} \, dr \quad (60)$$

provided $r_0 > 0$ is chosen large enough, where $V(r)$ is the potential defined in (31). Notice that condition (13) implies $\int_{r_0}^\infty V(r)^{-1/2} \, dr < \infty$. Making use of Lemma 4.1, let us first fix $\eta_0 = \frac{1}{4} \log 2 > 0$, then take $r_0 > 0$ large enough such that

$$e^{-\eta_0} \leq \frac{r^{(N-1)/2} \phi(r)}{c V(r)^{-1/4} \exp(\eta(r) - \int_{r_0}^r V(t)^{1/2} \, dt) \leq e^{\eta_0} \quad \text{for all } r > r_0. \quad (61)$$

Now let us abbreviate

$$E(r) \equiv \exp \left( 2 \int_{r_0}^r V(t)^{1/2} \, dt \right) \quad \text{for } r_0 \leq r < \infty.$$

We apply estimates (61) to formula (59) to obtain, using integration by parts, for $r_0 < R < \infty$:
\[
\int_{r_0}^{R} w^#_{\infty}(r) \, dr \leq 2 \int_{r_0}^{R} V(r)^{1/2} E(r) \left[ \int_{r}^{\infty} V(s)^{-1/2} E(s)^{-1} \, ds \right] \, dr
\]

\[
= \int_{r_0}^{R} \frac{d}{dr} E(r) \left[ \int_{r}^{\infty} V(s)^{-1/2} E(s)^{-1} \, ds \right] \, dr
\]

\[
= \left[ E(r) \int_{r}^{\infty} V(s)^{-1/2} E(s)^{-1} \, ds \right]_{r=r_0}^{r=R} + \int_{r_0}^{R} E(r) V(r)^{-1/2} E(r)^{-1} \, dr
\]

\[
= \left[ \int_{r}^{\infty} V(s)^{-1/2} \exp \left( -2 \int_{r}^{s} V(t)^{1/2} \, dt \right) \, ds \right]_{r=r_0}^{r=R} + \int_{r_0}^{R} V(r)^{-1/2} \, dr
\]

\[
\leq \int_{R}^{\infty} V(s)^{-1/2} \exp \left( -2 \int_{R}^{s} V(t)^{1/2} \, dt \right) \, ds + \int_{r_0}^{R} V(r)^{-1/2} \, dr = \int_{r_0}^{\infty} V(s)^{-1/2} \, ds.
\]

To summarize our estimates for the functions \( w^#_{0} : \mathbb{R}_+ \to \mathbb{R}_+ \) and \( w^#_{\infty} : (0, \infty) \to \mathbb{R}_+ \) in the inequalities \( |w| \leq w^#_{0} \) for \( r \leq r_0 \) and \( |w| \leq w^#_{\infty} \) for \( r > r_0 \), we observe that both functions \( w^#_{0} \) and \( w^#_{\infty} \) are continuously differentiable and satisfy the estimates

\[
|w(r)| \leq w^#_{0}(r) \leq C r \quad \text{for } 0 \leq r \leq r_0.
\]

where \( C > 0 \) is a constant, and

\[
\int_{r_0}^{\infty} |w(r)| \, dr \leq \int_{r_0}^{\infty} w^#_{\infty}(r) \, dr \leq \int_{r_0}^{\infty} V(r)^{-1/2} \, dr < \infty.
\]

Consequently, for \( g \) ranging over \( L^\infty(\mathbb{R}_+) \) with \( 0 \leq g \leq 1 \) in \( \mathbb{R}_+ \), the set of functions \( v = K\varphi g \in C^1(\mathbb{R}_+) \) defined above is uniformly equicontinuous on the compact metric space \( \mathbb{R}_+^* \), thanks to

\[
v(r) = v(0) + \int_{0}^{r} w(s) \, ds = v(\infty) - \int_{r}^{\infty} w(s) \, ds \quad \text{for } 0 \leq r < \infty.
\]

The limit \( v(\infty) = \lim_{r \to \infty} v(r) \in \mathbb{R}_+ \) exists by (63). Furthermore, owing to \( 0 \leq v(r) \leq (\Lambda - \lambda)^{-1} \) for \( r \in \mathbb{R}_+^* \), this set is also uniformly bounded on \( \mathbb{R}_+^* \). Thus, by Arzelà–Ascoli’s compactness criterion, the set \( K\varphi(\overline{B}_L^+(\mathbb{R}_+)) \) has compact closure in \( C(\mathbb{R}_+^*) \).
We have proved that the linear operator $K|_{X_{\text{rad}}}$ is compact on $X_{\text{rad}}$ and, moreover, its image satisfies $K(X_{\text{rad}}) \subset C(\mathbb{R}_+^*)$. 

