Carathéodory–Fejér interpolation in the ball

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Abstract

We use the theory of reproducing kernel Hilbert spaces to solve a Carathéodory–Fejér interpolation problem in the class of Schur multipliers of the reproducing kernel Hilbert space of functions analytic in the unit ball of $\mathbb{C}^N$ with reproducing kernel $1/(1 - \sum_{k=1}^{N} z_k w_k^*)$.

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1. Introduction

The reproducing kernel Hilbert space $\mathcal{H}(\mathbb{B}_N)$ of functions analytic in the ball

$$\mathbb{B}_N = \{ (z_1, z_2, \ldots, z_N) \mid \sum_{k=1}^{N} |z_k|^2 < 1 \}$$

and with reproducing kernel $1/(1 - \langle z, w \rangle)$, where

$$\langle z, w \rangle = \sum_{i=1}^{N} z_j w^*_j$$

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has played an important role in operator theory in recent years; see [1,12,20,21]. This
is due to the fact that Pick’s interpolation theorem holds there. The space \( \mathcal{H}(\mathbb{B}_N) \) is
contractively included in the Hardy space of the ball and the inclusion is strict. More-
over, it provides an interesting setting in which much of the analysis in the Hardy
space of the disk extends in a natural way. In previous works we studied various
aspects of this extension using the theory of reproducing kernel Hilbert spaces; see
[2,8,11]. Here we consider the case of the Carathéodory–Fejér interpolation problem.

Let us recall that a tangential version of the classical Carathéodory–Fejér interpo-
ation problem is

**Problem 1.1.** Given a point \( a \) in the open unit disk \( \mathbb{D} \), \( \xi \in \mathbb{C}^{p \times 1} \) and given vec-
tors \( \eta_j \in \mathbb{C}^{q \times q}, \ j = 1, 2, \ldots, n \), find all \( \mathbb{C}^{p \times q} \)-valued functions \( S(z) \) analytic and
contractive in \( \mathbb{D} \) and such that

\[
\xi^* \frac{S^{(k)}}{k!}(a) = \eta_k^*, \quad k = 0, 1, 2, \ldots, n. \tag{1.1}
\]

A solution to this problem using the theory of reproducing kernel Hilbert spaces
can be found in [18, Chapter 6, p. 69] and, in the case of the open upper half-plane
instead of the disk, in [5]. In the present paper we solve a similar problem in the class
of \( \mathbb{C}^{p \times q} \)-valued functions analytic and contractive in \( \mathbb{B}_N \) and such that the kernel

\[
I_p - S(z)S(w)^* \over 1 - \langle z, w \rangle \tag{1.2}
\]
is positive in \( \mathbb{B}_N \), i.e. such that all block hermitian matrices of the form

\[
\frac{I_p - S(w_z)S(w_j)^*}{1 - \langle w_z, w_j \rangle} \quad \ell, j = 1, \ldots, M \tag{1.3}
\]
are non-negative, where \( M = 1, 2, \ldots \) and where \( w_1, \ldots, w_M \) are arbitrary points
in \( \mathbb{B}_N \).

The positivity of the kernel (1.2) is equivalent to the fact that the operator of
multiplication by \( S \) on the left is a contraction from \( \mathcal{H}(\mathbb{B}_N)^{p \times 1} \) into \( \mathcal{H}(\mathbb{B}_N)^{p \times 1} \),
and we will call these functions Schur multipliers. The class of Schur multipliers is
included in the class of functions analytic and contractive in the ball. The inclusion
is strict. For instance, with \( N = 2 \) the functions

\[
p_m(z_1, z_2) = z_1 + c_1 z_2^2 + c_2 z_2^4 + \cdots + c_m z_2^{2m}, \]
where the \( c_j \) are defined via the Taylor series \( 1 - \sqrt{1 - t} = \sum_j c_j t^j \) (with \( |t| < 1 \))
are analytic and contractive in \( \mathbb{B}_2 \) but the corresponding kernels (1.2) are not positive;
see [11]. Another class of functions analytic and contractive in the ball and for which
the kernel (1.2) is not positive is given by the family of inner functions of the ball;
see [10].
The problem we solve is the following.

**Problem 1.2.** Let \( a \in \mathbb{B}_N \), \( \xi \in \mathbb{C}^p \) and \( \eta_0 = \eta_{0,j} \) and \( \eta_{k,j} \in \mathbb{C}^q \) where \( j = 1, 2, \ldots, N \) and \( k = 1, 2, \ldots, n \). Describe all \( \mathbb{C}^{p \times q} \)-valued Schur multipliers \( S \) on \( \mathbb{B}_N \) such that
\[
\xi^* \frac{1}{k!} \sum_{j \in \mathbb{Z}} \eta_{k,j}(a) = \eta_{k,j}^*, \quad k = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, N.
\]

This is not the most general one-sided interpolation problem of Carathéodory–Fejér type that one could think of. In particular it does not include conditions on mixed derivatives. In fact, after this paper was completed, we learned at the ILAS2001 conference in Haifa that Ball and Bolotnikov solved a general two-sided interpolation problem in the class of Schur multipliers. They use the method of extension of isometries (and in particular the Arov–Grossman formula; see [12, Section 4, p. 109] for an application of this method to the Nevanlinna–Pick interpolation problem). The methods used in the present paper are based on the theory of reproducing kernel Hilbert spaces and are quite different. The description of the set of all solutions is also based on a different type of linear fractional transformation.

In the reproducing kernel approach to interpolation there are three main steps (see [4–6] for the one variable case):

1. Build from the interpolation data a finite dimensional space of rational functions \( \mathcal{M} \). When endowed with an appropriate inner product this space has a reproducing kernel of the form
\[
J = \Theta(z) \bar{J} \Theta(w)^*,
\]
where
\[
J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \quad \text{and} \quad \bar{J} = \begin{pmatrix} \bar{I}_p & 0 \\ 0 & -I_q \end{pmatrix}
\]
with \( \bar{p} \geq p \).

