TOPOLOGICAL METHODS
FOR THE GINZBURG-LANDAU EQUATIONS

By Luís ALMEIDA and Fabrice BETHUEL

ABSTRACT. We consider the complex-valued Ginzburg-Landau equation on a two-dimensional domain $\Omega$, with boundary data $g$, such that $|g| = 1$.

We develop a variational framework for this equation: in particular we show that the topology of the level sets is related to a finite dimensional functional, the renormalized energy. As an application, we prove a multiplicity result of solutions for the equation, when $\varepsilon$ is small and the winding number of $g$ is larger or equal to 2. © Elsevier, Paris

I. Introduction

I.1. Let $\Omega$ be a smooth, bounded and simply connected domain in $\mathbb{R}^2$. Let $g$ be a smooth map from $\partial\Omega$ to $S^1 = \{z \in \mathbb{C}, |z| = 1\}$ (i.e. $|g| = 1$). We are interested in complex-valued solutions $u$ of the Ginzburg-Landau equation with Dirichlet boundary conditions

\begin{align*}
-\Delta u &= \frac{1}{\varepsilon^2} u \left(1 - |u|^2\right) \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial\Omega.
\end{align*}

Here $\varepsilon > 0$ is a parameter, homogeneous to a length, and which we will consider small in the sequel.

Solutions to (I.1)-(I.2) are critical points of the functional

\[ E_{\varepsilon}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2 \]

\[ = \frac{1}{2} \int_{\Omega} e_{\varepsilon}(u), \]

where we have used the notation:

\[ e_{\varepsilon}(u)(x) = |\nabla u|^2(x) + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2(x), \quad \forall \, x \in \Omega. \]
Clearly $E_\varepsilon$ is well-defined, for any $\varepsilon > 0$, and is a $C^1$-functional (satisfying the Palais-Smale condition) on the Sobolev space:

$$H^1_0(\Omega; \mathbb{R}^2) = \{ u \in H^1(\Omega; \mathbb{R}^2), \ u = g \text{ on } \partial \Omega \}.$$ 

In order to find solutions to (1.1)-(1.2), we might therefore try to apply the methods of calculus of variations. Since clearly $E_\varepsilon$ is positive, and hence bounded below, it is not difficult to see that $E_\varepsilon$ achieves its infimum, and therefore minimizers are solutions of the Ginzburg-Landau equation. Minimizers of the Ginzburg-Landau equation have been studied extensively by F. Bethuel, H. Brezis and F. Hélein in [BBH] where an asymptotic analysis, as $\varepsilon \to 0$, was carried out. A somewhat similar analysis ([BBH], Chapter X) was also derived for solutions which are not necessarily minimizing: this raised of course the question of existence of such solutions. The aim of this paper is to develop (partially) a variational framework in order to find non-minimizing solutions of the Ginzburg-Landau equation. More precisely, let

$$d = \deg(g, \partial \Omega) \in \mathbb{Z}$$

be the winding number of $g$ on $\partial \Omega$, as a map from $\partial \Omega$ to $S^1$. Without loss of generality we will assume throughout that

$$d \geq 0,$$

(this is of course not a restriction, since otherwise we could just reverse the orientation on $\mathbb{R}^2$). Our main result is the following:

**Theorem 1.** Assume that

$$d \geq 2.$$

There is some $\varepsilon_0$ (depending on $g$) such that if $\varepsilon \leq \varepsilon_0$, then equation (1.1)-(1.2) has at least 3 distinct solutions, among which one is not minimizing.

Theorem 1 was announced in [AB1]. Actually in the special case where $\Omega = D^2$, and $g = \exp i \theta$ on $\partial D^2$, one can construct solutions to the Ginzburg-Landau equation of the form $u_d(r, \theta) = f_\varepsilon(r) \exp i \theta$ (see for instance Berger and Chen [BC]) and if $\varepsilon$ is sufficiently small, then these solutions are not minimizing. In [AB2], we proved that in this special case equation (1.1)-(1.2) has at least $\mu|d|^3$ families of solutions (taking into account the fact that it is invariant under the action of rotations), where $\mu$ is some constant, and provided $\varepsilon$ is sufficiently small. Other multiplicity results have been given for special boundary values by F.-H. Lin [Li] (see also Felmer and Del Pino [FP]).

In order to prove Theorem 1, we use ideas from Morse theory, consider the level sets for $E_\varepsilon$,

$$E^a_\varepsilon = \{ v \in H^1_0(\Omega; \mathbb{R}^2), \ E_\varepsilon(v) < a \}$$

and study the changes of topology of $E^a_\varepsilon$, as $a$ varies. Indeed, by standard arguments, since $E_\varepsilon$ satisfies the Palais-Smale condition, if $E^a$ and $E^b$ have different topologies, then there
is a critical value \( a < c < b \), and hence there is a solution \( u \) to equation (1.1)-(1.2) such that \( E_\varepsilon(u) = c \). Let \( \mathcal{K}_\varepsilon \) be the infimum of \( E_\varepsilon \) on \( H^1_\varepsilon \) i.e.

\[
\mathcal{K}_\varepsilon = \inf \{ E_\varepsilon(v), \ v \in H^1_\varepsilon(\Omega) \},
\]

so that

\[
E_\varepsilon^n = \emptyset \text{ if } a \leq \mathcal{K}_\varepsilon.
\]

Let \( \chi > 0 \) be given; we will consider values of \( a \) of the form

\[
a = \mathcal{K}_\varepsilon + \chi,
\]

(note that if \( d \neq 0 \), \( \mathcal{K}_\varepsilon \to +\infty \) as \( \varepsilon \to 0 \), see below). The proof of Theorem 1 relies on the following:

**Theorem 2.** Assume \( d \geq 2 \). There exists \( \chi_1 > 0 \) such that if \( \chi \geq \chi_1 \), then there exists \( \varepsilon_1 \) (depending on \( \chi \) and \( g \)), such that for \( \varepsilon < \varepsilon_1 \), there exists a loop \( f_\varepsilon \) in \( F_\varepsilon \) i.e. a continuous map \( f_\varepsilon : S^1 \to E_\varepsilon^n \) (for a given by (1.4)), which is not contractible in \( E_\varepsilon^n \).

Let us first see how this implies the existence of a critical value \( c > \mathcal{K}_\varepsilon \) and hence of a non-minimizing solution. Since \( E_\varepsilon^\infty = H^1_\varepsilon(\Omega; \mathbb{R}^2) \), the loop \( f_\varepsilon \) is contractible in \( E_\varepsilon^\infty \) (which is an affine space) and \( E_\varepsilon^\infty \) and \( E_\varepsilon^n \) do not have the same topology for \( \chi > \chi_1 \) and \( \varepsilon \) sufficiently small. Hence there is some critical value \( c \), such that \( \mathcal{K}_\varepsilon < a < c < +\infty \). A somewhat similar argument yields a third solution.

Arguments based on the topology of level sets, and more precisely on loops in level sets can be found in a number of places (see for instance J.M. Coron [C], C. Taubes [T], Sibner, Sibner and Uhlenbeck [SSU], Sibner and Talvacchia [ST]). A common feature of the above quoted papers is that the study of the topology of \( E_\varepsilon \) can be reduced to a similar problem for a functional defined on a finite dimensional space: this holds also for our situation. Our analysis will rest on the notion of renormalized energy introduced in [BBH]. Let us recall next the main result of [BBH] which will be pertinent to our analysis.

1.2. Renormalized energies. One of the main observations in [BBH] is that the asymptotic behavior of solutions to (1.1)-(1.2) is governed by “charged vortices”, which behave according to a renormalized energy. Recall that, as \( \varepsilon \to 0 \), \( \mathcal{K}_\varepsilon \to +\infty \) (if \( d \neq 0 \)), so that the asymptotic behavior becomes singular in the limit: vortices are precisely the limiting singularities. To be more precise, let us sum up some of the results of [BBH].

**Theorem 3 ([BBH]).** Assume \( \Omega \) is starshaped, and let \( (v_\varepsilon) \) be a sequence of solutions to (1.1)-(1.2) for \( \varepsilon \to 0 \). We have:

a) \( (\text{a priori bound}) \). There is a constant \( C \) depending only on \( g \) such that

\[
E_\varepsilon(v_\varepsilon) \leq C(|\log \varepsilon| + 1).
\]

b) \( (\text{Asymptotics}) \). For some subsequence \( (\varepsilon_n)_{n \in \mathbb{N}} \), there exist \( \ell \) points \( a_1, \ldots, a_\ell \) in \( \Omega \) and map \( v_\varepsilon \in C^\infty(\bar{\Omega} \setminus \bigcup_{i=1}^{\ell} \{a_i\}; \mathbb{R}^2) \) such that:

- \( v_{\varepsilon_n} \to v_* \) in \( C^1(K) \) for every compact subset \( K \) of \( \overline{D^2} \setminus \bigcup_{i=1}^{\ell} \{a_i\} \);

\[\text{ JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES } \]
\[ v_\epsilon = \prod_{i=1}^{\ell} \left\{ \frac{z-a_i}{|z-a_i|} \right\}^{d_i} \exp i\varphi, \text{ where } \varphi \text{ is real harmonic function on } \Omega, \text{ and } d_i \text{ is the winding number of } v_\epsilon \text{ around the singularity } a_i, \text{ i.e. } d_i = \deg(v_\epsilon, \partial B(a_i, r)) \text{ for } r \text{ sufficiently small}; \]

- \( d_i \neq 0 \), for \( i = 1, \ldots, \ell \) and there exists a constant \( C \) such that \( \sum_{i=1}^{\ell} |d_i| \leq C \) (\( C \) depending only on \( g \)).

\[ \Delta = \{(x_1, \ldots, x_\ell) \in \Omega^\ell, \exists i \neq j : x_i = x_j\}. \]

More precisely, they are critical point of \( W_g((b_1, d_1), \ldots, (b_\ell, d_\ell)) \) given by, for \( (b_1, \ldots, b_\ell) \in \Omega^\ell \setminus \Delta \)

\[ W_g((b_1, d_1), \ldots, (b_\ell, d_\ell)) = -\pi \sum_{i \neq j} d_i d_j \log|b_i - b_j| + \frac{1}{2} \int_{\partial \Omega} \Phi \cdot g \times g_r - \pi \sum_{i=1}^{\ell} d_i R_0(b_i) \]

where \( \Phi \) is the solution of

\[ -\Delta \Phi = \sum_{i=1}^{\ell} 2\pi d_i \delta_a, \text{ on } \Omega. \]

\[ \frac{\partial \Phi}{\partial \nu} = g \times g_r, \text{ on } \partial \Omega. \]

Here \( \nu \) denotes the unit exterior normal to \( \partial \Omega \) and \( \tau \) is unit tangent to \( \partial \Omega \) oriented so that \( (\nu, \tau) \) is direct. Finally \( R_0 \) is given by

\[ R_0(x) = \Phi - \sum_{i=1}^{\ell} d_i \log|x - b_i|. \]

In the special (and important) case where \( v_\epsilon \) is actually a minimizer of \( E_{\epsilon_{\text{ren}}} \), it is proved in \([BBH]\) that

\[ d_i = +1 \]

and that the points \( a_i \) are minimizers of the renormalized energy \( W_g \). Moreover, we have

\[ K_{\epsilon_{\text{ren}}}(v_\epsilon) = \pi d|\log \epsilon| + W_g(a_1, \ldots, a_d) + d\nu_0 + o(1). \]

where \( o(1) \to 0 \) as \( \epsilon \to 0 \), and where \( \nu_0 \) is some universal constant [note that, in case \( d_i = +1, \ell = d \), we will simply note \( W_g(a_1, \ldots, a_d) \) instead of \( W_g((a_1, 1), \ldots, (a_d, 1)) \)].

From analogies with fluid dynamics the points \( a_i \) will be called vortices, and the numbers \( d_i \in \mathbb{Z} \) their topological charge. The renormalized energy is an interaction energy between vortices: vortices of charges of same sign repeal, whereas vortices of charges of opposite sign attract. The boundary condition confines the vortices inside the domain, and hence
repeals them. The leading term in the interaction of two vortices is coulombian (i.e. 
$-\log|a_i - a_j|$). Moreover, we clearly have

$$W_g((a_i, d_i)) \rightarrow +\infty \text{ if } |a_i - a_j| \rightarrow 0 \text{ and } \text{sgn } d_i = \text{sgn } d_j > 0,$$

for some vortices $a_i, a_j$ in \{a_1, ..., a_\ell\}, and if the other ones remain far apart from each other

$$W_g((a_i, d_i)) \rightarrow +\infty \text{ if } \text{dist}(a_j, \partial \Omega) \rightarrow 0 \text{ for some } 1 < j \leq \ell.$$

At this point we would like to emphasize that the “vortex structure”, as described in
Theorem 3, is only established for solutions to the Ginzburg-Landau equation, and not for
arbitrary maps in $H^1_g(\Omega; \mathbb{R}^2)$: this, anyway would not make sense for maps of very high
energy $E_\varepsilon$. However, one of the main tasks of our work will be to prove that part of the
above picture remains valid for maps of “reasonably small energy”, namely maps in $E^\alpha_\varepsilon$,
where $\alpha$ satisfies a bound of the form

$$K_\varepsilon \leq \alpha \leq K_1(|\log \varepsilon| + 1),$$

where $K_1$ is any fixed arbitrary constant (independent of $\varepsilon$). (Actually, in view of proving
Theorem 2, it would even suffice to assume only the bound (1.4): we expect our more
general assumption to be useful for further purposes). We will prove that (in a certain
sense), we may assign to each map in $E^\alpha_\varepsilon$ a collection \{(a_i, d_i)\} of charged vortices. In
order to do so, and also in order to use this idea for Morse theory, we are faced to a
number of difficulties:

a) Find a way to define the vortices and their charge,
b) Obtain bounds on the number of vortices,
c) Relate the energy of a map $u$ to the renormalized energy of the vortices.

For instance, even if $\alpha$ verifies a bound of the form (1.8), we can not exclude the
possibility that a map $u$ in $E^\alpha_\varepsilon$ possesses a large number of dipoles: that is, a collection of
pairs of “vortices of opposite charge”, which are very close (see for instance [A] for related
ideas). These dipoles should not appear in the collection $(a_i, d_i)$ since they presumably
do not affect the overall topological structure of the level sets $E^\alpha_\varepsilon$ (at least for what is
of interest for us in this paper).