7.2. Compactness on the entire space $X$

We keep the assumption $q(x) = Q(|x|)$ for all $x \in \mathbb{R}^N$. Recall $\lambda < \Lambda$ and $K = (A - \lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$. This time we will show first that the operator $K|_{X_{\text{rad} \circ}}$ is compact on $X_{\text{rad} \circ}$. We derive this result from the compactness of its restriction $K|_{X_{\text{rad}}} \subset X_{\text{rad} \circ}$ which we have already established in the previous paragraph.

For a function $u : \mathbb{R}^N \to \mathbb{R}$, we identify $u(x) \equiv u(r,x')$ if no confusion can arise. In the spherical coordinates, the Laplace operator becomes

$$\Delta = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \Delta_{S}, \quad (64)$$

where $\Delta_{S}$ denotes the Laplace–Beltrami operator on the sphere $\mathbb{S}^{N-1}$. A precise definition employing the tensor product

$$L^2(\mathbb{R}^N) = L^2(\mathbb{R}_+; r^{N-1} \, dr) \otimes L^2(\mathbb{S}^{N-1}; \, d\sigma(x'))$$

can be found in Reed and Simon [18, Section X.1, Example 4, p. 160]. We identify $L^2_{\text{rad}}(\mathbb{R}^N) \equiv L^2(\mathbb{R}_+; r^{N-1} \, dr)$ and denote by $P : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ the orthogonal projection of $L^2(\mathbb{R}^N)$ onto $L^2_{\text{rad}}(\mathbb{R}^N)$. It is easy to see that, for every $f \in L^2(\mathbb{R}^N)$,

$$P f(r) = \frac{1}{\sigma_{N-1}} \int_{\mathbb{S}^{N-1}} f(r,x') \, d\sigma(x'), \quad 0 < r < \infty. \quad (65)$$

We denote by $P|_X$ ($P|_{X^\circ}$, respectively) the restriction (extension) of $P$ to $X$ ($X^\circ$), both defined by (65).

Equation $u = K|_{X^\circ} f$, for $f,u \in X^\circ$, is equivalent with the partial differential equation

$$- \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u}{\partial r} \right) - \frac{1}{r^2} \Delta_{S} u + Q(r) u = \lambda u + f(r,x') \quad \text{in } X^\circ. \quad (66)$$

Applying the projection $P$ to this equation and using $P \Delta_{S} = \Delta_{S} P = 0$, we obtain the equation $u_{\text{rad}} = K|_{X^\circ} f_{\text{rad}}$ for the radially symmetric functions $f_{\text{rad}} = Pf$ and $u_{\text{rad}} = Pu$, i.e., $u_{\text{rad}} = K|_{X_{\text{rad}}} f_{\text{rad}}$.

In order to prove the compactness of $K|_{X^\circ}$, we will apply the well-known compactness criterion of Fréchet and Kolmogorov in the Lebesgue space $X^\circ = L^1(\mathbb{R}^N; \varphi \, dx)$; see Edwards [9, Theorem 4.20.1, p. 269] or Yosida [23, Chapter X, Section 1, p. 275].

**Lemma 7.4.** Given any $\varepsilon > 0$, there exists a number $R \equiv R(\varepsilon) \in (0, \infty)$ such that for every $f \in B_{X^\circ}$ and $u = K|_{X^\circ} f$ we have

$$\int_{|x| \geq R} |u(x)| \varphi(|x|) \, dx \leq \varepsilon. \quad (67)$$
Proof. Since $X^\odot$ is a Banach lattice, it suffices to verify (67) for every $f \in \overline{B}_X^\odot$ satisfying $f \geq 0$ a.e. in $\mathbb{R}^N$; consequently, also $u \geq 0$ a.e. in $\mathbb{R}^N$.

So let $f \in \overline{B}_X^\odot$. We have

$$\| f \|_{X^\odot} = \int_{\mathbb{R}^N} f(x) \varphi(|x|) \, dx = \int_0^\infty \left( \int_{\mathbb{S}^{N-1}} f(r, x') \, d\sigma(x') \right) \varphi(r) r^{N-1} \, dr$$

$$= \sigma_{N-1} \int_0^\infty f_{\text{rad}}(r) \varphi(r) r^{N-1} \, dr$$  \hspace{1cm} (68)

and similarly

$$\int_{|x| \geq R} u(x) \varphi(|x|) \, dx = \int_R^\infty \left( \int_{\mathbb{S}^{N-1}} u(r, x') \, d\sigma(x') \right) \varphi(r) r^{N-1} \, dr$$

$$= \sigma_{N-1} \int_R^\infty u_{\text{rad}}(r) \varphi(r) r^{N-1} \, dr.$$ \hspace{1cm} (69)