When \( N = 1 \) then we should have \( \bar{p} = p \) and \( \Theta \) is a rational \( J \)-inner function which, in the case of interpolation with points inside the open unit disk, is analytic in the closed unit disk.

2. Associate to a given solution \( S \) the reproducing kernel Hilbert space with reproducing kernel (1.2) (which will be denoted by \( \mathcal{H}(S) \) in the sequel) and check that the map \( \tau \) defined by
\[
\tau(F)(z) = (I_p - S(z))F(z)
\]
is a contraction from \( \mathcal{M} \) into \( \mathcal{H}(S) \). This forces a linear fractional transformation (of the form (2.6)) between \( S \) and \( \Theta \).
3. Check that all the solutions are given by the above-mentioned linear transformation when the parameter varies among all possible Schur multipliers (of appropriate size).

This is the strategy which is used in the present work. The space $\mathcal{M}$ is presented in Section 2, where we also give the main theorem of the paper; see Theorem 2.3. The map $\tau$ is studied in Section 3. There we also prove that the positivity of a certain matrix $P$ (the solution of the matrix equation (2.2)) is a necessary condition for the problem to have a solution. The description of the set of all solutions in terms of a linear fractional transformation is done in Section 4. The next section, Section 5, concludes the proof by proving that the positivity of $P$ is also a sufficient condition for the problem to have a solution. The fact that $P \succeq 0$ is a necessary and sufficient condition for Problem 1.2 to be solvable can also be seen using the commutant lifting theorem in [12]. Finally we briefly discuss in the last section a more general interpolation problem.

It should be recalled that the spaces which are used here as well as the map $\tau$ (defined by (1.4)) linking between them have been first introduced, in the one variable case, by de Branges and Rovnyak; see [13, Theorem 6, p. 85], [14, Theorem 11, p. 304], [15, pp. 24–46]. Still in the one variable case, the relationships between this approach and the approach based on the theory of extension of hermitian operators (based on Krein’s formula for the generalized resolvents of an hermitian operator) were explicitly given in [4–6]; see also [7] for the case of boundary points. A similar study in the present setting remains to be done.

2. **The space $\mathcal{M}$ associated to the interpolation problem**

We begin with some notation which will be familiar to the reader acquainted to the one variable version of the present problem. We set for $i = 1, \ldots, n$ and $j = 1, \ldots, N$:

$$v_{0,j} = v_0 = \begin{pmatrix} \xi \\ \eta_0 \end{pmatrix} \quad \text{and} \quad v_{i,j} = \begin{pmatrix} 0 \\ \eta_{i,j} \end{pmatrix}$$

and

$$\rho_a(z) = 1 - \langle z, a \rangle, \quad \phi_{i,j} = \frac{z_j}{\rho_a(z)^{i+1}}, \quad \text{and} \quad f_{i,j}(z) = \sum_{k=0}^{n} \phi_{k,j} v_{i-k,j}$$

(recall that $a$ is the interpolation point). Furthermore we define the matrices

$$\Psi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{(n+1) \times (n+1)}.$$
where $\delta_{i,j}$ denotes the Kronecker index,

$$B_k = \begin{pmatrix}
\Psi \delta_{1,k} & 0 & \cdots & 0 \\
0 & \Psi \delta_{2,k} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \Psi \delta_{N,k}
\end{pmatrix} \in \mathbb{C}^{(n+1)N \times (n+1)N},$$

and

$$\Gamma_k = \begin{pmatrix}
\xi & 0 & 0 & \cdots & 0 \\
\eta_0 & \eta_1,k & \eta_2,k & \cdots & \eta_{n,k}
\end{pmatrix} \in \mathbb{C}^{(p+q) \times (n+1)},$$

$$\Gamma = \begin{pmatrix}
\Gamma_1 & \Gamma_2 & \ldots & \Gamma_N
\end{pmatrix} \in \mathbb{C}^{(p+q) \times (n+1)N},$$

Finally, we set (with $a = (a_1 \ a_2 \ \cdots \ a_N)$)

$$A_k = a_k^* I_{(n+1)N} + B_k \quad \text{and} \quad F(z) = \Gamma \left( I_{(n+1)N} - \sum_{k=1}^{N} z_k A_k \right)^{-1}. \quad (2.1)$$

It is easy to verify that $A_k A_\ell = A_\ell A_k$.

**Proposition 2.1.** It holds that

$$\text{Span} \left\{ f_{i,j} \mid i = 0, \ldots, n, \ j = 1, \ldots, N \right\} = \left\{ F(z) \xi \mid \xi \in \mathbb{C}^{(n+1)N} \right\} \defeq \mathcal{M}.$$ 

**Proof.** We have

$$F(z) = \Gamma \begin{pmatrix}
\mathcal{P}_1 & 0 & \cdots & 0 \\
0 & \mathcal{P}_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathcal{P}_N
\end{pmatrix},$$

and a straightforward computation shows that

$$\mathcal{P}_k^{-1} = \begin{pmatrix}
\rho_a(z) & -z_k & 0 & \cdots & 0 \\
0 & \rho_a(z) & -z_k & \cdots & 0 \\
0 & 0 & \rho_a(z) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \rho_a(z)
\end{pmatrix}.$$
Hence we can write
\[
F(z) = I \left( \begin{array}{cccc}
\mathcal{P}^{-1} & 0 & \cdots & 0 \\
0 & \mathcal{P}^{-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \mathcal{P}^{-1}
\end{array} \right) - 1
\]
\[
= I \left( \rho_a(z) I_{(n+1)N} - \sum_{k=1}^{N} B_k z_k \right) - 1
\]
\[
= I \left( I_{(n+1)N} - \sum_{k=1}^{N} z_k (a_k^* I + B_k) \right) - 1
\]
and hence the result. □

Before stating the main theorem we need a definition.