In order to get rid of these unpleasant details in the structure of maps in $E^\alpha_\varepsilon$, we will use
a regularization technique, of parabolic type. We consider (as for instance in [K], [BCGS])
the following minimization problem, for $u \in H^1_g(\Omega; \mathbb{R}^2)$

$$\inf_{v \in H^1_g(\Omega; \mathbb{R}^2)} E_\varepsilon(v) + \int_\Omega \frac{|u - v|^2}{2h^2},$$

where $h$ is of the form

$$h = \varepsilon^\gamma$$

for some fixed constant $0 < \gamma < 1$, to be determined later. Clearly the infimum in (1.9)
is achieved by some map $u^h$. We do not claim uniqueness for $u^h$, and we may possibly
have to make a choice for \( u^b \). Nevertheless, once this (arbitrary) choice is made, we will set \( u^b = T(u) \). The map \( u^b \) satisfies the equation:

\[
\frac{u^b - u}{h^2} = \Delta u^b + \frac{u^b}{\varepsilon^2} \left( 1 - |u^b|^2 \right) \quad \text{on } \Omega.
\]

Moreover taking \( v = u \) as a testing map in (1.9), we see that

\[
E_\varepsilon(u^b) \leq E_\varepsilon(u^b) + \int_\Omega \frac{|u - u^b|^2}{2h^2} \leq E_\varepsilon(u),
\]

so that if \( u \in E^a \), \( u^b \in E^a \) also. By standard elliptic theory we see that \( u^b \in H^2(\Omega) \subset C^1(\Omega) \), so that \( u^b \) is (at least) a \( C^1 \) map.

The main advantage of considering \( u^b \) instead of \( u \) is that we may adapt the PDE methods which have led to Theorem 3, since \( u^b \) satisfies (up to a perturbation) an equation similar to the Ginzburg-Landau equation. Of course there is a price to pay: the map \( u \mapsto T(u) \) is certainly not continuous (since (1.9) might have many solutions). We will overcome this difficulty by introducing the notion of \( \eta \)-almost continuity (see definition III.1).

In Section II, we will study thoroughly the properties of \( u^b \). Using methods from \([BBH]\), and also the technique of local estimates introduced independently in \([BR]\) and \([Str]\), we will prove, following step by step the arguments of \([BR]\).

**Theorem 4.** Assume that \( a \) satisfies the bound (1.8) for some constant \( K_1 \) (i.e. \( a \leq K_1(|\log \varepsilon| + 1) \)). Let \( u \) be in \( E^a \) such that \( |u| \leq 1 \) on \( \Omega \). Then

\[
|u^b| \leq 1;
\]

(1.13)

(1.14) There is a constant \( N \in \mathbb{N} \), and a constant \( \lambda > 0 \), depending only on \( K_1, \gamma \) and \( g \), such that there are \( \ell \) points \( a_1, \ldots, a_\ell \) in \( \Omega \) such that

\[
\ell \leq N.
\]

(1.15)

\[
|u^b(x)| \geq \frac{1}{2} \quad \forall \ x \in \Omega \setminus \bigcup_{i=1}^{\ell} B(a_i, 4\lambda \varepsilon)
\]

(1.16)

and

\[
B(a_i, 4\lambda \varepsilon) \cap B(a_j, 4\lambda \varepsilon) = \emptyset \quad \text{if } i \neq j.
\]

(1.17)

This result, as in \([BBH]\), allows us to define vortices \( a_1, \ldots, a_\ell \) for \( u^b \), and their corresponding winding number or topological charge \( d_i \). Of course, moving slightly the points \( a_i \), the new positions would still match the requirements of Theorem 4. Hence the assignment of vortices for \( u^b \) requires also a choice, but the freedom in this choice is not too wild. We will denote by \( \Phi \) the map \( \Phi : E^a \to W, u \mapsto \Phi(u) = \{(a_1, d_1), \ldots, (a_\ell, d_\ell)\} \).
where $W$ is the configuration space of vortices (which will be defined and topologized in Section III), and where $d_i$ is the winding number of $u^h$ on $\partial B(a_i, \lambda \varepsilon)$.

Note that the parameter $\gamma$ represents somehow the precision with which we want details in the structure of $u$ to appear. In the limit $\gamma \to 1$ all these details are present and the number $N$ of vortices blows up. In the limit $\gamma \to 0$, $u^h$ tends to a minimizing map and all details are lost.

In Section III, we will prove that the map $\Phi$ is $\eta$-almost continuous, for $\eta \to 0$ as $\varepsilon \to 0$. We will define this notion of $\eta$-almost continuity, specially tailored for our purpose. The important point is that (provided $\eta$ is sufficiently small) homotopy classes can still be defined for $\eta$-almost continuous maps, and hence we may use tools from topology. These results, which are somewhat of independent interest, will be presented in Section IV.

In Section V, we turn back to maps in $E^n$. At this point, we have a clear picture of what are the vortices of maps in $E^n$, and of what kind of continuity properties we might expect. In order to relate the energy of $u$ to the renormalized energy $W_\gamma$ of the vortices, we need nevertheless to obtain a better control of the configuration; in particular, in the proof of Theorem 5, we would like to have a precise control of the number $d_i$, and avoid the unpleasant situation where vortices of opposite charges are present. We will use for that purpose a cluster technique, which was introduced in a preliminary version of [BBH] (but was finally removed from the book, replaced by simpler arguments). This yields:

**Theorem 5.** Let $u$ be in $E^n$, $a_i$ satisfying the bound (1.8), and let $0 < \mu < 1$ be given. There is a constant $0 < \varepsilon_0 < 1$ depending only on $K_1, \gamma, \mu$ and $\eta$, such that if $\varepsilon < \varepsilon_0$, there is some $\tilde{\ell} \leq \ell$, and a radius $\rho > 0$, such that (relabeling if necessary the points $\{a_1, \ldots, a_\ell\}$) the following holds

$$\mu \varepsilon \leq \varepsilon^\mu \leq \rho \leq \varepsilon^\tilde{\ell}, \quad \text{where } \tilde{\mu} = \mu^{N+1},$$

$$|u^h(x)| \geq \frac{1}{2} \quad \text{if } x \in \Omega \setminus \bigcup_{i=1}^{\tilde{\ell}} B(a_i, \rho)$$

$$|u^h(x)| \geq 1 - \frac{2}{|\log \varepsilon|^2} \quad \text{if } x \in \partial B(a_i, \rho),$$

$$\int_{\partial B(a_i, \rho)} e_\varepsilon(u^h) \leq \frac{K_1 C(N, \mu)}{\rho} \quad \text{for } i \in \{1, \ldots, \tilde{\ell}\},$$

where $C(N, \mu)$ is given by $C(N, \mu) = \frac{2N}{\mu^{N-\mu N+\ell}}$;

$$|a_i - a_j| > 8\rho, \quad \text{for } i \neq j, 1 \leq i \leq \tilde{\ell}, 1 \leq j \leq \tilde{\ell}.$$

We will denote by $\tilde{\Phi}_\mu$ (or simply $\tilde{\Phi}$) the map which assigns to $u \in E^n$ the set of vortices $\{(a_1, d_1), \ldots, (a_\tilde{\ell}, d_\tilde{\ell})\}$. As $\Phi$, we will show, in Section V, that $\tilde{\Phi}$ is $\eta$-almost continuous (for some suitable $\eta$, such that $\eta \to 0$ as $\varepsilon \to 0$).
In Section VI, we will specify the previous analysis in the case where \( a \) satisfies the additional property (1.4), i.e.

\[
a = \mathcal{K} + \chi,
\]

(for some fixed \( \chi \)). In this special situation, the cluster technique of Section V yields:

**Theorem 6.** Let \( \chi > 0 \) be given and \( a = \mathcal{K} + \chi \). Let \( \gamma = \frac{\gamma}{d+1} \), \( \mu = \frac{1}{d+1} \) in Theorem 5. Then there exists some \( \varepsilon_0 > 0 \), such that if \( \varepsilon < \varepsilon_0 \), then for \( u \in E^a \), we have

\[
\ell = d,
\]

(1.23)

\[
d_i = 1, \quad \text{for } i = 1, \ldots, d.
\]

Moreover there exists some constant \( \nu > 0 \) depending only on \( g, \chi \) such that

\[
|a_i - a_j| \geq \nu \quad \text{for } i \neq j, \quad 1 \leq i \leq d, \quad 1 \leq j \leq d.
\]

(1.25)

Note that a related result, and the level set \( E^a \) satisfying (1.4), have been considered by F.H. Lin [Li2] in the context of the heat flow related to (1.1). Our work was partially motivated by [Li2]. Note also that now we have established properties of maps in \( E^a \) (for \( a \) satisfying (1.4)) very close to that of minimizers, as in [BBH].

As a consequence of Theorem 6, we see that, if \( a \) satisfies (1.4), the map \( \tilde{\Phi} \) restricted to \( E^a \) is a map from \( E^a \) to \( \Omega^{d} \setminus \Delta \) (i.e. may replace the complicated configuration space by the simple space \( \Omega^{d} \setminus \Delta \), whose topology is more tractable, and where computations of renormalized energies are much simpler).

Section VII is devoted to the proof of Theorem 2, which involves both the results of Theorem 6 and the arguments of topological nature developed in Section IV. The starting point is:

**Proposition 1.** Assume \( d \geq 2 \). There is a loop \( f : S^1 \to \Omega^d \setminus \Delta \), which is not contractible in \( \Omega^d \setminus \Delta \).

**Proof.** We may assume without loss of generality that \( 0 \in \Omega \), and then there exists some \( r > 0 \) such that \( B(2r) = \{ x \in \Omega \mid |x| < 2r \} \subset \Omega \). We define \( f \) by

\[
f(\exp \theta, i\theta) = (0, r \exp i\theta, b_3, \ldots, b_d), \quad \text{for } \theta \in [0, 2\pi],
\]

where \( b_3, b_4, \ldots, b_d \) are \( d - 2 \) distinct points in \( \Omega \setminus B(2r) \). It is then clear that \( f \) is continuous, and not contractible in \( \Omega^d \setminus \Delta \).

Using the loop \( f \), we may construct some loop \( f_\varepsilon \) in \( H^1_\rho(\Omega; \mathbb{R}^2) \) such that the vortices of \( f_\varepsilon(\exp i\theta) \) are precisely given by \( f(\exp i\theta) \). Moreover, we can construct \( f_\varepsilon \) in such a way that \( f_\varepsilon \) is a loop in \( E^a \), \( a = \mathcal{K} + \chi \), where \( \chi \) is suitably chosen, depending on \( f \). The main point in the proof of Theorem 2 is then to prove that \( f_\varepsilon \) is not contractible in \( E^a \), i.e. that there does not exist a continuous function \( \psi : D^3 \to E^a \), such that \( \psi|_{\partial D^3} = f_\varepsilon \).

The proof of Theorem 1 is completed in Section VIII.
An interesting (and open) question is to determine what the vortices of the solutions obtained will be (in the limit $\varepsilon \to 0$). We will discuss this issue in Section IX: in particular we will point out (based on heuristic arguments) that for $d = 2$ the solution has either a vortex of charge $+2$, or a vortex of charge $-1$, and three vortices of charge $+1$.

In many parts of this paper we have been a little more general than what was actually needed for the proof of Theorem 1. Indeed the proof requires only understanding the topology of the level sets $E^u$, with $a$ satisfying (1.4), whereas our analysis gives a good insight of the behavior of level sets with a satisfying (1.8), a much weaker condition. We expect this analysis to be useful elsewhere, and that we will be able to generalize and extend the results of this paper in a forthcoming work.

II. Properties of $u^h$.

II.1. Preliminaries

In this section, we will give, for $u \in H^1_0(\Omega; \mathbb{R}^2)$ some properties of $u^h$, where $u^h$ is a minimizer of $E_\varepsilon(v) + \int_\Omega \frac{|u - v|^2}{h^2}$. We will assume throughout that $h = \varepsilon^\gamma$ (for some fixed $0 < \gamma < 1$), and

$$|u| \leq 1 \quad \text{on} \quad \Omega,$$

$$E_\varepsilon(u) \leq K_1(|\log \varepsilon| + 1).$$

Our main purpose is to establish Theorem 3, following step by step the method of [BR], where a somewhat similar perturbed problem was considered (the general strategy however goes back to [BBH]). We begin with:

**Lemma II.1.** We have $u^h \in H^3(\Omega)$ and

$$|u^h| \leq 1 \quad \text{on} \quad \Omega. \quad (II.1)$$

**Proof.** The fact that $u^h \in H^3$ follows from standard elliptic estimates. For the second assertion, consider the map $v$ defined on $\Omega$ by

$$v(x) = u^h(x) \quad \text{if} \quad |u^h(x)| \leq 1,$$

$$v(x) = \frac{u^h(x)}{|u^h(x)|} \quad \text{if} \quad |u^h(x)| > 1.$$}

Clearly

$$E_\varepsilon(v) < E_\varepsilon(u^h)$$

and

$$|u - v|^2 \leq |u - u^h|^2,$$

by convexity of the disc $D^2$. Hence we have

$$E_\varepsilon(v) + \int_\Omega \frac{|u - v|^2}{2h^2} \leq E_\varepsilon(u^h) + \int_\Omega \frac{|u - u^h|^2}{2h^2}$$

with equality if and only if $v = u^h$. Since $u^h$ is a minimizer for (1.9) $v = u^h$, and the conclusion follows.
LEMMA 11.2. - There is a constant $C$, depending only on $g$ (but not on $\gamma$)

\[(11.2) \quad \|\nabla u^h\| \leq \frac{C}{\varepsilon}.\]

Proof. - The proof is readily the same as the proof of Lemma A.2 in [BBH2]. Therefore we omit it.

II.2. Local Estimates

Recall that the proof of Lemma IV.2 in [BBH] (which is similar to our Theorem 3) relied heavily on Pohozaev's identity on $\Omega$. Here, in view of the perturbation we introduced, this does not yield interesting estimates if applied on the whole of $\Omega$. Instead a local version of it (as in [BR] or [Str]) leads to:

LEMMA 11.3. - Let $\alpha$ be given, such that $0 < \gamma < \alpha < 1$. Then, there is a constant $C$, depending only on $g$, $\gamma$ and $\alpha$, such that

\[(11.3) \quad \frac{1}{\varepsilon^2} \int_{\Omega \cap R(x_0, \varepsilon^\alpha)} \left( 1 - |u^h|^2 \right)^2 \leq C, \quad \forall x_0 \in \Omega.\]

Proof. - Translating the origin if necessary, we may assume that $x_0 = 0$. Since

\[(11.4) \quad E_\varepsilon(u^h) \leq K_1(\|\log \varepsilon\| + 1),\]

there is some $r_0 \in (\varepsilon^{-\frac{1}{2}}, \varepsilon^\alpha)$, such that (for $\varepsilon < \frac{1}{2}$)

\[\int_{\partial B(r_0) \cap \Omega} c_\varepsilon(u) \leq \frac{4K_1}{(\alpha - \gamma)r_0},\]

for otherwise, (11.4) would be contradicted. Next, we multiply equation (I.11) by the Pohozaev multiplier $\sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i}$ (which is in $H^2$) and integrate on $B(r_0) \cap \Omega$. We are led (as in [BDII], Chap. III) to

\[(11.5) \quad \int_{B(r_0) \cap \Omega} \left( \frac{w^h - u}{\varepsilon^2} \left( \sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i} \right) + \frac{1}{2\varepsilon^2} \left( 1 - |u^h|^2 \right)^2 \right) \leq \frac{1}{2} \int_{\partial B(r_0) \cap \Omega} (x \cdot \nu)c_\varepsilon(u^h)\]

\[\leq \frac{2K_1}{\alpha - \gamma},\]

(where $\nu$ denotes the unit normal exterior to $B(r_0) \cap \Omega$). Next we estimate the first term in the left-hand side of (11.5). Since $|u^h - u| \leq 2$ we have:

\[(11.6) \quad \left| \int_{B(r_0) \cap \Omega} \left( \frac{w^h - u}{\varepsilon^2} \left( \sum_{i=1}^2 x_i \frac{\partial u}{\partial x_i} \right) \right) \right| \leq \frac{2}{\varepsilon^2} \int_{B(r_0) \cap \Omega} |\nabla u^h|\]

\[\leq \frac{2}{\varepsilon^2} \left( \int_{B(r_0) \cap \Omega} |\nabla u^h|^2 \right)^{1/2}\]

\[\leq C \left( \frac{r_0}{\varepsilon} \right)^2 (1 + \log \varepsilon + 1)^{1/2}\]

\[\leq C \varepsilon^{\alpha - \gamma}(1 + \log \varepsilon + 1)^{1/2} \to 0 \quad \text{as} \quad \varepsilon \to 0,\]

where we have used the fact that $\gamma < \alpha$. Combining (11.5) and (11.6), we obtain (11.4).
The importance of Lemma II.3 becomes clear (as in [BBH], [BR], [Str]) in the light of the following

**Lemma II.4.** There exist positive constants \( \lambda_0, \mu_0 \) depending only on \( g \) such that if \( u^h \) verifies, for some \( x_0 \in \Omega \), and \( \ell > 0 \), such that \( \frac{\ell}{\varepsilon} > \lambda_0 \) and

\[
\frac{1}{\varepsilon^2} \int_{B(x_0, 2\ell)} \left( 1 - |u^h|^2 \right)^2 \leq \mu_0,
\]

then

\[
|u^h(x)| \geq \frac{1}{2} \quad \text{on} \quad \Omega \cap B(x_0, \ell).
\]

**Proof.** The argument relies on Lemma II.2 and is exactly the same as in the proof of Theorem III.3 of [BBH]. Therefore we omit it.