Here, the functions $f_{\text{rad}} = \mathcal{P} f$ and $u_{\text{rad}} = \mathcal{P} u$ are in $X^\odot_{\text{rad}}$ and satisfy $f_{\text{rad}} \geq 0$ and $u_{\text{rad}} = \mathcal{K}_{X^\odot} f_{\text{rad}} \geq 0$ a.e. in $\mathbb{R}_+$ together with

$$\int_0^\infty f_{\text{rad}}(r) \varphi(r) r^{N-1} \, dr \leq 1/\sigma_{N-1}. \hspace{1cm} (70)$$

The operator $\mathcal{K}_{X^\odot_{\text{rad}}}$ being compact on $X^\odot_{\text{rad}}$, by Lemma 7.3, there exists a number $R = R(\varepsilon) \in (0, \infty)$ depending on $\varepsilon$ such that

$$\int_R^\infty u_{\text{rad}}(r) \varphi(r) r^{N-1} \, dr \leq \varepsilon / \sigma_{N-1}.$$ \hspace{1cm} (71)

holds for $u_{\text{rad}} = \mathcal{K}_{X^\odot} f_{\text{rad}}$ whenever $f_{\text{rad}} \in X^\odot_{\text{rad}}$ satisfies $f_{\text{rad}} \geq 0$ a.e. in $\mathbb{R}_+$ and (70). Thus, we have verified inequality (67). □

Proof of Proposition 7.1. It suffices to prove that $\mathcal{K}_{X^\odot}$ is compact. Let $0 < R < \infty$. Since the restricted resolvent $\mathcal{R}_R : X^\odot \to L^1(B_R(0))$ is compact, by Proposition 6.1, so is $\mathcal{R}_R^\odot : X^\odot \to X^\odot : f \mapsto \chi_{B_R(0)} u$, where $u = \mathcal{K}_{X^\odot} f$ and $\chi_{B_R(0)}$ denotes the characteristic function of the open ball $B_R(0) \subset \mathbb{R}^N$. Moreover, applying Lemma 7.4, we get $\mathcal{R}_R^\odot \to \mathcal{K}_{X^\odot}$ uniformly on $\overline{B}_X^\odot$ as $R \to \infty$. We invoke a well-known approximation theorem (Edwards [9, Theorem 9.2.6, p. 622] or Yosida [23, Chapter X, Section 2, p. 278]) to conclude that also the limit operator $\mathcal{K}_{X^\odot} : X^\odot \to X^\odot$ must be compact.

The proof of Proposition 7.1 is finished. □
8. Compactness by comparison of two potentials

Let us consider two potentials, \( q_j : \mathbb{R}^N \to \mathbb{R} \) for \( j = 1, 2 \), each assumed to be a continuous function satisfying only conditions (2) in place of \( q \). We denote by \( \Lambda_j = \Lambda_{q_j} \) the principal eigenvalue of the Schrödinger operator

\[
A_j = A_{q_j} \overset{\text{def}}{=} -\Delta + q_j(x) \quad \text{on } L^2(\mathbb{R}^N).
\]  

(72)

The associated eigenfunction \( \varphi_j = \varphi_{q_j} \) is normalized by \( \varphi_j > 0 \) throughout \( \mathbb{R}^N \) and \( \| \varphi_j \|_{L^2(\mathbb{R}^N)} = 1 \). Finally, we write \( X_j = X_{q_j} \) and \( X_j^\varnothing = L^1(\mathbb{R}^N; \varphi_j \, dx) \).

The following comparison result is natural (and holds without any growth conditions other than (2)).

**Proposition 8.1.** Assume \( q_1 \leq q_2 \) in \( \mathbb{R}^N \). Then the following statements hold.

(a) \( 0 < \Lambda_1 \leq \Lambda_2 < \infty \).

(b) For each \( \lambda < \Lambda_1 \),

\[
f \geq 0 \quad \text{in } L^2(\mathbb{R}^N) \quad \Rightarrow \quad (A_{q_2} - \lambda I)^{-1}f \leq (A_{q_1} - \lambda I)^{-1}f \quad \text{in } L^2(\mathbb{R}^N).
\]

(c) Given any \( \lambda < \Lambda_1 \), if the restriction \( (A_{q_1} - \lambda I)^{-1}|_{X_1 : X_1 \to X_1} \) of the resolvent \( (A_{q_1} - \lambda I)^{-1} \) to \( X_1 \) is weakly compact, then \( (A_{q_j} - \lambda I)^{-1}|_{X_1} \) is also compact for \( j = 1, 2 \).