**Definition 2.2.** Let
\[
\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(s+q) \times (t+q)},
\]
where \((A, B, C, D) \in \mathbb{C}^{s \times t} \times \mathbb{C}^{s \times q} \times \mathbb{C}^{t \times q} \times \mathbb{C}^{q \times q}.
We set
\[
T_\Theta(\sigma) = (A \sigma + B)(C \sigma + D)^{-1}
\]
for every \(\sigma \in \mathbb{C}^{t \times q}\) for which the later is well defined.

The main results of this work are gathered in the following theorem. In the statement the symbol \(\mathcal{H}(\mathbb{B}_N)^{(p+q) \times 1}\) denotes the space \(\mathcal{H}(\mathbb{B}_N)^{(p+q) \times 1}\) endowed with the indefinite inner product
\[
(f, g)_{\mathcal{H}(\mathbb{B}_N)^{(p+q) \times 1}} = (f, Jg)_{\mathcal{H}(\mathbb{B}_N)^{(p+q) \times 1}}
\]
and \(\mathbb{P}^{-1}\) denotes the Moore–Penrose pseudo-inverse of \(\mathbb{P}\); see [22] and [16, Chapter 1] for the definition and main properties of the Moore–Penrose inverse.

**Theorem 2.3.** A necessary and sufficient condition for Problem 1.2 to be solvable is that \(\mathbb{P} \geq 0\) where \(\mathbb{P}\) is the unique solution of the matrix equation
\[
\mathbb{P} = \sum_{j=1}^{N} A_j^* \mathbb{P} A_j = I^* J I. \tag{2.2}
\]
Assume that \(\mathbb{P} \geq 0\) and that the condition
\[
\mathbb{P} \xi = 0 \implies F(z) \xi \equiv 0 \quad (\xi \in \mathbb{C}^{(n+1)N}) \tag{2.3}
\]
holds. Then the space $\mathcal{M}$ endowed with the $\mathcal{H}(\mathbb{B}_N)^{(p+q)_1}$ inner product is a finite dimensional reproducing kernel Hilbert space. Its reproducing kernel is of the form

$$K_M(z,w) = \frac{J - \Theta(z)\tilde{J}\Theta(w)^*}{1 - \langle z, w \rangle},$$

where

$$J = J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}, \quad \tilde{J} = J_{(n+1)N^2+p,q} = \begin{pmatrix} I_{(n+1)N^2+p} & 0 \\ 0 & -I_q \end{pmatrix}$$

and where $\Theta$ is the $C((n+1)N^2+p)_1$-valued function given by

$$\Theta(z) = (0, I_{p+q}) + \Gamma \left( I - \sum_{k=1}^{N} z_k A_k \right)^{-1} [p^{-1}]
\times \left( (z_1 I - A_1)\mathbb{B}^{1/2}, (z_2 I - A_2)\mathbb{B}^{1/2}, \ldots, (z_N I - A_N)\mathbb{B}^{1/2}, -I^a J \right).$$

In this expression, $I = I_{(n+1)N}$. Finally $S(z)$ is a solution to Problem 1.2 if and only if there exists a $C((n+1)N^2+p)_1$-valued Schur multiplier $\sigma$ such that

$$S = T_{\Theta}(\sigma).$$

As already mentioned, the proof of Theorem 2.3 is divided into the various sections of the paper. In the present section we prove formula (2.4). We begin with a number of remarks. First, when setting $N = 1$ in (2.5) one does not obtain back a $C(p+q)_1$-valued function $\Theta$. Formula (2.5) is not precise enough. If one were to solve Problem 1.2 in a recursive way (as was solved in [2] the Nevanlinna–Pick problem) the function $\Theta$ obtained that way would reduce to a $C(p+q)_1$-valued function when $N = 1$. Next, since $\sigma(A_k) = \{a_k^*\}$ and $\sum_{k=1}^{N} |a_k|^2 < 1$, Eq. (2.2) has a unique solution; see [19]. We note that condition (2.3) replaces here the more classical condition of strict positivity. This is due to the fact that we are given a generating set of the space $\mathcal{M}$ which is not a basis when one considers more than one chain. We also remark that $\mathbb{P}$ is the Gram matrix of $\mathcal{M}$ in the $\mathcal{H}(\mathbb{B}_N)^{(p+q)_1}$-inner product. Indeed since the $A_k$ commute we have

$$F(z) = \Gamma \left( \sum_{\alpha \in \mathbb{N}^N} z^\alpha A^\alpha \frac{[\alpha]!}{\alpha!} \right).$$

In this expression, $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ and we use the multi-index notations $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha_N}$, $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_N!$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_N$. Define $\mathbb{Q}$ by
\[ \eta^* Q \xi = (F(z) \xi, F(z) \eta) \]  
\[ = \left( J \Gamma \left( \sum_{\alpha \in \mathbb{N}^N} z^\alpha A^\alpha \frac{|\alpha|!}{\alpha!} \right) \xi, \Gamma \left( \sum_{\alpha \in \mathbb{N}^N} z^\alpha A^\alpha \frac{|\alpha|!}{\alpha!} \right) \eta \right) \]

where \( \xi, \eta \in \mathbb{C}^{(n+1)N \times 1} \). The monomials form an orthogonal set in \( \mathcal{H}(\mathbb{B}_N) \) and

\[ \|z^\alpha\|_{\mathcal{H}(\mathbb{B}_N)} = \frac{\alpha!}{|\alpha|!} \]

(are direct consequences of the power series expansion

\[ \frac{1}{1-(z, w)} = \sum_{\alpha \in \mathbb{N}^N} z^\alpha w^{\alpha} \frac{|\alpha|!}{\alpha!}, \]

see [12, p 120] if need be) we have

\[ \eta^* Q \xi = \eta^* \sum_{\alpha \in \mathbb{N}^N} \left( A^\alpha \right)^* \Gamma^* J \Gamma A^\alpha \frac{|\alpha|!}{\alpha!} \xi \]

for every \( \xi, \eta \). Hence

\[ Q = \sum_{\alpha \in \mathbb{N}^N} \left( A^\alpha \right)^* \Gamma^* J \Gamma A^\alpha \frac{|\alpha|!}{\alpha!}. \]