As in [BBH], combining Lemma II.3 and Lemma II.4, we obtain the following local version of Theorem 4.

**Proposition II.1.** Let \( 0 < \gamma < \alpha < 1 \) and let \( x_0 \in \Omega \) be given. There exists a constant \( \bar{N}_\alpha \) depending only on \( \alpha, \gamma \) and \( g \), and \( \ell \) points \( x_1, \ldots, x_\ell \) in \( B(x_0, \varepsilon_0^\alpha) \cap \Omega \) such that

\[
\ell \leq \bar{N}_\alpha
\]

and

\[
|u^h(x)| \geq \frac{1}{2} \quad \text{for} \quad x \in (B(x_0, \varepsilon_0^\alpha) \cap \Omega) \setminus \bigcup_{i=1}^{\ell} B(x_i, \lambda_0 \varepsilon),
\]

(\( \lambda_0 \) being the constant in Lemma II.4).

**Proof.** The proof is readily the same as the proof of Proposition IV.2 of [BR] (or Lemmas IV.1 and IV.2 of [BBH]). Therefore, we omit it.

Finally, as in [BR], we need to deal with vortices of degree zero, and prove that they do not occur. Set for \( x_0 \in \Omega \)

\[
I_{\rho, x_0}(u^h) = \int_{\partial[B(x_0, \rho) \cap \Omega]} e(x, u^h),
\]

(and simply denote \( I_{\rho} \) if no confusion is possible). We have

**Lemma II.5.** Let \( D > 0, 0 < \gamma < \beta < 1, \eta > 1 \) be given constants such that \( \eta \beta < 1 \). Let \( \rho < \varepsilon^\eta \) and assume that \( \rho \beta > \lambda_0 \varepsilon \) and that

\[
I_{\rho}(u^h) < \frac{D}{\rho},
\]

\[
|u^h(x)| \geq \frac{1}{2} \quad \text{on} \quad \partial[B(x_0, \rho) \cap \Omega]
\]

and

\[
\deg \left( \frac{u^h}{|u^h|}, \partial[B(x_0, \rho) \cap \Omega] \right) = 0.
\]
Then we have

\[ |u^h| \geq \frac{1}{2} \quad \text{on } B(x_0, \rho^2) \cap \Omega. \]

**Proof.** The proof follows closely the proof of Lemma IV.1 in [BR]. The main idea is to construct a comparison function for (1.9), adapting a construction of [BBH2]. For any measurable subset \( K \) of \( \Omega \), we set

\[ E^h(u^h; K) = \frac{1}{2} \int_K c^e(u^h) + \frac{|u - u^h|^2}{2h^2}. \]

We are going to prove that

\[ (11.7) \quad E^h(u^h; B(x_0, \rho)) \leq C(\rho I_p(u^h) + o(1)), \]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \). Let \( \Gamma = \partial[B(x_0, \rho) \cap \Omega] \). Since by assumption the degree of \( \hat{u}^h = \frac{u^h}{|u^h|} \) on \( \Gamma \) is equal to zero, we may write on \( \Gamma \)

\[ u^h = \exp i\Phi \quad \text{on } \Gamma, \]

where \( \Phi \) is in \( H^1(\Gamma) \) and hence is a continuous map from \( \Gamma \) to \( \mathbb{R} \). Next, we consider the complex-valued map \( \psi \) defined on \( \Omega \) by

\[ v = \mu \exp i\psi \quad \text{in } B(x_0, \rho) \cap \Omega, \]

\[ v = u^h \quad \text{on } \Omega \setminus B(x_0, \rho), \]

where \( \psi \) is defined on \( B(x_0, \rho) \cap \Omega \) and is the solution to

\[ \Delta \psi = 0 \quad \text{on } B(x_0, \rho) \cap \Omega, \]

\[ \psi = \Phi \quad \text{on } \Gamma = \partial[B(x_0, \rho) \cap \Omega], \]

and \( \mu \) is the solution to

\[ -\varepsilon^2 \Delta \mu + \mu = 0 \quad \text{on } B(x_0, \rho) \cap \Omega, \]

\[ \mu = |u^h| \quad \text{on } \partial[B(x_0, \rho) \cap \Omega]. \]

Arguing as in [BBH2] or [BR], we obtain:

\[ (11.8) \quad \frac{1}{2} \int_{B(x_0, \rho) \cap \Omega} c^e(v) \leq C(\rho I_p(u^h) + o(1)) \leq C. \]

On the other hand,

\[ (11.9) \quad \int_{B(x_0, \rho) \cap \Omega} \frac{|v - u|^2}{2h^2} \leq 2\pi \frac{\rho^2}{h^2} \leq 2\pi \varepsilon^2(\beta - \gamma) \to 0 \quad \text{as } \varepsilon \to 0, \]

since \( \beta - \gamma > 0 \). (11.7) follows from (11.8) and (11.9).
Finally, we will complete the proof of Lemma 5. We deduce from (11.7) that
\[ \int_{\rho_{\theta}, \rho_{\theta}'} \mathcal{P} \leq C \]
and that there is some \( r_0 \in [\rho_{\theta}, \rho] \) such that
\[ I_{r_0}(u^h) \leq \frac{C}{r_0|\log r_0|^{1/2}}. \]
Arguing as in the proof of (11.7) we obtain
\[ E^h(u^h; B(x_0, r_0) \cap \Omega) \leq \frac{C}{|\log \varepsilon|^{1/2}} + o(1) \]
and hence
\[ \frac{1}{\varepsilon^2} \int_{B(x_0, \varepsilon^2) \cap \Omega} \left( 1 - |u^h|^2 \right)^2 \leq \frac{C}{|\log \varepsilon|^{1/2}} + o(1) \rightarrow 0 \]
as \( \varepsilon \rightarrow 0 \). The conclusion then follows from Lemma 11.4.

Using Lemma 11.5, we may now prove:

**Proposition 11.2.** Let \( x_0 \in \Omega \) and let \( 0 < \gamma < \alpha < 1 \) be as above. Assume that

There exists a constant \( C_\alpha \) depending only on \( \alpha, \gamma \) and \( \varepsilon \) such that
\[ \int_{B(x, \varepsilon \alpha \gamma) \cap \Omega} |\nabla u^h|^2 \geq C_\alpha |\log \varepsilon|. \]

**Proof.** The proof is exactly the same as the proof of Proposition IV.3 in [BR] and relies on Lemma II.5 and Proposition II.1. Therefore we omit it.

**II.3. From local to global: proof of Theorem 4**

Assertion (I.13) follows from Lemma II.1. Assertions (I.14) to (I.16) can be deduced from Proposition II.2 and Proposition II.1 following exactly the same arguments as in proof of Theorem IV.1 of [BR].

Finally, in order to obtain condition (I.17) (i.e. the fact that vortices are far apart), we may use the “cluster” argument of [BBH]: this is stated in Theorem IV.1 of [BBH], and can be transposed without changes to our situation.

**Remark.** If, instead of (I.8) in Theorem 4, we had considered condition (I.4), then the proof would have been considerably simplified. Indeed a recent result of P. Felmer and M. Del Pino [FP] asserts that, if \( u \in E^a \), with \( a \) satisfying (I.4), then
\[ (\Pi.10) \quad \frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 \leq C, \]
where the constant $C$ depends only on $g$ and $\chi$. The proof of (II.10), though extremely ingenious, turns out to be quite simple. The proof of Theorem 4 under the more restrictive condition (I.4) could then be derived using exactly the same arguments as in [BBH] Chapter III and IV; as we already noticed, such a result would have been sufficient for proving Theorem 1.

We emphasize once again that we have established Theorem 4 (under condition (I.8)), and also introduced the space $W$ (see Section III), in order to construct a variational framework for our problem, that is to find solutions of higher energy and to describe their vortices (see Section IX, and the conjectures therein).

III. Continuity of the vortices

III.1. $\eta$-almost continuity

Recall that we have defined a map $T : H^1_g(\Omega; \mathbb{R}^2) \to H^1_g(\Omega; \mathbb{R}^2)$ such that $T(u) = u^h$, $\forall u \in H^1_g(\Omega; \mathbb{R}^2)$ where $u^h$ is a solution of the minimization problem (1.9). As already noticed in the introduction, we might have to make choice for $\eta u$, if problem (1.9) has more than one minimizer, and therefore we cannot expect $T$ to be continuous.

Next, consider the map

$$ P : H^1(\mathbb{R}^2; \mathbb{R}) \to H^1(\mathbb{R}; \mathbb{R}^*) $$

defined by

\[
\begin{cases}
Pu(x) = u(x) & \text{if } |u(x)| \leq 1 \\
Pu(x) = \frac{u(x)}{|u(x)|} & \text{if } |u(x)| > 1.
\end{cases}
\]

It is easy to see that $P$ is a continuous map, and $P \circ P = P$. Moreover

$$ E_\varepsilon(P(u)) \leq E_\varepsilon(u), $$

so that, for any $a \geq K_\varepsilon$, $P$ restricted to $E^a$ is a map with image in

$$ \tilde{E}^a = E^a \cap \{ v \in H^1_g(\Omega; \mathbb{R}^2), |v| \leq 1 \text{ a.e. on } \Omega\}. $$

Finally, if $u \in \tilde{E}^a$, and if $a$ satisfies condition (I.8) we may assign to $u^h$, by Theorem 4, the set of its “charged vortices” that is define a map $\psi$ from $\text{Im}(T(\tilde{E}^a))$ to $W$, the space of charged vortices (which will topologized below) such that $\psi : \text{Im}(T(\tilde{E}^a)) \to W$,

\[
(\text{III.2}) \quad u^h = T(u) \mapsto \psi(u^h) = \{(a_1, d_1), \ldots, (a_\ell, d_\ell)\},
\]

where $a_1, \ldots, a_\ell$ are the vortices provided by Theorem 4, $\ell \leq N$ and where $d_i \in \mathbb{Z}$, for $i = 1, \ldots, \ell$ is the “winding number” of $a_i$, i.e.

\[
(\text{III.3}) \quad d_i = \text{deg}\left(\frac{u^h}{|u^h|}, \partial B(a_i, \lambda \varepsilon)\right).
\]
Finally, composing $P$, $T$ and $\psi$, we define:

$$\Phi : E^a \to W$$

$$\Phi(u) = \psi(T(Pu)) = \psi \circ T \circ P(u).$$

In view of all the previous remarks, we cannot expect $\Phi$ to be a continuous (since for $T$ and $\psi$ we might have to make a choice). However we will prove that $\Phi$ is "almost" a continuous map. For that purpose, we will introduce a notion specially tailored for our problem.

**Definition III.1.** Let $F$ and $G$ be two metric spaces, and $\eta \geq 0$ be given; let $f$ be a map from $F$ to $G$. We say that $f$ is $\eta$-almost continuous at a point $u_0 \in F$, if and only if for any $\varepsilon > 0$, there exists $\delta > 0$, such that if $d(u, u_0) \leq \delta$, then

$$(111.4) \quad d(f(u), f(u_0)) \leq \eta + \varepsilon.$$

We say that $f$ is $\eta$-almost continuous if it is $\eta$-almost continuous at any point $u_0 \in F$.

**Remarks.**

1. Of course, if $\eta = 0$ we recover the usual notion of continuity.
2. Similar notions have certainly been considered elsewhere, but we have not been able to find in literature.

We will prove that $\Phi$ is $\eta$-almost continuous, for some $\eta$, dependent on $\varepsilon$, such that $\eta \to 0$ as $\varepsilon \to 0$ (see § III.3). Next we will define more carefully the space of charged vortices and introduce a distance on it.

**III.2. The space of charged vortices**

**III.2.1. The space $W$ is aimed to describe charged vortices.** Let $u_i$ be a vortex, as in Theorem 3 or Theorem 4. Let $d_i$ be its topological charge, given by (III.3). We will call $|d_i|$ the multiplicity of the vortex $u_i$. In order to define $W$, we will only consider vortices of multiplicity 1, i.e. of charge $+1$ or $-1$. Therefore, any vortex will be repeated according to its multiplicity, and vortices of multiplicity zero will be cancelled in our collection of vortices. Hence, instead of considering the collection

$$\{(a_1, d_1), (a_2, d_2), \ldots, (a_\ell, d_\ell)\},$$

we will consider the collection

$$\left\{ \begin{array}{c} (a_1, \varepsilon_1), (a_1, \varepsilon_1), \ldots, (a_1, \varepsilon_1) \ \text{times} \ |d_1| \\ (a_2, \varepsilon_2), (a_2, \varepsilon_2), \ldots, (a_2, \varepsilon_2), \ldots \ \text{times} \ |d_2| \end{array} \right\},$$

omitting vortices of degree zero, and setting

$$\varepsilon_i = \text{sgn } d_i, \ \text{for } i = 1, \ldots, \ell.$$

We have now vortices of positive charge, and vortices of negative charge. It is convenient to relabel the collection denoting $(n_1, n_2, \ldots, n_r)$ the vortices of negative charge and
the vortices of positive charge (recall that \( \Sigma d_i = d \)), hence the number of positive vortices is equal to the number of negative vortices plus \( d \). In view of Theorem 4, we have

\[ \ell \leq N \]

and, by Lemma II.2

\[ \| \nabla u^h \| \leq C_\varepsilon, \]

which implies

\[ |d_i| = \frac{1}{2\pi} \int_{\partial B(a,\lambda \varepsilon)} |u^h|^{-2} (u^h \times u_x) \leq C_1 \int_{B(a,\lambda \varepsilon)} \| \nabla u^h \| \leq C \]

where \( C \) depends only on \( \lambda \). Hence there is some constant \( m_0 \) depending only on \( g \) and \( \gamma \) (but not on \( \varepsilon \)) such that

\[ r \leq m_0, \]

that is, the number of vortices repeated according to their multiplicity is bounded independently of \( \varepsilon \).

