Europ. J. Combinatorics (1996) **17,** 485–491

Tiling with Bars and Satisfaction of Boolean Formulas

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Let F be a figure formed from a finite set of cells of the planar square lattice. We first prove that the problem of tiling such a figure with bars formed from 2 or 3 cells can be reduced to the logic problem 2-SAT. Afterwards, we deduce a linear-time algorithm of tiling with these bars.

1. Introduction

The problem of tiling a finite figure of the plane using copies of a finite set S of basic tiles is usually NP-hard even if the set of basic tiles is $\{h_2, v_3\}$, where h_2 denotes a horizontal 2×1 rectangle and v_3 denotes a vertical 1×3 rectangle. This fact has been proved by J. M. Robson [2] reducing the classical logic problem 3-SAT to the above tiling problem, in a polynomial time. (3-SAT is, given a finite set S of logical clauses, each of which contains at most 3 literals, to attribute a boolean value to each literal in such a way that the conjunction of the clauses of S is satisfied (or to indicate that this conjunction cannot be satisfied).)

Nevertheless, algebraic methods developed by C. and R. Kenyon [5] allow us to give a polynomial algorithm to tile a finite figure without a hole with copies of h_2 or v_3.

The present paper is devoted to the problem of tiling a figure of the plane when the set of basic tiles is $\{h_2, v_2, h_3, v_3\}$ (i.e. the set of the bars of length 2 or 3). A previous paper [8] has been devoted to this subject, but only the case of figures without holes was treated.

We prove by purely combinatorial methods the following result.

Main Result. *The problem of tiling a figure with bars of length* 2 *or* 3 *can be reduced in linear time to the logic problem* 2*-SAT.*

This result permits us to obtain a linear algorithm which, given a finite figure F of the plane, either exhibits a tiling of F or indicates that F cannot be tiled.

2. Definitions and Notation

A *cell* is a (closed) unit square of the plane $\mathbb{R}^2$, the center of which has integer co-ordinates. The cell (x, y) denotes the cell the center of which is the point (x, y). Integer x (respectively y) is called the *vertical* (respectively *horizontal*) *co-ordinate* of (x, y).

Cell C' is a *neighbor* of cell C if C and C' have a common edge. One canonically defines the *left, right, upper* and *lower* neighbors of a cell.

A *figure* is a finite union of cells. The area of a figure F is the number of cells included in F.

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An *isolated cell* of a figure F is a cell of F with no neighbor in F.

A *peak* of a figure F is a cell of F which has exactly one neighbor in F. A peak C of F is said to be *vertical* (respectively *horizontal, left, right, lower* or *upper*) if C is the vertical (respectively horizontal, left, right, lower or upper) neighbor of a cell of F.

A vertical (respectively horizontal) *bridge* of a figure F is a cell of F which has exactly two neighbors in F which are its vertical (respectively horizontal) neighbors.

Let m be an integer such that $m \geqslant 2$. An *m-bar* is a rectangle of length m and unit width, formed from the union of m neighboring cells. An m-bar B is *vertical* (respectively *horizontal*) if all the cells of B have the same horizontal (respectively vertical) co-ordinate.

3. Different Formulations of the Problem

A *tiling* Φ of a figure F is a set of 2-bars and 3-bars included in F such that each cell of F is included in exactly one element of set Φ.

A *packing* Π of a figure F is a set of 2-bars included in F such that each cell of F is included in at most one element of set Π. A *default* of a packing of F is a cell of F which is not an element of a bar of this packing. A default C of a packing Π of F is *pointed* if there exists a 2-bar B of Π such that $B \cup C$ is a 3-bar.

Proposition 3.1. *Let F be a finite figure. The following propositions are then equivalent*:
(i) *there exists a tiling Φ of F*;
(ii) *there exists a packing Π of F, all of the defaults of which are pointed.*
Moreover, a tiling of F can be obtained from a packing of F in linear time (*in the area of the figure*) *and a packing of F can be obtained from a tiling of F in linear time.*

Proof. Assume that figure F can be tiled and let Φ be a tiling. A packing Π of F can be naturally obtained from Φ replacing each 3-bar B by a 2-bar B' formed from two cells of B. Each default of Π is obviously pointed.