(c') Given any \( \lambda < \Lambda_1 \), if the extension \( (A_{q_1} - \lambda I)^{-1}|_{X_1^\varnothing : X_1^\varnothing \to X_1^\varnothing} \) of the resolvent \( (A_{q_1} - \lambda I)^{-1} \) to \( X_1^\varnothing \) is weakly compact, then \( (A_{q_j} - \lambda I)^{-1}|_{X_1^\varnothing} \) is also compact for \( j = 1, 2 \).

**Corollary 8.2.** Assume \( q_1 \leq q_2 \) in \( \mathbb{R}^N \). If the weak compactness condition (the “if” part) in (c) or (c'), Proposition 8.1, is satisfied, for some \( \lambda < \Lambda_1 \), then we have \( \sup_{\mathbb{R}^N}(\varphi_2/\varphi_1) < \infty \) or, equivalently, \( X_2 \hookrightarrow X_1 \) is a continuous embedding.

In Proposition 8.1, parts (c) and (c'), weak compactness implies also (strong) compactness in the norm topology. We remark that parts (c) and (c') are equivalent, since \( (A_{q_j} - \lambda I)^{-1}|_{X_1} \) is the adjoint of \( (A_{q_j} - \lambda I)^{-1}|_{X_1^\varnothing} \). Indeed, for equivalence in (strong) compactness we may apply Schauder’s theorem again (Edwards [9, Corollary 9.2.3, p. 621] or Yosida [23, Chapter X, Section 4, p. 282]), whereas equivalence in weak compactness is guaranteed by the Gantmacher–Nakamura theorem (Edwards [9, Corollary 9.3.3, p. 625]). We will prove part (c') below using the well-known compactness criteria of Fréchet and Kolmogorov for (strong) compactness (Edwards [9, Theorem 4.20.1, p. 269] or Yosida [23, Chapter X, Section 1, p. 275]) and Dunford and Pettis for weak compactness (Edwards [9, Theorem 4.21.2, p. 274]) in the Lebesgue space \( X_1^\varnothing = L^1(\mathbb{R}^N; \varphi_1 \, dx) \).

**Proof of Proposition 8.1.** Part (a). Follows immediately from the Rayleigh quotient (12) combined with \( q_1 \leq q_2 \) in \( \mathbb{R}^N \).

Part (b). Let us fix \( \lambda < \Lambda_1 \) and \( f \geq 0 \) in \( L^2(\mathbb{R}^N) \). Then \( u_j = (A_{q_j} - \lambda I)^{-1}f \), for \( j = 1, 2 \), is a weak solution of the Schrödinger equation

\[
-\Delta u_j + q_j(x)u_j = \lambda u_j + f(x) \quad \text{in } L^2(\mathbb{R}^N).
\]  

(73)
More precisely, we have \( u_j \in \mathcal{V}_{q_j} \) and Eq. (73) holds in the sense of distributions on \( \mathbb{R}^N \) valued in \( \mathcal{V}_{q_j}' \), the dual space of \( \mathcal{V}_{q_j} \) with respect to the duality induced by the natural inner product on \( L^2(\mathbb{R}^N) \). Notice that the embeddings of Hilbert spaces

\[
\mathcal{V}_{q_2} \hookrightarrow \mathcal{V}_{q_1} \hookrightarrow L^2(\mathbb{R}^N) \hookrightarrow \mathcal{V}_{q_1}' \hookrightarrow \mathcal{V}_{q_2}'
\]

are dense and continuous, by \( q_1 \leq q_2 \) in \( \mathbb{R}^N \). The weak maximum principle yields \( u_j \geq 0 \) a.e. in \( \mathbb{R}^N \). We need to show \( u_2 \leq u_1 \) a.e. in \( \mathbb{R}^N \). In other words, we have to prove that the function

\[
v = (u_2 - u_1)^+ \equiv \max\{u_2 - u_1, 0\}
\]

vanishes a.e. in \( \mathbb{R}^N \).