In view of the above discussion we introduce the space \( W_0 \) defined by:

\[ W_0 = \Omega^{2m_0+d} = \Omega^{m_0+d} \times \Omega^{m_0} = \{ (s,t) : s \in \Omega^{m_0+d}, t \in \Omega^{m_0} \} \]

and \( W \) will be defined as a quotient of \( W_0 \) by equivalence relations (we basically follow the method of Mac Duff [McD]). First in order to take into account that particles of same charge are indiscernable, and that ordering has no meaning, we define the equivalence relation \( \mathcal{R}_1 \)

\[ (s,t) \mathcal{R}_1 (s',t') \]

if and only if \( s \) and \( s' \) have the same elements (repeated with the same multiplicity) and \( t \) and \( t' \) have the same elements (repeated with the same multiplicity). We set

\[ \tilde{W}_0 = W / \mathcal{R}_1, \]

so that \( \tilde{W}_0 \) is the configuration space for \( m_0 + d \) vortices of charge +1, \( m_0 \) vortices of charge 1.

Next, we introduce, as in [McD] an equivalence relation which accounts for annihilation of vortices of opposite charges : this will also account for the fact that a configuration could have less than \( 2m_0 + d \) particles. Let \( (s,t) \) and \( (s',t') \) be in \( W_0 \), representative of two elements in \( \tilde{W}_0 \). We say that \( (s,t) \) is equivalent to \( (s',t') \) and denote \( (s,t) \sim (s',t') \) if and only if:
a) \( s - t = s' - t' \), and by this we mean that if we consider the set of positive vortices of \( s \), repeated according to their multiplicity and if we take off those which are also negative vortices (i.e. in \( t \)), repeated with multiplicity, and carry out the same procedure for \( (s', t') \), then the vortices which are left are the same (with multiplicity) in both cases.

b) \( t - s = t' - s' \), with a similar meaning. It is clear that this relation is compatible with \( \mathcal{R}_1 \) and hence is well-defined on \( \mathcal{W}_0 \). Our configuration will therefore finally be

\[
W = \mathcal{W}_0/\sim.
\]

It remains to check that the maps \( \psi \) and \( \Phi \) are well-defined from \( \text{Im}(\mathcal{E}^a) \) and \( E^a \) respectively to \( W \). We leave this to the reader.

### III.2.2. \( W \) is a metric space.

A distance can be defined on \( W \) using the notion of minimal connection introduced by Brezis, Coron and Lieb in [BCL]. For sake of completeness, we will recall its definition, in a more general setting.

Let \( G \) be a smooth bounded domain in \( \mathbb{R}^n \) (\( n = 2 \) for our situation) and let for some \( k \in \mathbb{N}_+ \), \( P_1, \ldots, P_k \) and \( N_1, \ldots, N_k \) be closed, regular subsets of \( G \) such that:

\[
\begin{align*}
(\text{III.6}) & \quad P_i \subset G; \quad P_i \cap P_j = \emptyset \text{ or } P_i = P_j, \\
(\text{III.7}) & \quad N_i \subset G; \quad N_i \cap N_j = \emptyset \text{ or } N_i = N_j, \\
(\text{III.8}) & \quad N_i \cap P_j = \emptyset \text{ or } N_i = P_j.
\end{align*}
\]

The last conditions in (III.6), (III.7) and (III.8) account for multiplicity and annihilation. The domains \( P_i, N_i \) represent charged holes. They could for instance be singletons, representing charged vortices as in III.2.1. The length of a minimal connection is then given by

\[
L(P_i, N_i) = \inf_{\sigma \in \mathcal{S}_k} \sum_{i=1}^k d(P_i, N_{\sigma(i)}),
\]

where \( \mathcal{S}_k \) denotes the set of permutations of \( \{1, \ldots, k\} \).

Next we consider two configurations \( c \) and \( c' \) in \( W \). Let \((p_1, \ldots, p_{m_0 + d}, n_1, \ldots, n_{m_0})\) and \((p'_1, \ldots, p'_{m_0 + d}, n'_1, \ldots, n'_{m_0})\) be representatives in \( \mathcal{W}_0 \) of \( c \) and \( c' \) respectively (here, as usual, the letter \( p \) stands for positive vortices, and the letter \( n \) for negative vortices). Let \( k = 2m_0 + d \), and

\[
\begin{align*}
(\text{III.9}) & \quad P_i = \{p_i\} \quad \text{if } i \leq m_0 + d \quad \text{and} \quad P_i = \{n'_{i-m_0-d}\} \quad \text{if } i > m_0 + d, \\
(\text{III.10}) & \quad N_i = \{p'_i\} \quad \text{if } i \leq m_0 + d \quad \text{and} \quad N_i = \{n_{i-m_0-d}\} \quad \text{if } i > m_0 + d.
\end{align*}
\]

In other words, we take all vortices of both \( c \) and \( c' \) and we revert the sign of the vortices of \( c' \) (whereas the sign of those of \( c \) is left unchanged). We then consider the length of the minimal connection related to the new configuration.

**Proposition III.1.** Let \( c \) and \( c' \) as above, and \( P_i, N_i \) defined by (III.9), (III.10). Set

\[
d(c, c') = L(P_i, N_i).
\]

Then \( d \) defines a distance on \( W \).
The proof is straightforward and we omit it (note that a related notion has been introduced in [BBC]).

Finally we will end this section by recalling some basic properties of the length of a minimal connection related to $S^1$-valued maps and the coarea formula. Let $G$ be a smooth, bounded and simply connected domain in $\mathbb{R}^2$, and let $P_1, \ldots, P_k, N_1, \ldots, N_k$ be charged holes in $G$, repeated with multiplicity, as above. We assume the holes to be simply connected. Set

$$G = G \setminus \bigcup_{i=1}^k (P_i \cup N_i)$$

and let $u : \tilde{G} \to S^1$ by a smooth map such that:

$$u \equiv Cte \text{ on } \partial G,$$

$$\deg(u, \partial P_i) = d_i \text{ where } d_i \text{ is the multiplicity of } P_i,$$

$$\deg(u, \partial N_i) = -d'_i \text{ where } d'_i \text{ is the multiplicity of } N_i.$$  

We have (see [BCL]).

**Proposition III.2.** - The length of a minimal connection connecting the $P_i$'s and $N_i$'s is given by the formula

$$L(u) = \sum_{i=1}^k L(P_i, N_i) = \frac{1}{2\pi} \sup \left\{ \int_{\tilde{G}} (u \times \nabla u) \cdot \nabla \xi, \ |\nabla \xi| \leq 1 \right\}.$$  

In particular

$$L(P_i, N_i) \leq \frac{1}{2\pi} \int_{\tilde{G}} |\nabla u|.$$  

The central role of the coarea of Federer-Fleming in this context has been emphasized by Almgren, Browder and Lieb [ABL]. Consider more generally a smooth compact riemannian manifold $\mathcal{N}$ of dimension $k$, and a smooth map $u$ from a $n$-dimensional domain $G$ to $\mathcal{N}$. For $y \in \mathcal{N}$ we consider the set

$$V(y) = u^{-1}(\{y\}).$$

By Sard's theorem, $V(y)$ is, for almost every $y \in \mathcal{N}$, a smooth submanifold of dimension $n - k$. The coarea formula yields

**Proposition III.3.** - Let $A$ be a measurable subset of $G$. We have

$$\int_{\mathcal{N}} \mathcal{H}^{n-k}(V(y) \cap A)dy = \int_A (J_k u)(x)dx,$$

where $J_k$ is the $k$-dimensional determinant of $u$ at $x$.

Recall that the $k$-dimensional Jacobian of $u$ at $x$ is the maximum $k$-volume of the image under $Du(x)$ of a unit $k$-dimensional cube. In particular if $\mathcal{N} = \mathbb{R}$, or $\mathcal{N} = S^1$, then $J_k u = |\nabla u|$.

Finally, going back to our map $u : \tilde{G} \to S^1$, it is proved in [ABL] that:

**Proposition III.4.** - For almost every $y \in S^1$

$$\mathcal{H}^1(V(y)) \geq L(u).$$

Combining this with the coarea formula, we recover the bound (III.13).
III.3. Continuity properties of the map $\Phi$

The aim of this section is to establish the following property of $\Phi$.

**Proposition III.5.** Assume $a$ satisfies (1.8). Then $\Phi$ is a $\eta$-almost continuous function from $E^a$ to $W$, for $\eta$ given by

\begin{equation}
\eta = C K_1 |\log \varepsilon|^\gamma,
\end{equation}

where $C$ is some universal constant (in particular $\eta \to 0$ as $\varepsilon \to 0$). More precisely we have

\begin{equation}
\text{dist}(\Phi(u), \Phi(v)) \leq (\eta + C |\log \varepsilon| ||u - v||_{H^k_0}),
\end{equation}

$\forall u, v \in \tilde{E}^a$, where $C$ is some constant depending only on $a$.

**Proof.** Let $u$ and $v$ be given in $\tilde{E}^a$, and let $u^h$ and $v^h$ be the corresponding solutions for the minimization problem (1.9), for $u$ and $v$ respectively, that is $u^h = T(u)$, $v^h = T(v)$.

Let $(a_1, \ldots, a_\ell)$ be the vortices for $u^h$, and $(a_{\ell+1}, \ldots, a_{\ell+s})$ be the vortices for $v^h$ with

$\ell \leq N$, $s \leq N$.

Let $\hat{\Omega} = \Omega \setminus \bigcup_{i=1}^{\ell+s} B(a_i, \lambda \varepsilon)$, so that, on $\hat{\Omega}$

$\frac{1}{2}, |v^h(x)| \geq \frac{1}{2}, \forall x \in \hat{\Omega}$.

We may therefore consider on $\hat{\Omega}$ the maps $\hat{u}$ and $\hat{v}$ defined by

$\hat{u} = \frac{u^h}{||u^h||}$ and $\hat{v} = \frac{v^h}{||v^h||}$

and consider on $\hat{\Omega}$, the map

$\psi = \hat{u} \hat{v}^{-1}$,  

where the product stands for complex multiplication and $\hat{v}^{-1}$ is the complex inverse of $\hat{v}$.

It follows that $\psi$ defined above is an $S^1$-valued map and that $\psi \equiv 1$ on $\partial \hat{\Omega}$. Moreover we have

$|\psi - 1| = |\hat{u} \hat{v}^{-1} - 1| = |\hat{u} - \hat{v}|$

and hence

$|\psi - 1| \leq \frac{1}{||u^h||} |u^h - v^h| + |v^h| \left( \frac{1}{||u^h||} - \frac{1}{||v^h||} \right) \leq \frac{1}{||u^h||} |u^h - v^h| + \frac{||u^h|| - ||v^h||}{||v^h||},$

i.e.

$|\psi - 1| \leq 4 |u^h - v^h|.$

This yields, integrating on $\hat{\Omega}$:
\[
\int_{\Omega} |\psi - 1|^2 \leq 16 \int_{\Omega} |u^h - v^h|^2 \\
\leq 48 \left( \int_{\Omega} |u - u^h|^2 + \int_{\Omega} |v - v^h|^2 + \int_{\Omega} |u - v|^2 \right)
\]
i.e.

\[
\int_{\Omega} |\psi - 1|^2 \leq C \left( K_1 |\log \varepsilon| \varepsilon^{2\gamma} + \|u - v\|^2_{H^1_0} \right)
\]

for some constant \(C\) depending only on \(K_1\).

The "vortices" of the map \(\psi\) are the holes \(B(a_i, \lambda \varepsilon)\) for \(1 \leq i \leq \ell + s\). We will assume for simplicity that \(B(a_i, \lambda \varepsilon) \cap B(a_j, \lambda \varepsilon) = \emptyset\) if \(i \neq j\) (this is always possible; otherwise, we may invoke the cluster argument of [BBH], Theorem IV.1, changing possibly \(\lambda\) by some larger constant, depending only on \(\rho, K_1\) and \(\gamma\)). Note that in this case we have:

\[
\deg(\psi, \partial B(a_i, \lambda \varepsilon)) = \deg(\tilde{u}; \partial B(a_i, \lambda \varepsilon)) - \deg(\tilde{u}; \partial B(a_i, \lambda \varepsilon)).
\]

In particular, in view of the definition of \(L\) and of the distance on \(W\), we are led to

\[
(\text{III.17}) \quad \int_{\Omega} |\psi - 1|^2 \leq C \left( K_1 |\log \varepsilon| \varepsilon^{2\gamma} + \|u - v\|^2_{H^1_0} \right)
\]

Here

\[
\mathcal{U}\{P_{k}\} = \{a_i, 1 \leq i \leq \ell + s \text{ if } \deg(\tilde{u}; \partial B(a_i, \lambda \varepsilon)) - \deg(\tilde{u}; \partial B(a_i, \lambda \varepsilon)) > 0\}
\]

and

\[
\mathcal{U}\{N_{k}\} = \{a_i, 1 \leq i \leq \ell + s \text{ if } \deg(\tilde{u}; \partial B(a_i, \lambda \varepsilon)) - \deg(\tilde{u}; \partial B(a_i, \lambda \varepsilon)) < 0\}.
\]

The factor \(4N\lambda \varepsilon\) on the right-hand side of \((\text{III.18})\) accounts for the fact that \(\psi\) is only defined on \(\widetilde{\Omega} = \Omega \setminus \bigcup_{i=1}^{\ell+s} B(a_i, \lambda \varepsilon)\) and not on \(\Omega \setminus \bigcup_{i=1}^{\ell+s} \{P_i\} \cup \{N_i\} = \Omega \setminus \bigcup_{i=1}^{\ell+s} \{a_i\}\). It represents an upper bound of the price to pay for connecting points of \(\partial B(a_i, \lambda \varepsilon)\) to \(a_i\).