Conversely, assume that there exists a packing Π of F such that each default of Π is pointed. For each default C of Π, chooose a 2-bar B_C of Π such that $C \cup B_C$ is a 3-bar. For each bar B of Π, let B' be the bar formed from cells of B and cells C such that $B_C = B$. Set X consists of bars B' with 2-bars, 3-bars and 4-bars. A tiling Φ of F is obtained from X by replacing each 4-bar B' of X by two 2-bars B_1 and B_2 such that $B' = B_1 \cup B_2$.

The transformations used can trivially be done in linear time. □

Informally, in this section, we have transformed the problem of tiling that we can solve into a packing problem.

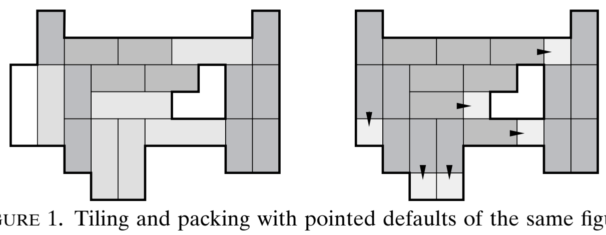

Figure 1. Tiling and packing with pointed defaults of the same figure.

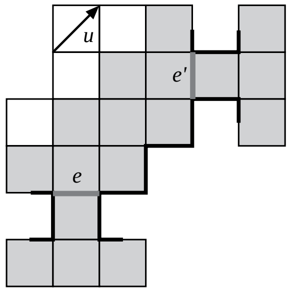

FIGURE 2. Conflict of edges.

4. NECESSARY CONDITIONS FOR A FIGURE TO BE TILEABLE

For the rest of this paper, F denotes a finite figure and α (respectively β) denotes the set of the edges e of the peaks (respectively bridges) of F which are not edges of the frontier of F.

DEFINITION 4.1. Let e and e' be two edges of $\alpha \cup \beta$. We say that e and e' create a *conflict* if there exists a (non-empty) sequence $(e_1, e_2, \dots, e_{2k})$ of edges of the frontier of F and a vector $u = (a, b)$, with $a^2 = b^2 = 1$, such that (see Figure 2):
(i) for each integer i, with $0 \leqslant i < 2k$, edges e_i and e_{i+1} have a common extremity;
(ii) for each integer i, with $0 \leqslant i < 2k - 1$, $e_{i+2} = t_u(e_i)$, where t_u denotes the translation of vector u;
(iii) edge e is vertical iff edge e_1 is vertical and e and e_1 have a common extremity which is not an extremity of e_2;
(iv) edge e' is vertical iff edge e_{2k} is vertical and e' and e_{2k} have a common extremity which is not an extremity of e_{2k-1}.

For each edge e of $\alpha \cup \beta$, we define a boolean variable x_e. We state the following rules.

RULES OF COMPATIBILITY. (i) If e is an element of α, then $x_e = 1$.
(ii) If e and e' are edges of the same bridge, then $x_e \vee x_{e'} = 1$.
(iii) If e and e' are in conflict in F, then $\neg x_e \vee \neg x_{e'} = 1$.

PROPOSITION 4.2. *If figure F can be tiled with bars, then the conjunction of rules of compatibility can be satisfied.*

PROOF. Let be such a tiling. Variable x_e takes value 1 when the two cells of F which share edge e are in the same bar (hence e is covered by a tile) and variable x_e takes value 0 otherwise (hence e is an edge of the boundary of a tile). □

REMARK 4.3. For each edge e of $\alpha \cup \beta$, variable x_e appears in at most 3 rules of compatibility.