First, from

\[
\nabla v(x) = \begin{cases} 
\nabla (u_2 - u_1)(x) & \text{if } u_2(x) > u_1(x); \\
0 & \text{if } u_2(x) \leq u_1(x),
\end{cases}
\]

for a.e. \( x \in \mathbb{R}^N \), and \( 0 \leq v \leq u_2 \) a.e. in \( \mathbb{R}^N \), we deduce that \( v \in \mathcal{V}_{q_2} \) (\( \subset \mathcal{V}_{q_1} \)). Furthermore, we have \( u_2 - u_1 \in \mathcal{V}_{q_1} \). Subtracting Schrödinger equations (73) (\( j = 1, 2 \)) from one another we arrive at

\[
-\Delta(u_1 - u_2) + q_1(x)(u_1 - u_2) = \lambda(u_1 - u_2) + (q_2(x) - q_1(x))u_2(x)
\]

in the sense of distributions valued in \( \mathcal{V}_{q_2}' \). We multiply this equation by \( v \in \mathcal{V}_{q_2} \) and then integrate over \( \mathbb{R}^N \), thus arriving at

\[
-\int_{\mathbb{R}^N} |\nabla v|^2 \, dx - \int_{\mathbb{R}^N} q_1(x)v^2 \, dx = -\lambda \int_{\mathbb{R}^N} v^2 \, dx + \int_{\mathbb{R}^N} (q_2 - q_1)u_2v \, dx.
\]

We combine this result with (12) and \( q_1 \leq q_2 \) in \( \mathbb{R}^N \) to get

\[
\lambda \int_{\mathbb{R}^N} v^2 \, dx \geq \int_{\mathbb{R}^N} |\nabla v|^2 \, dx + \int_{\mathbb{R}^N} q_1(x)v^2 \, dx + \int_{\mathbb{R}^N} (q_2 - q_1)u_2v \, dx \geq \Lambda_1 \int_{\mathbb{R}^N} v^2 \, dx.
\]

Since \( \lambda < \Lambda_1 \), this inequality is possible only if \( v = 0 \) holds a.e. in \( \mathbb{R}^N \). We have verified \( u_2 \leq u_1 \) a.e. in \( \mathbb{R}^N \).

As parts (c) and (c') are equivalent, we prove (c'). Let \( \lambda < \Lambda_1 \). It follows from part (b) that also \( (A_{q_2} - \lambda I)^{-1} \) possesses a unique extension \( (A_{q_2} - \lambda I)^{-1}|_{X_1^\ominus} \) to a bounded linear operator on \( X_1^\ominus \). More precisely, if \( 0 \leq f \in X_1^\ominus \) then \( u_j = (A_{q_j} - \lambda I)^{-1}|_{X_1^\ominus} f \), for \( j = 1, 2 \), satisfy \( 0 \leq u_2 \leq u_1 \) a.e. in \( \mathbb{R}^N \).

Assume that \( (A_{q_1} - \lambda I)^{-1}|_{X_1^\ominus} \) is weakly compact on \( X_1^\ominus \). We employ the Dunford–Pettis criterion (Edwards [9, Theorem 4.21.2, p. 274]) for weak compactness in the Lebesgue space.
$X_1^\circ = L^1(\mathbb{R}^N; \varphi_1 \, dx)$ to conclude that, given any $\varepsilon > 0$, there exists $R \equiv R(\varepsilon) \in (0, \infty)$ such that for every $f \in \mathcal{B}_{X_1^\circ} = \{ f \in X_1^\circ : \| f \|_{X_1^\circ} \leq 1 \}$, $f \geq 0$ in $\mathbb{R}^N$, we have

$$\int_{|x| \geq R} u_1(x)\varphi_1(|x|) \, dx \leq \varepsilon.$$ 

Recall that $u_j = (A_{q_j} - \lambda I)^{-1}|_{X_1^\circ} f$. Since $0 \leq u_2 \leq u_1$ holds a.e. in $\mathbb{R}^N$, we have

$$\int_{|x| \geq R} u_j(x)\varphi_1(|x|) \, dx \leq \varepsilon$$

for $j = 1, 2$. Consequently,

$$\int_{|x| \geq R} |u_j(x)|\varphi_1(|x|) \, dx \leq \varepsilon \quad \text{whenever } f \in \mathcal{B}_{X_1^\circ}, \quad (75)$$

owing to $|u_j| \leq (A_{q_j} - \lambda I)^{-1}|_{X_1^\circ}|f|$ which, in turn, follows from $\pm f \leq |f|$.