In order to get an upper bound for \(L(\psi)\) we will use Proposition III.4 and the coarea formula. Set, for \(y \in S^1\)

\[
V(y) - \psi^{-1}(\{y\}).
\]

Then, by Proposition III.4, we have

\[
(\text{III.19}) \quad L(\psi) \leq \mathcal{H}^1(V(y)) \text{ for almost every } y \in S^1.
\]

On the other hand, let

\[
\mathcal{N} = \left\{y \in S^1, \frac{1}{8} \leq |y - 1| \leq \frac{1}{4}\right\}
\]
and take $A = \psi^{-1}(\mathcal{N})$ in Proposition III.3. We obtain

$$
\int_C \mathcal{H}^1(V(y)) dy - \int_A |\nabla \psi| \leq \left( \int_A |\nabla \psi|^2 \right)^{1/2} (\text{meas } A)^{1/2}.
$$

(III.20)

In view of (III.17), we have

$$
(\text{meas } A) \leq 64 \int_\Omega |\psi - 1|^2 \leq C \left( K_1 |\log \varepsilon| e^{2\gamma} + \|u - v\|_{H^1}\right);
$$

(III.21)

on the other hand

$$
|\nabla \psi| \leq |\nabla \dot{u}| + |\nabla \dot{v}| \leq 2(|\nabla u^h| + |\nabla \dot{v}|),
$$

so that

$$
\int_\Omega |\nabla \dot{v}|^2 < 8 \int_\Omega |\nabla u^h|^2 + |\nabla \dot{v}|^2 < 8K_1 (|\log \varepsilon| + 1).
$$

(III.22)

Combining (III.20), (III.21) and (III.22) we are led to

$$
\int_C \mathcal{H}^1(V(y)) \leq CK_1 |\log \varepsilon| \left( e^{\gamma} + \|u - v\|_{H^1}\right).
$$

Therefore, there exist some $y_0 \in \mathcal{N}$ such that

$$
\mathcal{H}^1(V(y_0)) \leq CK_1 |\log \varepsilon| \left( e^{\gamma} + \|u - v\|_{H^1}\right).
$$

(III.23)

Combining (III.23) with (III.19) and (III.18) we finally obtain the desired conclusion.

### IV. Homotopy classes for $\eta$-almost continuous map

#### IV.1.

This section is of somewhat independent interest, and deals with general properties of $\eta$-almost continuous maps. Our motivation is to prove that homotopy classes can be defined for $\eta$-almost continuous maps, provided $\eta$ is sufficiently small, and provided we make suitable assumptions on the target and the domain. This will in turn allow us to use many tools from algebraic topology (degree theory, ...) for $\eta$-almost continuous maps.

Note that the idea is of course not new and homotopy classes for discontinuous maps have been defined in a number of places (see for instance Schoen-Uhlenbeck [SU], and White [W] for maps in the Sobolev class, Brezis-Nirenberg for maps in BMO ([BN])).
Our main ingredient (as also in the above quoted works) will be the fact that \( \eta \)-almost continuous maps can be approximated by continuous maps. The main assumption we have to add will be \( F \) to be compact. We will prove:

**Proposition IV.1.** Let \( F \) be a compact metric space and \( G \) be a Banach space. Let \( \eta > 0 \) be given and let \( f \) be an \( \eta \)-almost continuous map from \( F \) to \( G \). Then there exists a continuous map \( \tilde{f} \) from \( F \) to \( G \) such that:

\[
\text{dist}_G(f(u), \tilde{f}(u)) \leq 3\eta, \quad \forall \ u \in F.
\]

(IV.1)

The proof of Proposition IV.1 will be postponed. We begin with some preliminary results which will enter into the proof.

**IV.2. Preliminary results**

We will first give an extension of the Heine-Borel-Lebesgue property to \( \eta \)-almost continuous maps.

**Lemma IV.1.** Let \( F, G, f \) and \( \eta \) be as in Proposition IV.1. For any \( \delta > 0 \), there exists \( \theta > 0 \) such that, if \( \text{dist}_F(u_1, u_2) \leq \theta \), then

\[
\text{dist}_G(f(u_1), f(u_2)) \leq 2\eta + \delta.
\]

The proof is similar to the corresponding one for continuous maps \((\eta = 0)\) and relies on Lebesgue’s covering Lemma. We recall:

**Lemma IV.2.** (Lebesgue’s covering Lemma). For any open cover \((U_i)_{i \in I}\) of a compact metric space \( F \), there exists \( \rho > 0 \) (the Lebesgue number of the covering) such that for any \( u \in F \) the ball \( B(u; \rho) \) is included in at least one of the \( U_i \)'s.

Here

\[
B(u; \rho) = \{v \in F, \ \text{dist}_F(u, v) < \rho\}.
\]

For a proof, see for instance Kelley [Ke].

**Proof of Lemma IV.1.** Let \( x \in F \). Since \( f \) is \( \eta \)-almost continuous there exists a number \( \theta_x > 0 \) such that, if

\[
\text{dist}(x, y) \leq \theta_x, \quad \text{then} \quad \text{dist}(f(x), f(y)) \leq \eta + \frac{\delta}{2}.
\]

Consider the cover of \( F \) given by \((B(x, \theta_x))_{x \in F}\). Let \( \rho \) be the Lebesgue number of the covering. For any \( u \in F \), there exists in view of Lemma IV.2 some \( x \in F \) such that \( B(u; \rho) \subset B(x; \theta_x) \). In particular, if \( \text{dist}(u, v) < \rho \), then \( v \) belongs to \( B(x; \theta_x) \) and

\[
\text{dist}(f(u), f(v)) \leq \text{dist}(f(u), f(x)) + \text{dist}(f(x), f(v)) \leq 2\eta + \delta
\]

and the proof is complete.
IV.3. Proof of Proposition IV.1

Set $\delta = \eta$ and let $\theta$ be the number provided by Lemma IV.1 (corresponding to $\delta = \eta$). Consider the open cover of $F$ $(B(x, \theta))_{x \in F}$. Since $F$ is a compact space, we may extract a finite cover, i.e. there exists $\ell \in \mathbb{N}$, and $\ell$ points $x_1, x_2, ..., x_\ell$ in $F$ such that

$$F = \bigcup_{i=1}^{\ell} B(x_i, \theta).$$

Set

$$U_i = B(x_i, \theta).$$

Next we construct a partition of unity with respect to the cover. For that purpose, set

$$\varphi_i(x) = \text{dist}(x, F \setminus U_i) \text{ for } x \in F$$

and

$$\xi_i(x) = \frac{\varphi_i(x)}{\sum_{j=1}^{\ell} \varphi_j(x)}.$$ Clearly, $\varphi_i$ and $\xi_i$ are continuous functions on $F$, for $i = 1, ..., \ell$ and

$$\xi_i(x) = 0 \text{ if } x \not\in U_i, \quad \xi_i(x) \geq 0, \forall x \in F. \tag{IV.2}$$

$$\sum_{i=1}^{\ell} \xi_i(x) = 1, \forall x \in F. \tag{IV.3}$$

We set

$$\hat{f}(x) - \sum_{i=1}^{\ell} \xi_i(x)f(x_i)$$

and we are going to show that $\hat{f}$ satisfies (IV.1). We have by the triangle inequality and (IV.2)-(IV.3)

$$\left\|f(u) - \hat{f}(u)\right\| \leq \sum_{i=1}^{\ell} \xi_i(u)\|f(x_i) - f(u)\|.$$

Since $\xi_i(u) = 0$ if $\text{dist}(x_i, u) \geq \theta$ it follows from Lemma IV.1 that

$$\xi_i(u)\|f(x_i) - f(u)\| \leq \xi_i(u)[2\eta + \delta] \leq 3\eta \xi_i(u)$$

and hence

$$\left\|f(u) - \hat{f}(u)\right\| \leq 3\eta \left(\sum_{i=1}^{\ell} \xi_i(u)\right) = 3\eta,$$

which is the desired result.
IV.3. Homotopy classes

Next we will consider the case where the target is a smooth compact Riemannian manifold $N$. By Nash’s embedding theorem we may assume that $N$ is isometrically embedded in some finite dimensional euclidian space $G = \mathbb{R}^k$ (for some $k \in \mathbb{N}$). We are going to show how Proposition IV.1 allows us to define homotopy classes for $\eta$-almost continuous maps in $N$, provided $\eta$ is sufficiently small. First recall that we may define the nearest point projection, provided $\mu_0$ is sufficiently small:

$$
\Pi : \theta_{\mu_0} \to N
$$

$$
x \mapsto \Pi(x) \in N,
$$

where $\Pi(x)$ is uniquely defined by the condition $\text{dist}(x, \Pi(x)) = \text{dist}(x, N)$. Here $\theta_{\mu}$ denotes the subset of $G$ defined by

$$
\theta_{\mu} = \{x \in G. \text{dist}(x, N) < \mu\}.
$$

Moreover $\Pi$ is smooth. Next we have:

**Proposition IV.2.** Let $F$ be a compact metric space, $\eta > 0$ and $f$ be an $\eta$-almost continuous map from $F$ to $N$. There exists a constant $\eta_0$ depending only on $N$, such that if $\eta \leq \eta_0$, then there exists a continuous map $\hat{f}$ from $F$ to $N$ such that:

$$
\|f(u) - \hat{f}(u)\| \leq 4\eta, \forall u \in F.
$$

Moreover, if $\hat{f}_1$ and $\hat{f}_2$ are any two continuous maps from $F$ to $N$ such that (IV.4) holds, then they are homotopic.

**Proof.** Since $f$ is $\eta$-almost continuous as a map from $F$ to $G = \mathbb{R}^k$, there exists according to Proposition IV.1 a continuous map $\hat{f}$ from $F$ to $G$, such that

$$
\|f(u) - \hat{f}(u)\| \leq 3\eta, \forall u \in F.
$$

In particular, since $f(u) \in N$

$$
\hat{f}(u) \in \theta_{3\eta}, \forall u \in F.
$$

Next assume that $3\eta \leq \mu_0$, and set

$$
\hat{f}(u) = \Pi(\hat{f}(u)),
$$

so that $\hat{f}$ is a continuous map from $F$ to $N$, and we have for $u \in F$

$$
\|f(u) - \hat{f}(u)\| \leq \|f(u) - \hat{f}(u)\| + \|\hat{f}(u) - \hat{f}(u)\|
$$

$$
\leq 3\eta + \|\hat{f}(u) - \Pi(\hat{f}(u))\|.
$$

Next note that

$$
\text{Sup}\{y \in \theta_{\mu}, \|y - \Pi(y)\|\} \to 0 \text{ as } \mu \to 0,
$$

so that the conclusion (IV.4) follows easily.
For the second assertion, if for any \( u \in F \)
\[
\left\| f(u) - \tilde{f}_1(u) \right\| \leq 6\eta, \quad \left\| f(u) - \tilde{f}_2(u) \right\| \leq 6\eta,
\]
then
\[
\left\| f(u) - \tilde{f}_2(u) \right\| \leq 12\eta, \quad \forall u \in F.
\]
It is then standard that \( \tilde{f}_1 \) and \( \tilde{f}_2 \) are homotopic, provided \( \eta \) is sufficiently small.

Proposition III.2 leads to the definition:

**Definition IV.1.** -- Let \( f \) be a \( \eta \)-almost continuous map from \( F \) to \( N \), such that \( \eta \leq \eta_0 \). We will call homotopy class of \( f \) the homotopy class of any continuous map \( \tilde{f} \) from \( F \) to \( N \) satisfying (IV.4).

In the next paragraph, we will specify a little more our manifold \( N \), in view of our applications.

**IV.4. \( \eta \)-almost continuous map in \( \Omega^d \setminus \Delta \)**

Set \( \Sigma = \Omega^d \setminus \Delta \), where
\[
\Delta = \{ (x_1, \ldots, x_d) \in \Omega^d, \; x_i \neq x_j, \; \text{if } i \neq j \},
\]
so that \( \Sigma \) is a non-compact Riemannian manifold. For \( r > 0 \), consider the subset \( \Sigma_r \) of \( \Sigma \) defined by
\[
(IV.5) \quad \Sigma_r = \{ (x_1, \ldots, x_d) \in \Sigma, \; |x_i - x_j| \geq r, \quad \forall i \neq j \}.
\]
If \( r \) is sufficiently small, say \( r < r_0 \) for some constant \( r_0 \) depending only on \( \Omega \) and \( d \), then \( \Sigma_r \) is not empty and is a smooth manifold with boundary.

Next let \( r < r_0 \), and let \( f \) be a \( \eta \)-almost continuous map from \( \bar{D}^2 \) to \( \Sigma_r \). We have:

**Proposition IV.3.** -- There exist some constant \( \eta_1 \) depending only on \( r, \Omega \) and \( d \), such that if \( \eta < \eta_1 \), then \( f \) restricted to \( \partial D^2 \) is homotopic to a constant map.

**Proof.** -- The analysis of § IV.3 can be carried over for compact manifolds with boundaries and hence to \( \Sigma_r \). Hence, if \( \eta \) is sufficiently there is some continuous map \( \tilde{f} \) from \( F \) to \( \Sigma_r \) such that
\[
\left| f(u) - \tilde{f}(u) \right| \leq 4\eta, \quad \forall u \in \bar{D}^2.
\]
Since, by simple topological arguments, \( \tilde{f} \) restricted to \( \partial D^2 \) is homotopic to a constant map, it follows that the same holds also for \( f \), provided \( \eta \) is sufficiently small.

We close this paragraph by a result of a slightly different nature, which will also enter in the proof of Theorem 5. It shows that in some cases a map in \( W \) can be lifted to a map with values in \( \Omega^d \setminus \Delta \). To be more precise, assume \( 0 \in \Omega \), there is a natural embedding
\[
i : \Sigma = \Omega^d \setminus \Lambda \rightarrow W_0 = \Omega^{m_0+d} \times \Omega^{m_0},
\]
\[
(x_1, \ldots, x_d) \rightarrow (x_1, \ldots, x_d, 0, \ldots 0)
\]
and clearly passing to the quotients, this yields a continuous map $i : \Sigma \to W$. We denote by $i_\nu$ the restriction of $i$ to $\Sigma_\nu$, i.e. $i_\nu : \Sigma_\nu \to W$. We will show that under special assumptions, a map in $W$ can be lifted to a map in $\Sigma_\nu$.

Let $F$ be a compact metric space and $f : F \to W$ an $\eta$-almost continuous map such that $f$ has the following property:

(IV.6) There exists $\nu > 0$ such that for any $u \in F$, there exists an element $c \in \Sigma_\nu$ such that $f(u) = i(c)$.

Then we have:

**Proposition IV.4.** - If $\eta \leq \eta_\nu$, where $\eta_\nu$ depends only on $d$ and $\nu$, then there exists an $\eta$-almost continuous map $\tilde{f}$ from $F$ to $\Sigma_\nu$ such that $f(u) = i((\tilde{f}(u))) \quad \forall u \in F$.

**Proof.** - In the case where the map $f$ is continuous the argument relies on a standard lifting procedure. For $\eta$-almost continuous one may invoke the approximation property (Proposition IV.2), which reduces the problem to the previous case.

The next section deals with a completely different topic.

**V. Cluster of vortices**

This section is devoted to the proof of Theorem 5. As already mentioned, the material presented in this section originated in a preliminary version of [BBH], and has never been published so far. It is based on a “clustering” method, which might be of independent interest, and which we are going to present next in a more general setting. We will follow, almost word for word, the above mentioned version of [BBH].