We now compute \( Q - \sum_{j=1}^N A_j^* \Omega A_j \). In the expression giving \( Q \) the coefficient of the term \( \left( A^\alpha \right)^* \Gamma^* J \Gamma A^\alpha \) is \( \frac{|\alpha|!}{\alpha!} \). In the expression \( \sum_{j=1}^N A_j^* \Omega A_j \) the coefficient of the term \( \left( A^\alpha \right)^* \Gamma^* J \Gamma A^\alpha \) is

\[ \sum_{m=1}^N \frac{(|\alpha| - 1)!}{(\alpha_m - 1)! \prod_{j \neq m, j=1}^N \alpha_j!} = \frac{(|\alpha| - 1)! \prod_{m=1}^N \alpha_m!}{|\alpha|! \alpha!} = \frac{|\alpha|!}{\alpha!}. \]

Hence, in the subtraction \( Q - \sum_{j=1}^N A_j^* \Omega A_j \), the only term left is the first one, implying

\[ Q - \sum_{j=1}^N A_j^* \Omega A_j = \Gamma^* J \Gamma \]

and so \( Q = \mathbb{P} \). In particular we have

\[ F(z) \xi \equiv 0 \quad (\xi \in \mathbb{C}^{(n+1)N}) \implies \mathbb{P} \xi = 0. \]

Condition (2.3) expresses that the converse implication holds.

We now prove equality (2.5). The arguments are taken from [2]. There \( \mathbb{P} > 0 \).

Here there is a need to take the Moore–Penrose pseudo-inverse of \( \mathbb{P} \). When \( N = 1 \),
such reproducing kernel formulas involving a pseudo-inverse can be found in [3]; see [3, Formula (3.10), p. 85].

Denote

\[ \Upsilon(z) = \left( I_{(n+1)N} - \sum_{k=1}^{N} z_k A_k \right)^{-1} \]

and set

\[ \Omega(z) = \left( (z_1 I_{(n+1)N} - A_1)^{p^{1/2}}, (z_2 I_{(n+1)N} - A_2)^{p^{1/2}}, \ldots, \right. \]

\[ \left. (z_N I_{(n+1)N} - A_N)^{p^{1/2}}, -I^* J \right) \]

Then, with \( \Theta(z) \) as in (2.5) we can write

\[ \Theta(z) J \Theta(w)^* = F(z) T(z, w) F(w)^* \]

where

\[ T(z, w) = \left[ \Upsilon(w)^{-1} \right]^{p^{-1}} + \Upsilon(z)^{-1} \Omega(z) \Omega(w)^* p^{p^{-1}} \]

\[ = \left[ I_{(n+1)N} - \sum_{k=1}^{N} w_k^* A_k^* \right]^{p^{-1}} + \left( I_{(n+1)N} - \sum_{k=1}^{N} z_k A_k \right) p^{p^{-1}} \]

\[ - \left( \sum_{k=1}^{N} (z_k I_{(n+1)N} - A_k^*)^p (w_k^* I_{(n+1)N} - A_k) + I^* J F \right) \times p^{p^{-1}}. \]

Since \( p^{p^{-1}} p^{p^{-1}} = p^{p^{-1}} \) we have

\[ T(z, w) = \left[ p^{p^{-1}} (1 - (z, w)) - \sum_{k=1}^{N} z_k A_k p^{p^{-1}} - \sum_{k=1}^{N} w_k^* p^{p^{-1}} A_k^* \right. \]

\[ + \sum_{k=1}^{N} z_k p^{p^{-1}} A_k p^{p^{-1}} + \sum_{k=1}^{N} w_k^* p^{p^{-1}} A_k^* p^{p^{-1}} \]
\[
\begin{align*}
\sum_{k=1}^N z_k \left( p^{[-1]} \{ p - I_{(n+1)N} \} A_k p^{[-1]} \right) \\
+ \sum_{k=1}^N w_k^* p^{[-1]} A_k^* \left( p^{[-1]} p - I_{(n+1)N} \right).
\end{align*}
\]

We now prove that
\[
\left( p^{[-1]} p - I_{(n+1)N} \right) A_k p^{[-1]} = p^{[-1]} A_k^* \left( p^{[-1]} p - I_{(n+1)N} \right) = 0. \tag{2.7}
\]

It is enough to prove the first one; the other is its adjoint. Using (2.1) we have
\[
\begin{align*}
\left( p^{[-1]} p - I_{(n+1)N} \right) A_k p^{[-1]} &= \left( p^{[-1]} p - I_{(n+1)N} \right) \left( a_k^* \right) I_{(n+1)N} + B_k \right) p^{[-1]} \\
&= \left( p^{[-1]} p - p^{[-1]} \right) a_k^* + \left( p^{[-1]} p - I_{(n+1)N} \right) B_k p^{[-1]} \tag{2.8}.
\end{align*}
\]

The first part of (2.8) vanishes by the definition of the pseudo-inverse. To compute the second term, we note the following: since the columns of \( F(z) \)—aside from those of index \( k(n+1) + 1, \ k = 0, \ldots, N - 1 \)—are linearly independent, we have that for \( \xi = (\alpha_1, \ldots, \alpha_{(n+1)N})^\top \), \( F(z)\xi \equiv 0 \) if and only if
\[
\sum_{k=0}^{N-1} \alpha_{k(n+1)+1} = 0 \quad \text{and} \quad \alpha_i = 0 \quad \text{for} \quad i \neq k(n+1) + 1.
\]

Since (2.3) is in force,
\[
\ker P = \{ \xi \in C^{(n+1)N \times 1} \mid F(z)\xi \equiv 0 \}
\]
and so
\[
\begin{align*}
\ran P &= \left\{ (\alpha_1, \ldots, \alpha_{(n+1)N})^\top \ \left| \sum_{k=0}^{N-1} \alpha_{k(n+1)+1} = 0 \quad \text{and} \quad \alpha_i = 0 \right. \right. \\
&\left. \left. \quad \text{for} \ i \neq k(n+1) + 1 \right\} \subseteq \ran P \right. \subseteq \ran P.
\end{align*}
\]
Since \( \text{ran } P = \text{ran } [P^{[-1]}] \) (see [16, Theorem 1.2.2, p. 12]) we obtain that 
\[
\left( [P^{[-1]}]P - I_{(n+1)N} \right) B_k = 0
\]
which allows to conclude.