Since the restricted resolvent $R_{j,R} : X_1^\circ \to L^1(B_R(0)) : f \mapsto u_j|_{B_R(0)}$ is compact, by Proposition 6.1, so is $R_{j,R}^\circ : X_1^\circ \to X_1^\circ : f \mapsto \chi_{B_R(0)}u_j$, where $u_j = (A_{q_j} - \lambda I)^{-1}|_{X_1^\circ} f$ and $\chi_{B_R(0)}$ denotes the characteristic function of the open ball $B_R(0) \subset \mathbb{R}^N$. Moreover, applying inequality (75), we get $R_{j,R}^\circ \to (A_{q_j} - \lambda I)^{-1}|_{X_1^\circ}$ uniformly on $\mathcal{B}_{X_1^\circ}$ as $R \to \infty$. We invoke a well-known approximation theorem (Edwards [9, Theorem 9.2.6, p. 622] or Yosida [23, Chapter X, Section 2, p. 278]) to conclude that also the limit operator $(A_{q_j} - \lambda I)^{-1}|_{X_1^\circ} : X_1^\circ \to X_1^\circ$ must be compact. This proves part (c').

**Proof of Corollary 8.2.** Parts (c) and (c') of Proposition 8.1 being equivalent, assume that (c) holds. Then the conclusion of our corollary follows immediately from Lemma 4.3, with $X_1$ in place of $X$ and $(A_{q_2} - \lambda I)^{-1}|_{X_1}$ in place of $T$. Indeed, since the spectra of $(A_{q_2} - \lambda I)^{-1}$ on $L^2(\mathbb{R}^N)$ and $(A_{q_2} - \lambda I)^{-1}|_{X_1}$ on $X_1$ coincide, equation $(A_{q_2} - \lambda I)^{-1}\varphi_2 = (A_2 - \lambda)^{-1}\varphi_2$ for $\varphi_2 \in L^2(\mathbb{R}^N)$ forces $\varphi_2 \in X_1$. \(\square\)

**9. Positivity and compactness for $q(x)$ nonradial**

The results of the previous section allow us to finally remove the restriction that $q$ be radially symmetric, i.e., we consider a potential $q : \mathbb{R}^N \to \mathbb{R}$ that satisfies hypothesis (H$_q$).

**9.1. Compactness of $\mathcal{K}|_X$ for $q$ nonradial**

**Proof of Theorem 3.2.** Part (a). According to hypothesis (H$_q$), potentials $q$, $Q_1$, and $Q_2$ satisfy (19), that is,

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_{12}Q_1(|x|) \quad \text{for all } x \in \mathbb{R}^N.$$
Consequently, Remark 2.1 guarantees that these potentials satisfy also conditions (2) in place of \( q \). We denote by \( \Lambda_q , A_{Q_1} , \) and \( A_{Q_2} \) the principal eigenvalues of the Schrödinger operators \( A_q , A_{Q_1} , \) and \( A_{Q_2} \) with potentials \( q , Q_1 , \) and \( Q_2 \), respectively. The associated eigenfunctions \( \varphi_q , \varphi_{Q_1} , \) and \( \varphi_{Q_2} \) are normalized by being positive throughout \( \mathbb{R}^N \) and having the \( L^2(\mathbb{R}^N) \) norm = 1.

First, we have \( 0 < \Lambda_{Q_1} = \Lambda_q < \Lambda_{Q_2} < \infty \), by Proposition 8.1, part (a). From Proposition 7.1 we infer that, given any \( \lambda < \Lambda_{Q_1} \), the restriction \( (A_{Q_1} - \lambda I)^{-1}|_{X_{Q_1}} : X_{Q_1} \to X_{Q_1} \) of the resolvent \( (A_{Q_1} - \lambda I)^{-1} \) to \( X_{Q_1} \) is compact (hence, also weakly compact). By Proposition 8.1, part (c), the same is true of the restrictions \( (A_q - \lambda I)^{-1}|_{X_q} \) and \( (A_{Q_2} - \lambda I)^{-1}|_{X_{Q_2}} \) to \( X_q \) and \( X_{Q_2} \), respectively. The associated eigenfunctions \( \varphi_{Q_1} \), \( \varphi_q \), and \( \varphi_{Q_2} \) are normalized by being positive throughout \( \mathbb{R}^N \) and having the \( L^2(\mathbb{R}^N) \) norm = 1.

Second, denoting

\[
V_j(r) = Q_j(r) - \Lambda + \frac{(N - 1)(N - 3)}{4r^2}
\]

for \( r > r_0 , \ j = 1 , 2 , \) (76)

where \( 0 < r_0 < \infty \) is large enough, such that \( V_j(r) > 0 \) for all \( r > r_0 \), we take advantage of Lemma 4.1 for \( \varphi_{Q_j}(r) \), formula (32), to obtain

\[
\frac{\varphi_{Q_1}(r)}{\varphi_{Q_2}(r)} = c_{12} \left( \frac{V_2(r)}{V_1(r)} \right)^{1/4} \exp \left( \eta_{12}(r) + \int_{r_0}^r \left[ \frac{\varphi_{Q_1}(t)}{\varphi_{Q_2}(t)} \right] \frac{\varphi_{Q_2}(t)}{\varphi_{Q_1}(t)} \right) \]