Let $G \subset G'$ be two bounded domains in $\mathbb{R}^2$ such that $G \subset G'$. Let $\eta = \text{dist}(G, \partial G')$ and assume $\eta > 0$. Let $K_0 > 0$, $\lambda > 0$, $N \in \mathbb{N}^*$ be arbitrary constants. We consider a small parameter $\epsilon > 0$ and we assume that there is a smooth map $\nu$ from $G'$ to $\mathbb{C}$ and a collection of points $\{x_i\}_{i \in J}$, $x_i \in G$. $\forall i \in J$ and such that the following conditions hold

(V.1) \quad \nu \in \operatorname{Card} J \leq N,$

(V.2) \quad |v(x)| \leq 1. \quad \forall x \in G',

(V.3) \quad |v(x)| = 1. \quad \forall x \in \partial G',

(V.4) \quad \frac{1}{2} \int_{G'} v(x) \leq K_0 (|\log \epsilon| + 1),

(V.5) \quad |v(x)| > \frac{1}{2} \text{ on } G' \setminus \bigcup_{i \in J} B(x_i, \lambda \epsilon).

Then we have:

**Theorem V.1.** - ([BBH], Preliminary version 1992). Assume that (V.1)-(V.5) hold, and let $0 < \mu < 1$ be a given constant. There is some constant $0 < \epsilon_0 < 1$ depending only
on $K_1, \lambda, \eta, N$ and $\mu$ such that if $\varepsilon < \varepsilon_0$, we may choose a subset $J' \subset J$ and a radius $\rho > 0$ such that the following properties hold:

\begin{align*}
(V.6) & \quad \lambda \varepsilon \leq \varepsilon^u \leq \rho \leq \varepsilon^\mu \leq \eta \text{ with } \mu = \mu^{N+1}. \\
(V.7) & \quad |v(x)| \geq \frac{1}{2} \text{ if } x \in G \setminus \bigcup_{i \in J'} B(x_i, \rho), \\
(V.8) & \quad |v(x)| \geq 1 - \frac{2}{|\log \varepsilon|^2} \quad \text{if } x \in \partial B(x_i, \rho) \text{ for } i \in J', \\
(V.9) & \quad \int_{\partial B(x_i, \rho)} e_\varepsilon(v) \leq \frac{K_1 C(N, \mu)}{\rho} \text{ for } i \in J', \\
(V.10) & \quad |x_i - x_j| \geq 8\rho \quad \text{for any } i \neq j \in J'.
\end{align*}

We postpone the proof of Theorem V.1 for a little moment, and we will state a preliminary result, which will be used in the course of the proof. For $\xi \in [0, 1]$, set:

$$V(\xi) = \{x \in G', \ |v(x)| = \xi\}.$$  

By Sard's theorem, $V$ is a finite union of smooth curves in $G'$ for almost every $\xi \in [0, 1]$. We have:

**Lemma V.1.** There is some $\xi_0 \in \left(1 - \frac{2}{|\log \varepsilon|^2}, 1 - \frac{1}{|\log \varepsilon|^2}\right)$ such that:

\begin{equation}
\mathcal{H}^1(V(\xi_0)) \leq C K_1 |\log \varepsilon|^2 \varepsilon
\end{equation}

and $V(\xi_0)$ is a finite union of smooth curves. Here $C$ denotes some universal constant.

We give the proof of Lemma V.1 after completion of the proof of Theorem V.1.

**Proof of Theorem V.1.** The proof is by induction, using a finite number of steps (at most $N$).

**Step 1.** If $\varepsilon$ is sufficiently small, we may find a radius $\rho_1$ such that

\begin{align*}
(V.12) & \quad \lambda \varepsilon \leq \varepsilon^u \leq \rho_1 \leq \varepsilon^\mu \\
(V.13) & \quad \int_{\partial B(x_i, \rho_1)} e_\varepsilon(v) \leq \frac{K_1 C_1(\mu)}{\rho_1} \text{ with } C_1(\mu) = \frac{2N}{\mu - \mu^2} \\
(V.14) & \quad |v(x)| \geq 1 - \frac{2}{|\log \varepsilon|^2} \quad \text{if } x \in \partial B(x_i, \rho_1).
\end{align*}

**Proof of Step 1.** Assume first that:

$$\varepsilon < \varepsilon_1 = \min\left\{1, \eta^{1/\mu^2}, \lambda^{1/\lambda^{1/\mu}}\right\},$$

so that

$$\lambda \varepsilon \leq \varepsilon^u.$$
and

\[ B(x_i, \rho) \subset B(x_i, \varepsilon^{n^i}) \subset G' \text{ for } \rho < \varepsilon^{n^i}. \]

Let

\[ B_i = \left\{ \rho \in (\varepsilon^n, \varepsilon^{n^i}) \text{ such that } \inf_{x \in \partial B(x_i, \rho)} |v(x)| \leq \xi_0 \right\}, \]

where \( \xi_0 \) is given by Lemma V.1, and let

\[ A_i = (\varepsilon^n, \varepsilon^{n^i}) \setminus B_i. \]

We claim that

(V.15) \[ \text{meas } B_i \leq \text{meas } (\mathcal{H}^1(V(\xi_0))) \leq CK_1 \log \varepsilon^3. \]

Indeed the set \( V(\xi_0) \) consists of a finite union closed curves; they do not intersect the boundary since we have assumed \( |v(x)| = 1 \) on \( \partial G' \). We keep only the maximal curves in \( V(\xi_0) \), i.e. if one curve encloses another one, we keep only the exterior one. Let \( V_k \) be the collection of maximal curves in \( V(\xi_0) \), and let \( W^k \) be the domain enclosed by \( V_k \). We have

\[ |v(x)| \geq \xi_0, \quad \forall x \in G' \setminus \bigcup_k W^k. \]

Let \((a^k, b^k)\) be the smallest interval such that:

\[ W^k \subset B(x_i, b^k) \setminus B(x_i, a^k) \]

so that

\[ B_i \subset \bigcup_k (a^k, b^k) \]

and hence

\[ b^k - a^k \leq \mathcal{H}^1(V^k), \]

from which we deduce the claim (V.15).

We complete next the proof of Step 1. We argue by contradiction and assume that, for every \( \rho \) in \( A = \bigcap_{i \in J} A_i \), we have

(V.16) \[ \int_{\partial B(x_i, \rho)} e_\varepsilon(v) > K_1 \frac{C_1(\mu)}{\rho}. \]

Summing over \( J \) and integrating over \( A \), we are led to

(V.17) \[ \int_A \left( \sum_{i \in J} \int_{\partial B(x_i, \rho)} e_\varepsilon(v) \right) \geq K_1 C_1(\mu) \int_A \frac{d\rho}{\rho}. \]
On the other hand, we have

\begin{equation}
\int_A \frac{dp}{\rho} = \int_{x^2}^{x^2} \frac{dp}{\rho} - \int_B \frac{dp}{\rho} = (\mu - \mu^2)|\log \varepsilon| - \int_B \frac{dp}{\rho},
\end{equation}

where we have set \( B = \bigcup_{i \in J} B_i \).

Since the function \( \frac{1}{\rho} \) is non-increasing, we have

\begin{equation}
\int_B \frac{dp}{\rho} \leq \int_{x^2 + \text{meas } B} \frac{dp}{\rho} \leq \int_{x^2 + \sum_{i \in J} \text{meas } B_i} \frac{dp}{\rho} \leq \log \left(1 + NCK_1 \varepsilon^{1-\mu} |\log \varepsilon|^5\right) \leq CKN_1 \varepsilon^{1-\mu} |\log \varepsilon|^5,
\end{equation}

where we have used (V.15). Consequently there is some constant \( \varepsilon_2 \leq \varepsilon_1 \) such that if \( \varepsilon < \varepsilon_2 \), then

\begin{equation}
\sum_{i \in J} \int_{B_i} \frac{dp}{\rho} \leq \frac{\mu - \mu^2}{2} |\log \varepsilon|.
\end{equation}

Combining this last inequality with (V.18) we obtain

\begin{equation}
\int_A \frac{dp}{\rho} \geq \frac{\mu - \mu^2}{2} |\log \varepsilon|
\end{equation}

and returning to (V.17), we find, using Fubini’s Theorem:

\begin{align*}
N \int_{G^n} e_\varepsilon(v) &> \sum_{i \in J} \left( \int_{\partial B_{(x_i, p)}} e_\varepsilon(v) \right) dp \\
&> \int_A \left( \sum_{i \in J} \int_{\partial B_{(x_i, p)}} e_\varepsilon(v) \right) dp \\
&> K_1 C_1(\mu) \frac{\mu - \mu^2}{2} |\log \varepsilon| \quad \text{by (V.17) and (V.19)} \\
&> K_1 N |\log \varepsilon|,
\end{align*}

a contradiction with assumption (V.4). Hence (V.16) does not hold, i.e. there is some \( p_1 \in A \) such that

\begin{equation}
\int_{\partial B_{(x_i, p_1)}} e_\varepsilon(v) \leq K_1 \frac{C_1(\mu)}{p_1}, \quad \forall i \in J
\end{equation}

and

\begin{equation}
|x(v)| \geq \xi_0 \geq 1 - \frac{2}{|\log \varepsilon|^2}, \quad \forall x \in \partial B_{(x_i, p_1)}, \ i \in J.
\end{equation}

We turn now our attention to condition (V.10) in Theorem V.1 (all other conditions are, at this stage, fulfilled with \( \rho = p_1 \) and \( J' = J \)). We have the following alternative:
Case 1. - (V.10) is satisfied for all $i \neq j$ in $J$, and in this case the balls $B(x_i, \rho_1)$ fulfill the conditions of the Theorem, and the proof is completed.

Case 2. - (V.10) is not satisfied. Hence labelling the points by $J = \{1, \ldots, \ell\}$, there are at least two of them, say $x_1$ and $x_{\ell-1}$, such that

(V.10) \[ |x_i - x_{\ell-1}| \leq 8\rho_1. \]

In particular, since $\rho_1 \leq \varepsilon\mu^2$, we have

\[ |x_{\ell} - x_{\ell-1}| \leq 8\varepsilon\mu^2. \]

We are going to eliminate the point $x_\ell$ from our collection $\{x_1, \ldots, x_\ell\}$ and use the same arguments with $\ell - 1$ points.

Step 2. - We first remark that, by (V.22),

(V.22) \[ B(x_\ell, \lambda \varepsilon) \subseteq B(x_{\ell-1}, \rho) \]

provided

\[ \rho \geq 9\varepsilon\mu^2 \geq \lambda \varepsilon + 8\varepsilon\mu^2. \]

Arguing as in Step 1, we see that if $\varepsilon$ is sufficiently small, we may find some new radius $\rho_2$, such that for $i = 1, \ldots, \ell - 1$, we have:

(V.23) \[ 9\varepsilon\mu^2 < \rho_2 < \varepsilon\mu^3. \]

(V.24) \[ \int_{\partial B(x_i, \rho_2)} e_\varepsilon(v) \leq K_1 \frac{C_2(\mu)}{\rho_2} \text{ where } C_2(\mu) = \frac{2N}{\mu^2 - \mu^3}. \]

(V.25) \[ |v(x)| > 1 - \frac{2}{|\log \varepsilon|^2} \text{ if } x \in \partial B(x_i, \rho_2). \]

It follows from (V.23)-(V.26) that assumptions (V.6)-(V.9) are satisfied for the balls $B(x_i, \rho_2)$, for $i \in J' = \{1, \ldots, \ell - 1\}$. It remains to verify (V.10). As in Step 1 we distinguish two cases.

Case 1. - (V.10) is satisfied and the proof of Theorem V.1 is complete with $J' = \{1, 2, \ldots, \ell - 1\}$ and $\rho = \rho_2$.

Case 2. - (V.10) is not satisfied. Relabelling the points, we may assume that

(V.10) \[ |x_{\ell-2} - x_{\ell-1}| \leq 8\rho_2. \]

We proceed as in Step 1 and remove the point $x_{\ell-1}$ from the collection. Then in Step 3 we use the same arguments with $\ell - 2$ points.

More generally at step $q$ we are left with $\ell - q + 2$ points $x_1, \ldots, x_{\ell-q+2}$ such that

\[ |x_{\ell-q+1}, \ldots, x_{\ell-q+2}| \leq 8\varepsilon\mu^2. \]
Arguing as in Step 1 and Step 2, we see that if $\varepsilon$ is sufficiently small, we may find some radius $\rho_q$, such that for any $i = 1, \ldots, \ell - q + 1$, we obtain:

\begin{align}
\text{(V.27)} \quad 9\varepsilon^{\mu^q} \leq \rho_q \leq \varepsilon^{\mu^{q+1}}, \\
\text{(V.28)} \quad \int_{\partial B(x_i, \rho_q)} e_\varepsilon(v) \leq K_1 \frac{C_q(\mu)}{\rho_q} \quad \text{where} \quad C_q(\mu) = \frac{2N}{\mu^q - \mu^{q+1}}, \\
\text{(V.29)} \quad |v(x)| \geq 1 - \frac{2}{|\log \varepsilon|^2} \quad \text{if} \quad x \in \partial B(x_i, \rho_q).
\end{align}

Clearly (V.8) and (V.9) hold for $1 \leq i \leq \ell - q + 1$, by (V.28) and (V.29) for $\rho = \rho_q$.

We claim that (V.7) is satisfied, that is

\begin{align}
\text{(V.30)} \quad |u(x)| \geq \frac{1}{2} \quad \text{if} \quad x \in G' \setminus \bigcup_{i=1}^{\ell-q+1} D(x_i, \rho_q).
\end{align}

It suffices indeed, in view of (V.5) and (V.6), to check that

\begin{align}
\text{(V.31)} \quad \bigcup_{j=\ell-q+2}^{\ell} B(x_i, \lambda \varepsilon) \subset B(x_{\ell-q+1}, \rho_q).
\end{align}

For this purpose assume that $x \in \bigcup_{j=\ell-q+2}^{\ell} B(x_i, \lambda \varepsilon)$ i.e.

\begin{align}
|x - x_j| \leq \lambda \varepsilon \quad \text{for some} \quad j, \quad \ell - q + 2 \leq j \leq \ell.
\end{align}

Then we have by (V.27)

\begin{align}
|x - x_{\ell-q+1}| &\leq |x_{\ell-q+1} - x_{\ell-q+2}| + \ldots + |x_{j+1} - x_j| + |x - x_j| \\
&\leq 8 \left( \varepsilon^{\mu^q} + \varepsilon^{\mu^{q+1}} + \ldots + \varepsilon^{\mu^q} \right) + \lambda \varepsilon.
\end{align}

Hence

\begin{align}
|x - x_{\ell-q+1}| \leq \varepsilon^{\mu^q},
\end{align}

provided

\begin{align}
8\varepsilon^{\mu^q-1} \leq \varepsilon^{\mu^q},
\end{align}

which is satisfied for sufficiently small $\varepsilon$. This proves (V.31).

Turning to (V.10) we distinguish two cases.

Case 1. - (V.10) is satisfied. The proof of Theorem V.1 is complete with $J' = \{1, 2, \ldots, \ell - q + 1\}$ and $\rho = \rho_q$.

Case 2. - (V.10) is not satisfied. We eliminate $x_{\ell-q+1}$ from the collection and repeat the argument at Step $q + 1$.

Since at each step, we eliminate one point, we complete the proof in at most $N$ steps (since $\ell \leq N$).
Next we turn to the proof of Lemma V.1.