**Remark 2.4.** Condition (2.7) seems to be an obstruction to extend the present methods to an arbitrary set of commuting operators \( A_1, \ldots, A_N \). See Section 6.

3. The linear fractional transformation

In this section we assume that Problem 1.2 has a solution \( S \). We prove that \( P \geq 0 \).

Under the hypothesis (2.3) we prove that \( S \) is of the form (2.6).

Let thus \( S(z) \) be a solution of Problem 1.2. We can now consider two reproducing kernel Hilbert spaces: the space \( \mathcal{H} = \mathcal{H}(\Theta) \) defined above, and the reproducing kernel Hilbert space \( \mathcal{H}(S) \) with reproducing kernel
\[
K_S(z, w) = \frac{I_p - S(z)S(w)^*}{1 - \langle z, w \rangle}.
\]

**Proposition 3.1.** The map \( \tau : f \mapsto (I_p - S(z)) f \) is an isometry from the space \( \mathcal{H} \) endowed with the \( \mathcal{H}(\mathbb{B}_N)^{(p+q) \times 1} \) inner product into \( \mathcal{H}(S) \).

**Proof.** We first remark that the function
\[
\frac{\partial^{i} K_S(z, w)}{\partial w_j^*} \xi
\]
belongs to \( \mathcal{H}(S) \) for every choice of \( w \in \mathbb{B}_N \) and \( \xi \in \mathbb{C}^{p \times 1} \) since the kernel \( K_S(z, w) \) is analytic in the variables \( z_j \) and \( w_j^* \). See e.g. [5] for a proof of this general fact on reproducing kernel spaces. We will prove by induction that
\[
\tau(f_{i,j}) = (I_p - S(z)) \sum_{k=0}^{i} \frac{z_j^k}{(1 - \langle z, a \rangle)^{k+1}} v_{i-k,j} \xi.
\]

Indeed, for \( i = 0 \) we have
\[
\tau(f_{0,j}) = (I_p - S(z)) f_{0,j} = (I_p - S(z)) \frac{\xi}{1 - \langle z, a \rangle} = \frac{\xi - S(z)\eta_0}{1 - \langle z, a \rangle}.
\]

Since \( S(z) \) solves the interpolation problem, the last implies \( \tau(f_{0,j}) = K_S(z, a) \xi \).

Now we look at \( \tau(f_{i,j}) \):
\[
\tau(f_{i,j}) = (I_p - S(z)) f_{i,j} = (I_p - S(z)) \sum_{k=0}^{i} \frac{z_j^k}{(1 - \langle z, a \rangle)^{k+1}} v_{i-k,j}.
\]
\[ \begin{align*}
&= -S(z)\eta_{i,j} \frac{1}{1 - \langle z, a \rangle} + -S(z)\eta_{i-1,j} \frac{1}{(1 - \langle z, a \rangle)^2} z_j + -S(z)\eta_{i-2,j} \frac{1}{(1 - \langle z, a \rangle)^3} z_j^2 + \cdots \\
&\quad + -S(z)\eta_{i,j} \frac{1}{(1 - \langle z, a \rangle)^i} z_j^{i-1} + -S(z)\eta_{i-1,j} \frac{1}{(1 - \langle z, a \rangle)^{i+1}} z_j^i 
\end{align*} \]

We have that
\[
\left( \frac{\partial}{\partial w_j} S(w) \right)^* = \frac{\partial}{\partial w_j} S(w)^*.
\]

Thus differentiating with respect to \( w_j^* \) and applying the interpolation conditions yields
\[
i! \frac{\partial}{\partial w_j} \tau(f_{i,j}) = -S(z) \frac{\partial^{i+1}}{\partial w_j^{i+1}} S(w)^* \bigg|_{w = a} + -S(z) \frac{\partial^i}{\partial w_j^i} S(w)^* \bigg|_{w = a} \\
&\quad + i \frac{S(z) \frac{\partial^i}{\partial w_j^i} S(w)^*}{(1 - \langle z, a \rangle)^2} z_j \xi + 2i \frac{S(z) \frac{\partial^{i-1}}{\partial w_j^{i-1}} S(w)^*}{(1 - \langle z, a \rangle)^3} z_j^2 \xi \\
&\quad + (i - 1)i \frac{S(z) \frac{\partial^{i-2}}{\partial w_j^{i-2}} S(w)^*}{(1 - \langle z, a \rangle)^4} z_j^3 \xi \\
&\quad \vdots \\
&\quad + i! \frac{S(z) \frac{\partial^1}{\partial w_j^1} S(w)^*}{(1 - \langle z, a \rangle)^{i+1}} z_j^i \xi \\
&\quad + (i + 1)i! \frac{S(z) \frac{\partial^{i+1}}{\partial w_j^{i+1}} S(w)^*}{(1 - \langle z, a \rangle)^{i+2}} z_j^{i+1} \xi.
\]

Using
\[
\frac{i!}{(i - k)!} + k \frac{i!}{(i - k + 1)!} = \frac{(i + 1)!}{(i - k + 1)!}
\]
we have
\[
\frac{\partial}{\partial w_j} \tau(f_{i,j}) = (i + 1) \tau(f_{i,j}).
\]