for all \( r > r_0 \), where \( c_{12} > 0 \) is a constant and \( \eta_{12}(r) \to 0 \) as \( r \to \infty \). With regard to Remark 4.2, and thanks to conditions (19) and (20) (in the form of (8)), this formula yields \( \sup_{\mathbb{R}^N}(\varphi_{Q_1}/\varphi_{Q_2}) < \infty \) or, equivalently, \( X_{Q_1} \hookrightarrow X_{Q_2} \) is a continuous embedding. Here, we have used the fact that \( V_2 - V_1 = Q_2 - Q_1 \) implies the identity

\[
V_2(t)^{1/2} - V_1(t)^{1/2} = \frac{Q_2(t)^{1/2} + Q_1(t)^{1/2}}{V_2(t)^{1/2} + V_1(t)^{1/2}} \left[ Q_2(t)^{1/2} - Q_1(t)^{1/2} \right].
\]

Finally, let us rewrite the equation \( A_q \varphi_q = \Lambda_q \varphi_q \) for \( \varphi_q \in X_{Q_1} = X_{Q_2} \) as

\[
-\Delta \varphi_q + Q_2(|x|)\varphi_q = f(x) \quad \text{in } X_{Q_2}^\circ,
\]

where

\[
f(x) = \left[ Q_2(|x|) - q(x) + \Lambda_q \right] \varphi_q(x) \geq \Lambda_q \varphi_q(x) > 0 , \quad x \in \mathbb{R}^N ,
\]

by condition (19). Notice that

\[
\sup_{\mathbb{R}^N}(\varphi_q/\varphi_{Q_1}) < \infty , \quad \sup_{\mathbb{R}^N}(\varphi_{Q_2}/\varphi_{Q_1}) < \infty , \quad \text{and } \sup_{\mathbb{R}^N}(\varphi_{Q_1}/\varphi_{Q_2}) < \infty ,
\]

combined with \( \int_{\mathbb{R}^N} Q_2 \varphi_{Q_2}^2 \ dx < \infty \), yield \( \int_{\mathbb{R}^N} Q_2 \varphi_q \varphi_{Q_2} \ dx < \infty \), that is, \( Q_2 \varphi_q \in X_{Q_2}^\circ = L^1(\mathbb{R}^N; \varphi_{Q_2} \ dx) \). Consequently, also \( Q_2 - q \varphi_q \in X_{Q_2}^\circ \) which guarantees \( f \in X_{Q_2}^\circ \). We apply Proposition 5.1 with \( Q = Q_2 \) and \( \lambda = 0 < \Lambda_Q \) to conclude that \( \inf_{\mathbb{R}^N}(\varphi_q/\varphi_{Q_2}) > 0 \) or, equivalently, \( X_q \hookrightarrow X_{Q_2} \) is a continuous embedding.
Summarizing the results proved in this section for \( \varphi_q, \varphi_{Q_1}, \) and \( \varphi_{Q_2}, \) we arrive at \( X_q = X_{Q_1} = X_{Q_2}, \) i.e., \( \gamma_1 \varphi_q \leq \varphi_{Q_1}, \varphi_{Q_2} \leq \gamma_2 \varphi_q \) everywhere in \( \mathbb{R}^N, \) where \( 0 < \gamma_1 \leq \gamma_2 < \infty \) are some constants. As we already know that the restriction \( (A_q - \lambda I)^{-1}|_{X_{Q_1}} \) to \( X_{Q_1} \) is compact, part (a) follows immediately.

**Part (b).** In the remaining part of the proof we abbreviate \( A = A_q, \Lambda = \Lambda_q, \) and \( X = X_q. \) Let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( A, \) that is, \( Av = \lambda v \) for some \( v \in L^2(\mathbb{R}^N), v \neq 0. \) Since \( A \) is positive definite and selfadjoint on \( L^2(\Omega), \) its inverse \( A^{-1} \) is bounded on \( L^2(\mathbb{R}^N). \) Property (2) implies that \( A^{-1} \) is also compact. Consequently, \( \lambda \in \mathbb{R} \) and \( \lambda \geq \Lambda > 0. \) Given \( v \in L^2(\mathbb{R}^N), v \neq 0, \) it follows that equation \( Av = \lambda v \) is equivalent with \( A^{-1}v = \lambda^{-1}v. \) By part (a), also the restriction \( A^{-1}|_X \) to \( X \) is compact. Now we can apply Lemma 4.3 with \( T = A^{-1}|_X \) compact on \( X \) to obtain the conclusion of part (b).