5. SUFFICIENCY OF THE CONDITIONS

THEOREM 5.1. *Let F be a figure* (*with no isolated cell*) *such that the rules of compatibility of F can be simultaneously satisfied. There exists a packing of F such that each default is pointed.*

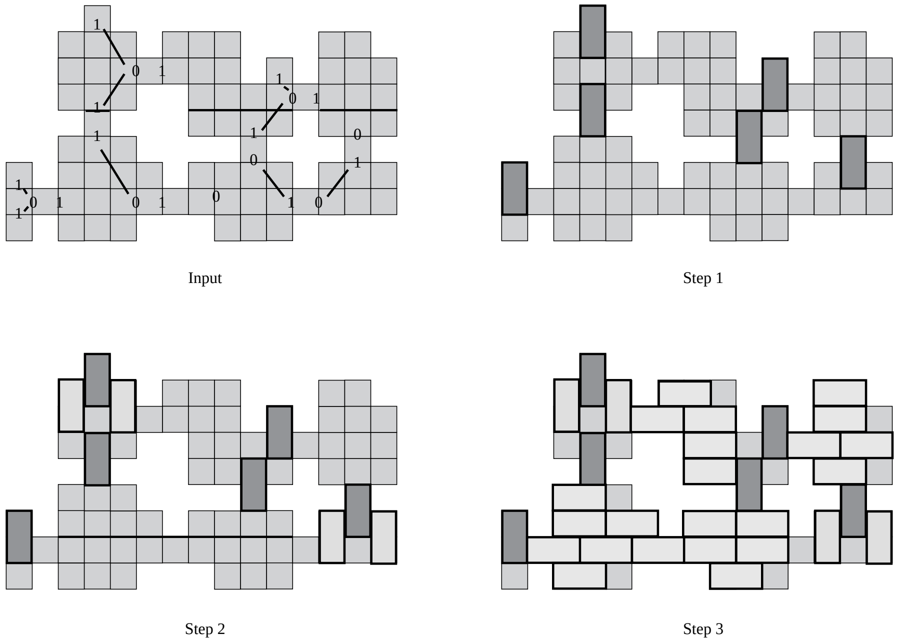

FIGURE 3. The packing algorithm.

PROOF. Assume that, for each edge e of $\alpha \cup \beta$, a value of x_e is given in such a way that all the rules of compatibility are satisfied. Consider the following algorithm (see Figure 3).

PACKING ALGORITHM. *Initialization*: construct a list λ of the vertical bridges and the vertical peaks of F, mark the edges of the frontier of F and construct a list Λ of cells with two vertical marked edges and a horizontal marked edge.

Moreover, Π denotes a set of 2-bars. For initialization, Π is empty.

Step 1: successively take each cell A of λ.

If A is either pointed or covered by a 2-bar which has been previously put, take the successive cell.

Otherwise, let A' and A'' be, respectively, the upper and lower neighbors of A, and let e' and e'' be, respectively, the upper and lower edges of A. If e' is an edge of $\alpha \cup \beta$ and $x_{e'} = 1$, put the vertical 2-bar $B_1 = A \cup A'$ in set Π and mark the vertical edges of A'. Otherwise, put bar $B_2 = A \cup A''$ in set Π, and mark the vertical edges of A''. In the two cases, update list Λ.

Step 2: take the first cell C of Λ.

If C is either pointed or covered by a 2-bar which has been previously put, delete C from list Λ.

Otherwise, let e denote the unmarked edge of C. Put vertical 2-bar $B = C \cup C'$ in set Π, where C and C' are the cells of F which share edge e, mark the vertical edges of C' and update list Λ.

This instruction is repeated until list Λ is empty.

Step 3: successively take each cell D of F.

If C is either pointed or covered by a 2-bar which has been previously put, take the successive cell.

Otherwise, let L and R, respectively, denote the left and right neighbors of D. If L is a cell of F which has not been previously covered by a 2-bar, put bar $B' = D \cap L$ in set Π; otherwise put bar $B'' = D \cup R$.

Proposition 5.2. *The above algorithm gives a packing Π of F with 2-bars such that each default is pointed.*

Proof. We have to prove the following.

(i) In Step 1, cells A' and A'' have not been previously covered by a 2-bar. This is obvious, since A is not pointed,

(ii) In Step 2, cell C' is a cell of F which has not been previously covered by a 2-bar. This is obvious, since edge e is unmarked (which proves that C' is in F) and C is not pointed.