Using the induction condition implies that
\[
\tau(f_{i,j}) = \frac{1}{i!} \frac{\partial^i}{\partial w_j^i} K_S(z, w) \xi \bigg|_{w = a}.
\]

and so \( \tau(f_{i,j}) \in \mathcal{H}(S) \).
To complete the proof we show that for every \( f, g \in \mathcal{H} \)

\[
\langle f, g \rangle_{\mathcal{H}(B)^{(p+q)\times 1}} = \langle \tau(f), \tau(g) \rangle_{\mathcal{H}(S)}.
\]  

(3.1)

By definition

\[
\langle \tau(f_{i,j}), \tau(f_{\ell,k}) \rangle_{\mathcal{H}(S)}
= \langle (I_p - S(z)) f_{i,j}, (I_p - S(z)) f_{\ell,k} \rangle_{\mathcal{H}(S)}
\]

\[
= \frac{1}{\ell!} \frac{\partial_i}{\partial w^i_j} K_S(z, w) \xi \left|_{w=a} \right. , \frac{\partial_{\ell}}{\partial w^\ell_k} K_S(z, w) \xi \left|_{w=a} \right. \mathcal{H}(S)
\]

\[
= \xi^* \frac{1}{\ell!} \frac{\partial_{\ell}}{\partial z^\ell_k} K_S(z, w) \xi \left|_{w=a} \right. (a).
\]

where we have used that, for \( f \in \mathcal{H}(S) \),

\[
\left. f, \frac{\partial_{\ell}}{\partial w^\ell_k} K_S(z, w) \xi \right|_{w=a} = \left( \frac{\partial_{\ell}}{\partial z^\ell_k} f \right)(a).
\]

Using Newton's formula we then have

\[
\langle \tau(f_{i,j}), \tau(f_{\ell,k}) \rangle_{\mathcal{H}(S)}
\]

\[
= \xi^* \frac{1}{\ell!} \frac{\partial_{\ell}}{\partial z^\ell_k} \left( \sum_{m=1}^{i} \binom{i}{m} \frac{\partial^m}{\partial w^m_j} (I_p - S(z)S(w)) \frac{\partial^{i-m}}{\partial w^{i-m}(1 - \langle z, w \rangle)} z_{a} \right)
\]

\[
= \xi^* \frac{1}{\ell!} \frac{\partial_{\ell}}{\partial z^\ell_k} \left( \sum_{m=1}^{i} \binom{i}{m} (\xi^* \delta_{0,m} - \xi^* S(z) \eta_{m,j} m!) \frac{z_j^{i-m} \xi}{(1 - \langle z, w \rangle)^{i-m+1}(i - m)!} \right) z_{a}
\]

\[
= \frac{1}{\ell!} \frac{\partial_{\ell}}{\partial z^\ell_k} \left( \sum_{m=0}^{i} \binom{i}{m} (\xi^* \delta_{0,m} - \xi^* S(z) \eta_{m,k}) \frac{z_j^{i-m} \xi}{(1 - \langle z, w \rangle)^{i-m+1}} \right) z_{a}
\]

\[
= \frac{1}{\ell!} \sum_{m=0}^{i} \sum_{t=0}^{\ell} \binom{i}{t} \frac{\partial^{t}}{\partial z^t_k} (\xi^* \delta_{0,m} - \xi^* S(z) \eta_{m,k}) \frac{z_j^{i-m} \xi}{(1 - \langle z, w \rangle)^{i-m+1}} \right) z_{a}
\]

\[
= \sum_{m=0}^{i} \sum_{t=0}^{\ell} \frac{1}{(\ell - t)!} \delta_{i,k}^{p,q} \eta_{m,k} \frac{\partial^{t}}{\partial z^t_k} (1 - \langle z, w \rangle)^{i-m+1} \right) z_{a}.
\]
On the other hand, we have

\[
\left\langle \sum_{m=0}^{i} \frac{z_{m}^{i} v_{i-m,j}}{(1 - \langle z, a \rangle)^{m+1}} \sum_{t=0}^{\ell} \frac{z_{k}^{\ell} v_{\ell-t,k}}{(1 - \langle z, a \rangle)^{t+1}} \right\rangle_{\mathcal{H}(B_N)_{p+q} \times 1} = \sum_{m=0}^{i} \sum_{t=0}^{\ell} v_{i-t,k} J_{p,q} v_{i-m,j} \times \left( \left( \frac{z_{j}^{m}}{(1 - \langle z, a \rangle)^{m+1}} \frac{1}{t!} \left( \frac{\partial^{t}}{\partial w_{k}^{t}} 1 - \langle z, w \rangle \right) \right)_{w=a} \right)_{\mathcal{H}(B_N)_{p+q} \times 1}
\]

and hence equality (3.1) holds. □

Hence \( \tau \) is an isometry and the operator \((I_{\mathcal{H}(S)} - \tau \tau^*)\) is positive. Thus the kernel

\[
(I_{\mathcal{H}(S)} - \tau \tau^*) K_{S}(z, w)
\]

is positive in \( B_{N} \).

Since the Gram matrix of \( \mathcal{M} \) in \( \mathcal{H}(B_N)_{p+q} \times 1 \) is the matrix \( P \) it follows that \( P \geq 0 \) is a necessary condition for the problem to have a solution. Assume now that (2.3) is in force. We have \( \mathcal{M} = \mathcal{H}(\Theta) \). We prove now the existence of a \( \mathcal{C}^{(n+1)N^2 + p} \times q \)-valued Schur multiplier \( \sigma \) such that \( S = T_{\Theta}(\sigma) \). We leave to the reader to check that for every

\[
f(z) = \sum_{k=1}^{m} \frac{I_{p} - S(z)S(w_{k})^{*}}{1 - \langle z, w_{k} \rangle} \xi_{k} \in \mathcal{H}(S)
\]

it holds that

\[
\tau^*(f(z)) = \sum_{k=1}^{m} \frac{J - \Theta(z)J\Theta(w_{k})^{*}}{1 - \langle z, w_{k} \rangle} \left( \frac{I_{p}}{-S(w_{k})^{*}} \right) \xi_{k}.
\]