**Proof of Lemma V.1.** - The proof relies on the coarea formula, applied to the function $|v|$. Setting:

$$A = \left\{ x \in G', \, 1 - \frac{2}{|\log \varepsilon|^2} \leq |v(x)| \leq 1 - \frac{1}{|\log \varepsilon|^2} \right\}$$

we have

$$\int_{1-\frac{2}{|\log \varepsilon|^2}}^{1-\frac{1}{|\log \varepsilon|^2}} \mathcal{H}^1(V(\xi)) d\xi = \int_A |\nabla|v||$$

(V.32)

$$\leq \left( \int_{G'} |\nabla v|^2 \right)^{1/2} (\text{meas } A)^{1/2}.$$ 

Using the bound (V.4) we have

$$\frac{1}{4} \int_{G'} (1 - |v|^2)^2 \leq K_1 |\log \varepsilon|^2;$$

we deduce that

$$(\text{meas } A) \leq C K_1 |\log \varepsilon|^2 \varepsilon^2$$

for some constant $C$. Inserting into (V.32) we find

$$\int_{1-\frac{2}{|\log \varepsilon|^2}}^{1-\frac{1}{|\log \varepsilon|^2}} \mathcal{H}^1(V(\xi)) d\xi \leq C K_1 |\log \varepsilon|^2 \varepsilon.$$ 

Hence there is, by the mean value formula some $\xi_0 \in \left( 1 - \frac{1}{|\log \varepsilon|^2}, 1 - \frac{2}{|\log \varepsilon|^2} \right)$ which satisfies (V.11).

Finally, we end this section by showing briefly how Theorem V.1 implies Theorem 5.

**Proof of Lemma 5.** - Let $\eta > 0$ and consider the domain

$$\Omega_\eta = \{ x \in \mathbb{R}^2, \, \text{dist} (x, \Omega) < \eta \} \supset \Omega.$$ 

If $\eta$ is sufficiently small, then $\Omega_\eta$ is a smooth domain. We may construct on $\Omega_\eta \setminus \Omega$ a smooth complex valued map $\tilde{g}$ such that

$$|\tilde{g}| = 1 \text{ on } \Omega_\eta \setminus \Omega,$$

$$\tilde{g} = g \text{ on } \partial \Omega.$$ 

Next, define on $\Omega_\eta$ the map $\tilde{u}_h$ by

$$\tilde{u}_h = u^h \text{ on } \Omega,$$

$$\tilde{u}_h = \tilde{g} \text{ on } \Omega_\eta \setminus \Omega,$$
so that \( \hat{u}^b \) is continuous and in \( H^1(\Omega_n) \). We may then apply Theorem V.1 to \( G = \Omega \), \( G' = \Omega_n \) and the map \( \hat{u}^b \) (or rather some smooth approximation of \( \hat{u}^b \), since we have assumed that \( v \) is smooth). This yields the conclusions of Theorem 5.

Next we consider the map \( \Phi_\mu \):

\[
\Phi_\mu : E^a \rightarrow W
\]

where \( a_1, \ldots, a_\ell \) are the vortices given by Theorem 5 for \( u^b \) (if \( u \in \hat{E}^a \) that is \( Pu = u \)), and otherwise for \( T(P(u)) \), if \( u \in E^a \). We have:

**Proposition V.1.** The map \( \Phi_\mu \) is \( \eta \)-almost continuous from \( E^a \) to \( W \) for

\[
\eta = C \left( \varepsilon^N\eta^\alpha + \varepsilon\eta^\gamma \right)
\]

where the constant \( C \) depends only on \( \gamma, \mu, g \) and \( K_1 \).

**Proof.** Recall that \( \Phi : E^a \rightarrow W \), \( \Phi = \psi(T(P(u))) \) is \( \eta \)-almost continuous with \( \eta = C\varepsilon^\gamma\eta^\gamma \log \varepsilon \) (see Proposition III.5, and (III.1), (III.2) for definitions of \( \psi, T, P \)). Let \( u \) be a map in \( E^a \), and let

\[
c = \Phi(u) = ((a_1, d_1), \ldots, (a_\ell, d_\ell))\)
\[
c' = \Phi_\mu(u) = ((a_1, \bar{d}_1), \ldots, (a_\ell, \bar{d}_\ell)).
\]

We claim that

\[
(V.33)\quad d(c, c') = d(\Phi(u), \Phi_\mu(u)) \leq C\varepsilon^\gamma N\alpha + \varepsilon\eta^\gamma \log \varepsilon
\]

Indeed we have

\[
\bigcup_{i=1}^\ell \{a_i\} \subset \bigcup_{i=1}^\ell \{a_i\},
\]

(from the construction in Theorem 5), and if some \( i, a_i \notin \bigcup_{i=1}^\ell \{a_i\} \), then there exists \( j \in \{1, \ldots, \ell\} \) such that

\[
(V.34)\quad |a_i - \bar{a}_j| \leq \rho \leq \varepsilon^\gamma N\alpha.
\]

The claim (V.33) can then be deduced using the definition of the length of minimal connection. Finally let \( u \) and \( v \) be in \( E^a \); we have, by the triangle inequality

\[
d\left(\Phi_\mu(u), \Phi_\mu(v)\right) \leq d(\Phi(u), \Phi(v)) + d\left(\Phi_\mu(u), \Phi(u)\right) + d\left(\Phi_\mu(v), \Phi(v)\right)
\]

\[
\leq C\varepsilon^\gamma N\alpha + d(\Phi(u), \Phi(v))
\]

and the proof of Proposition V.1 can be completed using the \( \eta \)-almost continuity of \( \Phi \) for \( \eta = C\varepsilon^\gamma \log \varepsilon \).
VI. Proof of theorem 6

Let $\chi > 0$ be a given constant. Recall that we consider here maps in $E^a$ for

$$a = K_\epsilon + \chi,$$

where $K_\epsilon$ is given by (I.3). In order to prove Theorem 6, we will begin by the following preliminary result.

**Lemma VI.1.** Let $D^2$ be the unit disc in $\mathbb{R}^2$, and let $C_1 > 0$ be some fixed constant. Let $v \in H^1(D^2; \mathbb{C})$ such that

(VI.1) $\int_{\partial D^2} e_{\epsilon}(v) \leq C_1,$

(VI.2) $1 \geq |v(x)| \geq 1 - \frac{1}{\log \epsilon}$ on $\partial D^2$.

Then we have

(VI.3) $E_\epsilon(v) \geq \pi |d||\log \epsilon| + C_2,$

where $C_2$ is a constant depending only on $C_1$ and $d = \deg(l_{\epsilon}^{-1}v, \partial D^2)$.

**Proof.** We are going to use a related result, Theorem IX.3 of [BBH]. This result asserts that if $G$ is some starshaped smooth bounded domain in $\mathbb{R}^2$ and if $g$ is some smooth map from $\partial G$ to $S^1$, then

(VI.4) $E_\epsilon(v) \geq \pi |d||\log \epsilon| + C,$

for any $v \in H^1_b(\Omega; \mathbb{C})$, where $d = \deg(g, \partial G)$ and where $C$ is a constant depending only on $g$. Here the situation considered is slightly more general, since we do not assume that $v$ is prescribed on $\partial D^2$; instead we assume only (V.1) and (V.2). However we are going to reduce our problem to the previous one. For that purpose we shall consider the larger disc $D^2(2) = \{x \in \mathbb{R}^2, |x| < 2\}$, and construct first a map $\bar{v}$ on $\Sigma = D^2(2) \setminus D^2$ such that

(VI.5) $\bar{v} = \frac{v}{|v|}$ on $\partial D^2$,

(VI.6) $\bar{v} = \exp i\theta$ on $\partial D^2(2),$

(VI.7) $|\bar{v}| = 1$ on $\Sigma,$

(VI.8) $\int_{\Sigma} |\nabla \bar{v}|^2 \leq C_3,$

where $C_3$ depends only on $C_1$. 

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Construction of $\tilde{v}$. In view of (VI.1) and (VI.2), we see that

\begin{equation}
|d| = \frac{1}{2\pi} \int_{\partial D^2} |v^{-1} v \times v_\theta| \leq \left( \frac{1}{\pi} \int_{\partial D^2} |v_\theta|^2 \right)^{1/2} \leq \left( \frac{C_1}{\pi} \right)^{1/2}
\end{equation}

provided $\varepsilon$ is sufficiently small. Moreover by (VI.1), $v \in H^1(\partial D^2)$ and therefore is continuous on $\partial D^2$. We may write on $\partial D^2$

\[ v(0) = |v| \exp(i\theta + \varphi(\theta)) \quad \theta \in [0, 2\pi]. \]

where $\varphi(\theta)$ is a real-valued function from $\partial D^2 = S^1$ to $\mathbb{R}$ which is in $H^1(\partial D^2)$ and hence continuous. The function $\varphi$ is uniquely defined in $H^1(\partial D^2)$ up to a multiple of $2\pi$. In particular we may assume that:

\[ \varphi(0) \in [0, 2\pi]. \]

Moreover, we have

\[ |\varphi'(\theta)| \leq \frac{1}{|v|} \left( \left| \frac{\partial v}{\partial \theta} \right| + d \right) \leq 2 \left( \left| \frac{\partial v}{\partial \theta} \right| + d \right) \quad \text{if } \varepsilon \text{ is small enough.} \]

Integrating over $\partial D^2$, we obtain using (VI.9)

\[ \int_{\partial D^2} |\varphi'(\theta)|^2 \leq 24C_1 \]

and hence, since $\varphi(0) \in [0, 2\pi]$, we are led to

\begin{equation}
||\varphi||^2_{H^1(\partial D^2)} \leq K(1 + C_1),
\end{equation}

where $K$ is some absolute constant.

Next let $\psi$ be the solution of

\[ \begin{align*}
\Delta \psi &= 0 & \text{on } \Sigma, \\
\psi &= \varphi & \text{on } \partial D^2, \\
\psi &= 0 & \text{on } \partial D^2(2).
\end{align*} \]

By (III.10) and elliptic estimates we have:

\[ \int_{\Sigma} |\nabla \psi|^2 + |\psi|^2 \leq K(1 + C_1) \]

for some absolute constant $K$. We set

\[ \tilde{v} = \exp(i\theta + \psi) \quad \text{on } \Sigma \]

and we easily verify that $\tilde{v}$ satisfies (IV.5)-(VI.8).
Proof of Lemma VI.1 completed. – We consider the map \( \tilde{v} \) defined on \( D^2(2) \) by

\[
\tilde{v} = \tilde{v} \quad \text{on } \Sigma = D^2(2) \setminus D^2(1),
\]

\[
\tilde{v}(x) = \begin{cases} 
\frac{v(x)}{|v|} & \text{for } x \in D^2, \text{ if } |v(x)| \geq 1, \\
\frac{1}{|\log |v||} & \text{if } |v(x)| \leq 1 - \frac{1}{|\log |v||}.
\end{cases}
\]

We easily verify that \( \tilde{v} \in H^1(D^2(2)) \) and that

\[
\tilde{v}(x) = \exp id\theta, \quad \text{for } x \in \partial D^2(2).
\]

Consequently we may use (VI.4) for \( \tilde{v} \), with \( G = D^2(2) \) and \( g = \exp id\theta \) that is

(VI.11) \[ E_\varepsilon(\tilde{v}) \geq \pi |d||\log \varepsilon| + C. \]

where \( C \) depends only on \( d \), hence on \( C_1 \). Next we note that by (VI.7)

(VI.12) \[ \int_{D^2(2)} (1 - |\tilde{v}|^2)^2 = \int_{D^2} (1 - |\tilde{v}|^2)^2 \leq \int_{D^2} (1 - |v|^2)^2. \]

On the other hand, we verify that

(VI.13) \[ \int_{D^2} |\nabla \tilde{v}|^2 \leq \left( \int_{D^2} |\nabla v|^2 \right) \left( 1 - \frac{1}{|\log |v||} \right)^{-2}, \]

so that, combining (VI.11), (VI.12), (VI.13) with (VI.8)

\[ E_\varepsilon(v) \geq \left( 1 - \frac{1}{|\log \varepsilon|} \right)^2 (E_\varepsilon(\tilde{v}) - C_3), \]

which yields (VI.3) and completes the proof of the Lemma.

Now we are able to turn to the proof of Theorem 6.

Proof of Theorem 6. – Let \( u \in \tilde{E}^a \), with \( a \) satisfying (I.4) i.e.

\[ a = K_\varepsilon + \chi \]

for some fixed \( \chi > 0 \). Since \( u \in E^a \), we see that \( u^h \in E^a \). For \( i = 1, \ldots, \ell \), let \( \bar{a}_1, \ldots, \bar{a}_\ell \) be the vortices provided by Theorem 5, and let \( \bar{d}_i \) be the corresponding winding numbers, i.e.

\[ \bar{d}_i = \deg \left( \left| u^h \right|^{-1} u^h, \partial B(\bar{a}_i, \rho) \cap \Omega \right). \]

Recall also that \( u^h \) satisfies (I.18) to (I.21). We will divide the proof of Theorem 6 into several steps.
Step 1. - We have
\[ \ast \cdot \tilde{d}_i \neq 0 \text{ for } i = 1, \ldots, \ell, \text{ if } \varepsilon \text{ is sufficiently small.} \]

Proof. - This is a consequence of Lemma II.5. Indeed, from the construction of \( \tilde{a}_i \), we have
\[
(\text{VI.14}) \quad |u^h(\tilde{a}_i)| < \frac{1}{2}, \quad \forall i = 1, \ldots, \ell.
\]
Assume by contradiction that \( \tilde{d}_i = 0 \); then we may apply Lemma II.5 to the ball \( B(\tilde{a}_i, \rho) \), with \( \beta = \mu^{N+1} \) and for some fixed \( \eta \leq \mu^{-(N+1)} \). This yields
\[
|u^h| \geq \frac{1}{2} \text{ on } B(\tilde{a}_i, \lambda \varepsilon),
\]
a contradiction with (VI.14).

Step 2. - We have for \( i = 1, \ldots, \ell \)
\[
(\text{VI.15}) \quad \int_{B(\tilde{a}_i, \rho)} e_\varepsilon(u^h) \geq \pi|d_i| \log \frac{\varepsilon}{\rho} + C,
\]
where \( C \) is some constant depending only on \( g \).

Proof. - The proof relies on Lemma VI.1. We assume for simplicity that \( B(\tilde{a}_i, \rho) \subset \Omega \) (otherwise it would suffice to consider \( \Omega_\rho \) and extend \( u^h \) to \( \Omega_\rho \)). Consider the change of variables defined by \( \hat{x} = \frac{x - \tilde{a}_i}{\rho} \), so that, if \( x \in B(\tilde{a}_i, \rho) \), \( \hat{x} \in D^2 \). We have in the new variables
\[
(\text{VI.16}) \quad \int_{B(\tilde{a}_i, \rho)} e_\varepsilon(u^h) = \int_{D^2} e_\varepsilon(\hat{u}^h),
\]
where \( \hat{\varepsilon} = \frac{\varepsilon}{\rho} \) and \( \hat{u}^h(\hat{x}) = u^h(\rho \hat{x} + \tilde{a}_i) \). By Lemma VI.1 we have, since by (1.20) and (1.21), (VI.1) and (IV.2) are satisfied for \( \hat{u}^h \) and \( \hat{\varepsilon} \)
\[
\int_{D^2} e_\varepsilon(u^h) \geq \pi|d_i| \log \hat{\varepsilon} + C = \pi|d_i| \left| \log \frac{\varepsilon}{\rho} \right| + C,
\]
where \( C \) depends only on \( \mu, \gamma \) and \( g \). The conclusion follows by (VI.16) (recall that \( \mu = \frac{1}{4d+1}, \gamma = \frac{2}{4d+1} \)).