(iii) In step 3, at least one of the cells L and R is in F and has not been previously covered by a 2-bar. This needs considerable attention.

Let $(B_1, B_2, \ldots, B_q)$ denote the sequence of the vertical 2-bars used in the execution of step 1 and let $(B_{q+1}, B_{q+2}, \ldots, B_p)$ denote the sequence of the vertical 2-bars used in the execution of step 2, in the order of placing. For each integer i such that $1 \leqslant i \leqslant p$, let S_i denote the set of marked edges after having put bars B_j, with $j \leqslant i$ (let S_0 denote the edges of the frontier of F), and let C_i denote the cell which has forced us to put B_i (i.e. the bridge or peak for step 1, the cell with three marked edges, one horizontal and two vertical for step 2).

Assume that, during the execution of step 3, a non-pointed cell D of F, the horizontal neighbors L and R of which both are either not included in F or previously covered by a 2-bar, arises. Since D is not a bridge, one can assume, without loss of generality, that cell L is in F.

Since cell D is not pointed, cell L is covered by a vertical 2-bar B_{i_0}, with $1 \leqslant i_0 \leqslant p$. Moreover, cell L cannot be C_{i_0}, since the right edge of L is not in S_{i_0-1} (since D is not covered by a bar B_j with $j < i_0$). Assume that C_{i_0} is the lower neighbor of L (the case in which C_{i_0} is the upper neighbor of L is treated in a similar way).

Let D' denote the lower neighbor of D; if D' is in F and is covered during step 1 or step 2, then D is pointed, which is a contradiction; if D is in F and is not covered after step 2, then the right edge of C_{i_0} is not in S_{i_0-1}, which is a contradiction. Thus, necessarily, cell D' is not in F.

Let D'' denote the upper neighbor of D. If D'' is in F and is covered during step 1 or step 2, then D is pointed, which is a contradiction; if D'' is in F and is not covered after step 2, then we remark that the right edge of D is necessarily marked at the end of step 2 (otherwise R is in F and is covered by a horizontal 2-bar, which yields the result that D is not pointed). This is a contradiction, since D has three marked edges, which contradicts the emptiness of list Λ after step 2. Thus cell D'' is not in F.

Let e and e', respectively, denote the left and right edges of D. The above remarks prove that e is an edge of $\alpha \cup \beta$. ∎

Lemma 5.3. *The equality $x_e = 0$ holds.*

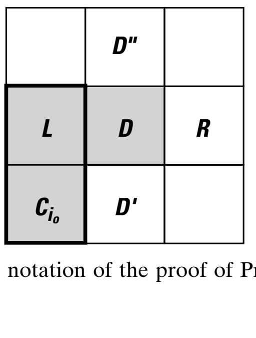

Figure 4. The notation of the proof of Proposition 5.2.

The proof of this lemma is given below. From this lemma, cell D is a horizontal bridge of F and e' is an edge of $\alpha \cup \beta$. As for x_e, one can obtain $x_{e'} = 0$, which contradicts part (ii) of the rules of compatibility. Thus the existence of D is impossible, which proves the validity of the algorithm. □

PROOF OF LEMMA 5.3. Assume that $x_e = 1$.

If $1 \leq i_0 \leq q$, then C_{i_0} is either a peak of a bridge of F. Let e_{i_0} be the upper edge of C_{i_0}. Edge e_{i_0} is an edge of $\alpha \cup \beta$, and $x_{e_{i_0}} = 1$. Thus edges e and e_{i_0} create a conflict, which is a contradiction. Thus we have $q < i_0 \leq p$.

This yields the result that the lower edge of C_{i_0} is marked; thus the lower neighbor of C_{i_0} is not in F (we remark that each horizontal marked edge is an edge of the frontier of F). Moreover, the left neighbor L_{i_0} of C_{i_0} is in F (since C_{i_0} is not a peak of F) and the left edge e'_{i_0} of C_{i_0} is an element of s_{i_0-1}; thus there exists a vertical 2-bar B_{i_1}, with $i_1 \leq i_0$, such that B_{i_1} contains L_{i_0}.