Let \( c \in \mathbb{C}^{p \times 1} \). Then,

\[
\left\{ (I_{\mathcal{H}(S)} - \tau \tau^*) K_{S}(z, w)c \right\} = (I_{p} - S(z)) \left( \Theta(z)J\Theta(w_{k})^{*} \left( c \right) \right) \geq 0.
\]
Writing
\[ \Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
we thus have that the kernel
\[ \frac{(A - SC)(z)(A - SC)(w)^* - (B - SD)(z)(B - SD)(w)^*}{1 - \langle z, w \rangle} \]
is positive in \( \mathbb{B}_N \). By Leech’s theorem in the present setting (see [2] and see
[23, p. 107] for the one variable case), the last equation implies that there exists a
\( \mathbb{C}^{(n+1)N^2+p}\times q \)-valued Schur multiplier \( \sigma \) such that
\( (B - SD)(z) = (A - SC)(z) \sigma(z) \). To conclude the proof, we need to check that \( \det (C \sigma + D) \neq 0 \). This follows
from the fact that \( \Theta(z)J \Theta(z)^* = J \) on the sphere.

4. Carathéodory–Fejér interpolation

In this section we assume that \( P \geq 0 \) and that (2.3) holds and we prove that the
linear fractional transformation (2.6) describes the set of all solutions of the interpo-
lation problem 1.2 when \( \sigma \) varies among all \( \mathbb{C}^{(n+1)N^2+p}\times q \)-valued multipliers \( \sigma \).
The relationship \( S = T_\sigma(\sigma) \) implies that
\[ \frac{I_p - S(z)S(w)^*}{1 - \langle z, w \rangle} = (I_p - S(z))J - \Theta(z)J \Theta(w)^* \frac{I_p}{1 - \langle z, w \rangle} \left( -S(w)^* \right) \]
\[ + (A(z) - S(z)C(z)) \frac{I_{(n+1)N^2+p}\times q}{} - \sigma(z)\sigma(w)^* \frac{A(w) - S(w)C(w)^*}{1 - \langle z, w \rangle}. \]
This decomposition of the positive kernel \( (I_p - S(z)S(w)^*/(1 - \langle z, w \rangle) \) into a sum of two positive kernels implies, as in the case \( N = 1 \) (see e.g. [14], [9, p. 128]; the
arguments are identical here) that the map \( \tau \) sends \( \mathcal{H}(\Theta) \) into \( \mathcal{H}(S) \). In particular
the function
\[ (I_p - S(z))f_{i,j}(z) \]
begins to \( \mathcal{H}(S) \) for every \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, N\} \). These functions are in
particular analytic in a neighborhood of \( a \) and this forces the interpolation conditions
on \( S \).

5. Conclusion of the proof of Theorem 2.3

In this section we use an approximation argument and prove that the condi-
tion \( P \geq 0 \) (without the supplementary condition (2.3)) is a sufficient condition for
Problem 1.2 to be solvable. Let \( \epsilon \) be a strictly positive number and consider the interpolation problem obtained by replacing \( \xi \) by \( \xi^\epsilon = (1 + \epsilon) \xi \):

\[
(1 + \epsilon) \frac{1}{k!} \frac{\partial^k S}{\partial z^k} (a) = \eta_{k,j}^\epsilon, \quad k = 0, 1, \ldots, n, \quad j = 1, 2, \ldots, N.
\]

We will refer to this problem as the modified problem. It corresponds to a space \( \mathcal{M}_\epsilon \) generated by the columns of the matrix function \( F_\epsilon(z) = C_\epsilon (I_{(n+1)N} - \sum z_k A_k)^{-1} \), where

\[
\Gamma_\epsilon = \Gamma + \epsilon \begin{pmatrix}
\xi & 0 & \cdots & 0 & \xi & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}.
\]

In the above matrix the vector \( \xi \) appears at the places \( 1, 1 + (n + 1), 1 + 2(n + 1), \ldots, 1 + (N - 1)(n + 1) \). We denote by \( \mathbb{P}_\epsilon \) the solution of the equation

\[
\mathbb{P}_\epsilon - \sum_{k=1}^N A_k^* \mathbb{P}_\epsilon A_k = \Gamma^*_\epsilon J \Gamma_\epsilon
\]

and prove that condition (2.3) holds for the modified problem:

\[
\mathbb{P}_\epsilon c = 0 \implies F_\epsilon(z)c = 0.
\]

We have

\[
\Gamma^*_\epsilon J \Gamma_\epsilon = \Gamma^* \Gamma + (\epsilon^2 + 2\epsilon)(\xi^* \xi) \text{diag} (1, 0, \ldots, 0, 1, \ldots, 0, 1, \ldots),
\]

where the non zero elements in the above diagonal matrix are at the places \( 1, 1 + (n + 1), 1 + 2(n + 1), \ldots, 1 + (N - 1)(n + 1) \). Thus, using once more the multi-index notation,

\[
\mathbb{P}_\epsilon = \mathbb{P} + \sum_{\alpha \in \mathbb{N}^N} (A^\alpha)^* \text{diag} (1, 0, \ldots, 0, 1, \ldots, 0, 1, \ldots) A^\alpha
\]

and in particular

\[
\mathbb{P}_\epsilon c = 0 \iff \mathbb{P} c = 0 \quad \text{and} \quad \text{diag} (1, 0, \ldots, 0, 1, \ldots, 0, 1, \ldots) c = 0.
\]

As already noticed for \( \epsilon = 0 \), we have that \( F_\epsilon(z)\xi = 0 \) if and only if

\[
\alpha_{k(n+1)+1} = 0 \quad \text{and} \quad \alpha_i = 0 \quad \text{for} \quad i \neq k(n+1) + 1.
\]

Thus (2.3) holds for the modified problem and for every \( \epsilon > 0 \) there exists a solution \( S_\epsilon \) to the modified problem. By Montel’s theorem there exists a sequence \( \epsilon_1, \ldots \) such that the functions \( S_{\epsilon_1}, \ldots \) converge pointwise and uniformly on compact subsets of the ball to a function \( S \) analytic and contractive in the ball. This function is moreover a Schur multiplier, as is checked using the positivity of the various Pick matrices (1.3) with \( S_\epsilon \) in place of \( S \) and letting \( \epsilon \to 0 \). This completes the proof. \( \square \)
6. The general case