**Part (c).** Assume that \( \lambda \in \mathbb{C} \) is not an eigenvalue of \( A. \) With regard to part (a) we may restrict ourselves to the case \( \lambda \notin (-\infty, \Lambda). \) Hence, by the Riesz–Schauder theory applied to \( A^{-1}, \) which is compact on \( L^2(\mathbb{R}^N), \) \( \lambda \) is in the resolvent set of \( A \) and the resolvent \( \mathcal{K} = (A - \lambda I)^{-1} \) is compact on \( L^2(\mathbb{R}^N). \) We refer to Edwards [9, Theorem 9.10.2, p. 679] or Yosida [23, Chapter X, Theorem 5.1, p. 283] for the Riesz–Schauder theorem. Consequently, the following identities hold on \( L^2(\mathbb{R}^N): \)

\[
\mathcal{K}(\lambda^{-1} I - A^{-1}) = (\lambda^{-1} I - A^{-1})\mathcal{K} = \lambda^{-1} A^{-1} - I. \tag{78}
\]

In particular, \( \lambda^{-1} \) cannot be an eigenvalue of \( A^{-1}. \) So \( \lambda^{-1} \) is not an eigenvalue of \( A^{-1}|_X \) either. The restriction \( A^{-1}|_X \) being compact on \( X, \) by part (a), we may apply Lemma 4.3 with \( T = A^{-1}|_X \) again to conclude that the restriction \( \lambda^{-1} I - A^{-1}|_X \) of \( \lambda^{-1} I - A^{-1} \) to \( X \) has a bounded inverse, say, \( L = (\lambda^{-1} I - A^{-1}|_X)^{-1}. \) Hence, from (78) we deduce \( \mathcal{K}|_X = \lambda^{-1} L(A^{-1}|_X) \) which shows that also \( \mathcal{K}|_X \) is compact on \( X \) as claimed.

The proof of Theorem 3.2 is now complete. \( \square \)

### 9.2. Positivity for a nonradial potential \( q(x) \)

**Proof of Theorem 3.1.** Let \( -\infty < \lambda < \Lambda_q \) and \( u = (A_q - \lambda I)^{-1}|_{X_q^\circ} f. \) Since \( 0 \leq f \in X_q^\circ, \) we may apply the weak maximum principle (as in the proof of Proposition 5.1) to get \( 0 \leq u \in X_q^\circ. \) Hence, it suffices to prove our theorem for \( g = \min\{f, \varphi_q\} \) in place of \( f, \) that is, \( 0 \leq f \leq \varphi_q \) a.e. and \( f \neq 0 \) in \( \mathbb{R}^N. \) This forces also \( 0 \leq u \leq (A_q - \lambda)^{-1} \varphi_q \) a.e. and \( u \neq 0 \) in \( \mathbb{R}^N, \) by the weak maximum principle again.

Similarly as in the proof of Theorem 3.2, part (a) above, let us rewrite the equation \( A_q u = \lambda u + f \) for \( u \in X_q, \) with \( f \in X_q, X_q = X_{Q_1} = X_{Q_2}, \) as

\[
-\Delta u + Q_2(|x|)u = \lambda u + g(x) \quad \text{in } X_{Q_2}^\circ,
\]

where

\[
g(x) = [Q_2(|x|) - q(x)]u(x) + f(x) \geq f(x), \quad x \in \mathbb{R}^N,
\]

by condition (19) and \( u \geq 0 \) a.e. in \( \mathbb{R}^N. \) Again, we combine Corollary 3.3 with \( \int_{\mathbb{R}^N} Q_2 \varphi_{Q_2}^2 \, dx < \infty \) to get \( \int_{\mathbb{R}^N} Q_2 u \varphi_{Q_2} \, dx < \infty, \) that is, \( Q_2 u \in X_{Q_2}^\circ = L^1(\mathbb{R}^N; \varphi_{Q_2} \, dx). \) Consequently, also \( (Q_2 - q)u \in X_{Q_2}^\circ \) which guarantees \( g \in X_{Q_2}^\circ. \) We apply Proposition 5.1 with \( Q = Q_2 \) and
\[ \lambda < A_q \leq \Lambda Q = \Lambda Q_2 \]

to conclude that \( \inf_{\mathbb{R}^N} (u/\varphi_{Q_2}) > 0 \) or, equivalently, \( u \geq c \varphi_q \) a.e. in \( \mathbb{R}^N \), with some constant \( c \equiv c(f) > 0 \). \( \square \)

References