Edge e'_{i_0} is not an element of S_{i_1-1}, since neither L_{i_0} nor C_{i_0} are covered by a 2-bar B_j such that $j \leq i_1 - 1$. Thus, we have $L_{i_0} \neq C_{i_1}$.

Moreover, C_{i_1} is not the upper neighbor of L_{i_0}, since L is not covered by a 2-bar B_j such that $j \leq i_1 - 1$. Thus, cell C_{i_1} is the lower neighbor of L_{i_0}.

As for i_0, we have $i_1 > q$, since otherwise the upper edge e_{i_1} of cell C_{i_1} is such that $x_{e_{i_1}} = 1$ and e and e_{i_0} create a conflict. Thus, as for C_{i_0}, the lower neighbor of C_{i_1} is not in F and the left neighbor L_{i_1} of C_{i_1} is covered by a 2-bar B_{i_2}, with $i_2 < i_1$, and C_{i_2} is the lower neighbor of L_{i_1}. This kind of argument can be repeated *ad infinitum*; thus we have an infinite sequence $(C_{i_j})_{j \in \mathbb{N}}$ of cells of f such that, for each integer j, $C_{i_{j+1}}$ is the lower neighbor of the left neighbor of C_{i_j}, which is a contradiction. Thus the assumption $x_e = 1$ is false. □

6. COMPLEXITY

In this section, we recall all the steps of an algorithm to tile a figure F, given for input, and we study its complexity. Let n denote the area of a figure F and let m denote the number of edges of cells of F. Notice that $m \leq 4n$ and $n \leq 2m$.

Step 1. Verify that F has no isolated cell. This costs at most $O(n)$ time units.

Step 2. Construct the list of conditions of compatibility. This can easily be constructed by successively exploring the closed curves of $\mathbb{R}^2$ which are the connected components of the frontier of the figure. Thus, this list can be constructed in $O(n)$ time units.

Step 3. The set of rules of compatibility gives an instance of the logic problem 2-SAT, which has a complexity in $O(n' + m')$, where n' (respectively m') is the number of literals (respectively m') used (this is a classical result of complexity theory (see [1] for details)). Thus, a solution can be found in at most $O(m)$ time units (since $n' \leq n$ and $m' \leq 3m$ from Remark 4.2).

If no solution exists, the algorithm stops: F cannot be tiled. Otherwise, the algorithm executes the algorithm of Section 5.

Step 4. execution of the packing algorithm. Clearly, each step of this algorithm needs at most $O(n)$ time units, since each cell of F appears at most once in each list. At the end of this step, a packing Π of F with 2-bars is constructed.

TABLE 1

Tiling complexity	Figures without hole	Figures which may have a hole
$\{h_2, v_2\}$	Linear [9]	Polynomial ($n^{\frac{3}{2}}$) [4]
$\{h_2, v_3\}$	Linear [5]	NP-complete [2]
$\{h_2, v_2, h_3, v_3\}$	Linear [8]	Linear
$\{h_2, v_2, h_3\}$	Linear [10]	

Step 5. Construct a tiling from the packing. This needs at most $O(n)$ time units, from Proposition 3.1.

We can conclude that we have a linear (in m or n) algorithm of tiling.

7. Open Problems

All of the results of the complexity obtained about tiling with bars give a strange situation, as shown in Table 1.

A direction for research is the problem of tiling figures with bars of length at least m (integer m being fixed). The problem treated in this paper is clearly equivalent to the problem of tiling figures with bars of length at least 2.

Another question which has not previously been studied is the problem of tiling a figure of the planar hexagonal lattice with 2-bars and 3-bars. In this lattice, the problem of tiling a simply connected figure with 2-bars can be solved in the linear time (see [6]),and the problem of tiling a figure with 3-bars seems to be very difficult, since only very partial results have been obtained (see [3], [7] and [9]), and the tile group is very complex.

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Received 20 February 1995 and accepted 20 June 1995

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