The Carathéodory–Fejér interpolation can be viewed as a special case of a more general interpolation problem. First a definition: let $F$ and $G$ be, respectively, $\mathbb{C}^{p \times q_1}$-valued and $\mathbb{C}^{p \times q_2}$-valued functions with entries in $\mathcal{H}(\mathbb{B}_N)$ and with power series expansions $F(z) = \sum_{\alpha \in \mathbb{N}^N} F_{\alpha} z^\alpha$ and $G(z) = \sum_{\alpha \in \mathbb{N}^N} G_{\alpha} z^\alpha$ we set

$$[F, G] = \sum_{\alpha \in \mathbb{N}^N} \frac{|\alpha|!}{\alpha} \mathbb{C}^{q_2 \times q_1}.$$ (6.1)

When $q_1 = q_2 = q$ we have that $\text{Tr} [F, G] = \langle F, G \rangle_{\mathcal{H}(\mathbb{B}_N)^{p \times q}}$. We note that the entries of a Schur multiplier are in the space $\mathcal{H}(\mathbb{B}_N)$ since constants belong to $\mathcal{H}(\mathbb{B}_N)$. Therefore the following problem makes sense:

Given matrices $C \in \mathbb{C}^{p \times q}$, $A_i \in \mathbb{C}^{n \times n}$, and $B \in \mathbb{C}^{n \times q}$, where the matrices $A_j$ are such that the entries of $C(I_n - \sum_{k=1}^{N} z_k A_k)^{-1}$ are in $\mathcal{H}(\mathbb{B}_N)$, find all $\mathbb{C}^{p \times q}$-valued Schur multipliers $S$ such that:

$$\begin{bmatrix} S(z), C \left(I - \sum_{k=1}^{N} z_k A_k \right)^{-1} \end{bmatrix} = B.$$ (2.7)

This problem includes Carathéodory–Fejér interpolation problems with mixed derivatives. To solve the general left-sided interpolation problem with the method presented here one follows the three steps mentioned in the introduction. The details will be presented in [17]. We mention that the space $\mathcal{H}$ is the span of the columns of the matrix-function $C(I_n - \sum_{k=1}^{N} z_k A_k)^{-1}$ and that the condition of solvability is that the corresponding Gram matrix

$$\mathbb{P} = \begin{bmatrix} C \left(I_n - \sum_{k=1}^{N} z_k A_k \right)^{-1}, C \left(I_n - \sum_{k=1}^{N} z_k A_k \right)^{-1} \end{bmatrix}$$

is non-negative.

A main difficulty (which we did not solve yet) is to check when (2.7) holds. A partial answer is given in the next theorem:

**Theorem 6.1.** Assume that the $A_j$ pairwise commute. Let $\mathcal{A}$ be the maximal subspace of $\ker C$ that is $A_j$ invariant for $j = 1, \ldots, N$. Equality (2.7) holds if and only if $\mathcal{A}^\perp$ is $A_j$ invariant for $j = 1, \ldots, N$.

**Proof.** We first notice that the two following conditions are equivalent:

1. $C(I_n - \sum_{j=1}^{N} z_j A_j)^{-1} \xi$.
2. $CA_\alpha^{\alpha} \xi = 0$ for every $\alpha \in \mathbb{N}^N$. 
Thus $\xi \in \mathcal{A}$ if and only if $C(I_n - \sum_{j=1}^{N} z_j A_j)^{-1}\xi \equiv 0$. In view of the definition of $\mathcal{P}$, this will hold if and only if $\xi^* \mathcal{P} \xi = 0$. Since $\mathcal{P} \geq 0$, this in turn is equivalent to $\mathcal{P} \xi = 0$. So we have $\mathcal{A} = \ker \mathcal{P}$. Let

$$\mathbb{C}^n = \mathcal{A} \oplus \mathcal{V} \oplus (\ker C)^\perp$$

where $\mathcal{V}$ is the orthogonal complement of $\mathcal{A}$ in $\ker C$.

Since $\mathcal{A}$ is $A_j$ invariant, we have the matrix decomposition of $A_j$

$$A_j = \begin{pmatrix} A_{1,j} & A_{2,j} & A_{3,j} \\ 0 & A_{4,j} & A_{5,j} \\ 0 & A_{6,j} & A_{7,j} \end{pmatrix}$$

with respect to the above orthogonal sum. Since $I - [p^{-1}] P$ is the orthogonal projection on $\mathcal{A}$, we have in a similar way

$$I - [p^{-1}] P = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$ 

Since $\mathcal{V} \oplus (\ker C)^\perp = (\ker \mathcal{P})^\perp$ and $(I - [p^{-1}] P) [p^{-1}] = 0$, we have

$$(p^{-1}) \mathbb{C}^n \rightarrow \mathcal{A} \oplus \mathcal{V} \oplus (\ker C)^\perp,$$

where $P_1$ and $P_2$ are full rank. Thus:

$$(p^{-1}) [p^{-1}] A_j [p^{-1}] = \begin{pmatrix} 0 \\ A_{2,j} P_1 \\ A_{3,j} P_2 \end{pmatrix}.$$ 

Hence equality (2.7) is equivalent to $A_{2,j} = 0$ and $A_{3,j} = 0$, which implies that $(\ker \mathcal{P})^\perp = \mathcal{A}^\perp$ is $A_j$ invariant.

That the condition is sufficient follows from the following argument: by definition of $[p^{-1}]$ we have $(I - [p^{-1}] P) [p^{-1}] = 0$. Hence all columns of $[p^{-1}]$ are included in $\mathcal{A}^\perp$. Thus, assuming that $\mathcal{A}^\perp$ is $A_j$ invariant will imply (2.7). □

References