Step 3. - We have \( \ell = d \) and
\[
\tilde{d}_i = +1, \text{ for } i = 1, \ldots, d.
\]

Proof. - By step 2, we have
\[
\int_{\bigcup_{i=1}^{\ell} B(\tilde{a}_i, \rho)} e_\varepsilon(u^h) \geq \pi \sum_{i=1}^{\ell} \tilde{d}_i \left| \log \frac{\varepsilon}{\rho} \right| + C.
\]
Since $u^h$ is in $F^a$ with $a$ satisfying (1.4) we also have

$$\int_{\Omega} v_\varepsilon(u^h) \leq a d \log \varepsilon + C,$$

where $C$ is a constant depending on $\chi$ and $g$. We therefore deduce combining the previous inequalities that

(VI.17) \[
\pi \sum_{i=1}^l |d_i| \left| \log \frac{\varepsilon}{\rho} \right| \leq \pi d \log \varepsilon + C,
\]

where $C$ is a constant depending on $g, \gamma_1$ and $\chi$. Since $\varepsilon^\mu \leq \rho \leq \varepsilon^{N+1}$, we have

\[
\left| \log \frac{\varepsilon}{\rho} \right| \geq \left| \log \frac{\varepsilon}{\varepsilon^\mu} \right| = (1 - \mu) \log \varepsilon
\]

and hence (VI.17) yields

(VI.18) \[
(1 - \mu) \pi \sum_{i=1}^l |\tilde{d}_i| \log \varepsilon \leq \pi d \log \varepsilon + C.
\]

If $\varepsilon$ is sufficiently small, we obtain

(VI.19) \[
(1 - \mu) \sum_{i=1}^l |d_i| \leq d + \frac{1}{4}.
\]

Recall that $\sum_{i=1}^l d_i = d$, so that

(VI.19) \[
\sum_{i=1}^l |d_i| \geq d.
\]

Combining (VI.18) and (VI.19) we see that

(VI.20) \[
d \leq \sum_{i=1}^l |d_i| \leq \left( d + \frac{1}{4} \right)(1 - \mu)^{-1}.
\]

With our choice

$$\mu = \frac{1}{4d + 1},$$

we obtain

$$\left( 1 - \mu \right)^{-1} \left( d + \frac{1}{4} \right) \leq d + \frac{2}{3} < d + 1.$$

and returning to (VI.20) we find, since \( d_i \in \mathbb{Z} \) and \( d \in \mathbb{N} \),

\[
\sum_{i=1}^{t} \vert d_i \vert = d,
\]

which together with step 1 yields the desired conclusion.

At this point, we have established (1.23) and (1.24). The remainder will be devoted to the proof of (1.25).

**Step 4.** We have

\[
\int_{\Omega_{\rho}} \vert \nabla u^h \vert^2 \geq \pi d \log \rho + W_{g}(\bar{a}_1, ..., \bar{a}_d) + o(1),
\]

where \( o(1) \to 0 \) as \( \varepsilon \to 0 \), and where \( \Omega_{\rho} = \Omega \setminus \bigcup_{i=1}^{t} B(a_i, \rho) \).

**Proof.** Consider on \( \Omega_{\rho} \) the map \( \bar{u} = \vert u^h \vert^{-1} u^h \), so that \( \bar{u} \) is \( S^1 \)-valued. As in [BBH], Section I introduce the function \( \Phi \), solution to:

\[
\begin{align*}
\Delta \Phi &= 0 \quad \text{on } \Omega_{\rho}, \\
\Phi &= C^{iu} = C^{i} \quad \text{on } \partial B(a_i, \rho), \quad \forall \; i = 1, ..., d, \\
\int_{\partial B(a_i, \rho)} \frac{\partial \Phi}{\partial \nu} &= 2\pi, \quad \forall \; i = 1, ..., d, \quad \nu \text{ being the exterior normal}, \\
\int_{\partial \Omega_{\rho}} \frac{\partial \Phi}{\partial \nu} &= g \times g_{\tau} \quad \text{on } \partial \Omega, \quad (\tau, \nu) \text{ being a direct orthonormal frame on } \partial \Omega.
\end{align*}
\]

[Here we have implicitly assumed that \( \text{dist}(a_i, \partial \Omega) \geq 2\rho \) : if this is not satisfied replace \( \Omega \) by \( \Omega' = \{ x \in \mathbb{R}^2, \text{dist}(x, \Omega) \leq 2\rho \} \), and extend \( u^h \) by \( \bar{g} \) as in the proof of Theorem 51.]

From the analysis of [BBH] it turns out that

\[
\int_{\Omega_{\rho}} \vert \nabla \Phi \vert^2 = \pi d \log \rho + W_{g}(\bar{a}_1, ..., \bar{a}_d) + O(\rho^2).
\]

As in [BBH], we may next write the Hodge-de-Rham decomposition

\[
\bar{u} \times \nabla \bar{u} = \nabla^{T} \Phi + \nabla H,
\]

where \( H \in C^{1}(\Omega_{\rho}) \) is such that \( H = 0 \) on \( \partial \Omega \) and \( \nabla^{T} \Phi = (\Phi_{x_2}, \Phi_{x_1}) \). In view of (VI.26) and (II.2) we have

\[
\frac{\nabla H}{\varepsilon} \leq \frac{C}{\varepsilon} \quad \text{and} \quad \int_{\Omega_{\rho}} \vert \nabla H \vert^2 \leq C(\log \varepsilon + 1).
\]
We verify that
\[\int_{\Omega_{\rho}} |\nabla \tilde{u}|^2 = \int_{\Omega_{\rho}} |\nabla \Phi|^2 + \int_{\Omega_{\rho}} |\nabla H|^2 + 2 \int_{\Omega_{\rho}} \nabla^T \Phi \cdot \nabla H.\]

The last term on the right-hand side is the integral of a Jacobian: it hence reduces to boundary terms. Since \( \Phi = C_{10} \) on \( \partial B(1, \rho) \) and \( H = 0 \) on \( \partial \Omega \) it is equal to zero, so that:

\[\int_{\Omega_{\rho}} |\nabla \tilde{u}|^2 - \int_{\Omega_{\rho}} |\nabla \Phi|^2 + |\nabla H|^2 \geq \int_{\Omega_{\rho}} |\nabla \Phi|^2.\]

Next we observe that \( |\nabla u_h|^2 \geq |u_h|^2 |\nabla \tilde{u}|^2 \), so that:

\[\int_{\Omega_{\rho}} |\nabla u_h|^2 \geq \int_{\Omega_{\rho}} |u_h|^2 |\nabla \tilde{u}|^2 \geq \int_{\Omega_{\rho}} |u_h|^2 |\nabla \Phi|^2 + 2 \int_{\Omega_{\rho}} |u_h|^2 \nabla^T \Phi \cdot \nabla H \]

\[\geq \int_{\Omega_{\rho}} |\nabla \Phi|^2 - R_1 - R_2,\]

where
\[R_1 = \int_{\Omega_{\rho}} \left(1 - |u_h|^2\right) |\nabla \Phi|^2\]

and
\[R_2 = \int_{\Omega_{\rho}} \left(1 - |u_h|^2\right) |\nabla \Phi||\nabla H|.

We estimate \( R_1 \) and \( R_2 \). We have

\[R_1 \leq \rho^{-2} \int_{\Omega} \left(1 - |u_h|^2\right) \leq C \varepsilon^{-2} \left[\int_{\Omega} \left(1 - |u_h|^2\right)^2\right]^{1/2}\]

\[\leq C \varepsilon^{1-2\rho} |\log \varepsilon|^{1/2} \to 0.\]

For \( R_2 \), we decompose \( \Omega_{\rho} \) into \( \Omega_{\rho} = \Omega_1 \cup \Omega_2 \), where:

\[\Omega_1 = \left\{ x \in \Omega_{\rho}, \|u_h(x)\| \geq 1 - \frac{2}{|\log \varepsilon|^2}\right\},\]

\[\Omega_2 = \left\{ x \in \Omega_{\rho}, \|u_h(x)\| < 1 - \frac{2}{|\log \varepsilon|^2}\right\}.

We claim that

\[\text{meas } \Omega_2 \leq C \varepsilon^2 |\log \varepsilon|^{10}.

Indeed let \( \xi_0 \) be as in Lemma V.1. We see that \( \Omega_2 \subset W(\xi_0) \) where \( W(\xi_0) \) is the domain bounded by \( V(\xi_0) \). Using the isoperimetric inequality we have \( \text{meas}(W(\xi_0)) \leq (2\pi)^{-1} [\mathcal{H}^1(V(\xi_0))]^2 \), and the conclusion follows from Lemma V.1.
Next we write
\[ \int_{\Omega_1} \left( 1 - |u_h|^2 \right) |\nabla \Phi| |\nabla H| \leq \frac{2}{2 \log \varepsilon^2} \int_{\Omega_1} |\nabla \Phi| |\nabla H| \leq \frac{C |\log \varepsilon|}{|\log \varepsilon|^2} \to 0, \quad \text{as } \varepsilon \to 0, \]
and by (VI.26) and (VI.28)
\[ \int_{\Omega_2} \left( 1 - |u_h|^2 \right) |\nabla \Phi| |\nabla H| \leq C \rho^{-1} \int_{\Omega_2} \left( \frac{1 - |u_h|^2}{\varepsilon} \right) \leq C \rho^{-1} \left[ \int_{\Omega_2} \left( \frac{1 - |u_h|^2}{\varepsilon^2} \right)^{1/2} \right] \leq C e^{-\mu |\log \varepsilon|^{1/2} |\log \varepsilon|^5} \to 0 \quad \text{as } \varepsilon \to 0, \]
so that \( R_2 \to 0 \) as \( \varepsilon \to 0 \). Hence going back to (VI.30) we obtain
\[ \int_{\Omega_2} |\nabla u_h|^2 \geq \int_{\Omega_2} |\nabla \Phi|^2 + o(1), \quad o(1) \to 0 \quad \text{as } \varepsilon \to 0, \]
which combined with (VI.27) yields (VI.21).

**Step 5.** We have
\[ (VI.33) \quad \int_{\Omega} |\nabla u_h|^2 \geq \pi d |\log \varepsilon| + W_\delta(\bar{a}_1, ..., \bar{a}_d) + C \]
for some constant \( C \) depending only on \( g \).

**Proof.** It suffices to combine (VI.15), the fact that \( d_i = +1, \) and (VI.21).

**Step 6.** Proof of Theorem 6 completed. Since \( a \) satisfies (I.4),
\[ \int_{\Omega} |\nabla u_h|^2 \leq \pi d |\log \varepsilon| + C, \]
we deduce from (VI.33) that
\[ W_\delta(a_1, ..., a_d) \leq C, \]
where \( C \) depends only on \( g \) and \( \chi \). Since
\[ W_\delta(b_1, ..., b_d) \to +\infty \quad \text{if } |b_i - b_j| \to 0 \quad \text{for some } i \neq j, \]
the conclusion (I.25) follows.
VII. Proof of Theorem 2

We will divide the proof into two steps.

Step 1. Construction of the loop $f_\varepsilon$ in $E^0$. Consider first the loop $f$, constructed in Proposition 2, from $S^1$ to $\Omega^d \setminus \Delta$

$$f(\exp i\theta) = (0, r\exp i\theta, b_2, \ldots, b_d)$$

for $\theta \in [0, 2\pi]$, assuming $0 \in \Omega$ and $B(2r) \subset \Omega$. Here $b_2, \ldots, b_d$ are $d - 2$ distinct points in $\Omega \setminus B(2r)$. We set, for $\theta \in [0, 2\pi]$

$$b_1(\theta) = 0, \quad b_2(\theta) = r \exp i\theta, \quad b_i(\theta) = b_i$$

for $i = 2, \ldots, d$.

Consider next the domain, depending on $\theta$ and $\varepsilon$

$$\Omega_\varepsilon(\theta) = \Omega \setminus \bigcup_{i=1}^d B^2(b_i(\theta), \varepsilon)$$

and the map $f_\varepsilon$ from $S^1$ to $H^1(\Omega_\varepsilon; S^1)$ defined by:

$$f_\varepsilon(\exp i\theta)(z) = \prod_{i=1}^d \frac{z - b_i(\theta)}{|z - b_i(\theta)|} \exp i\varphi_\theta(z), \quad \text{for } z \in \Omega_\varepsilon(\theta).$$

Here $\varphi_\theta$ denotes the map defined on $\Omega$ by

$$\Delta \varphi_\theta = 0 \quad \text{on } \Omega,$$

$$\exp i\varphi_\theta(z) \prod_{i=1}^d \frac{z - b_i(\theta)}{|z - b_i(\theta)|} = g \quad \text{on } \partial \Omega.$$

For a given $\theta$ the map $\varphi_\theta$ is uniquely defined, up to a multiple of $2\pi$. We claim that we may choose this constant such that the map $\theta \mapsto \varphi_\theta$ is continuous. It is indeed easy to prove the assertion in the case $r$ is sufficiently small. For any $z \in \partial \Omega$, the map $\theta \mapsto \exp i\varphi_\theta(z)$ ($z$ fixed) takes values in a small interval of $S^1$, so that we may use a standard lifting argument. The more general case could be deduced by continuation.

Next, we define $f_\varepsilon(\exp i\theta)(z)$, for $z \in \bigcup_{i=1}^d B(b_i(\theta), \varepsilon)$. Let $f_\varepsilon(\exp i\theta)$ be such that

$$\Delta f_\varepsilon(\exp i\theta) = 0 \quad \text{on } B^2(b_i(\theta), \varepsilon),$$

$$f_\varepsilon(\exp i\theta)(z) = \prod_{i=1}^d \frac{z - b_i(\theta)}{|z - b_i(\theta)|} \exp i\varphi_\theta(z), \quad \text{for } z \in \partial B^2(b_i(\theta), \varepsilon).$$

It is then easy to verify that $f_\varepsilon$ is a continuous map from $S^1$ to $H^1_{\phi}(\Omega, \mathbb{R}^2)$. Moreover computing the energy of $f_\varepsilon(\exp i\theta)$, we have, for sufficiently small $\varepsilon$, for $\theta \in [0, 2\pi]$ (we use the analysis of [BBH], Section I)

$$\int_\Omega e_\varepsilon(f_\varepsilon(\exp i\theta)) \leq \int_\Omega |\nabla f_\varepsilon(\exp i\theta)|^2 + \sum_{i=1}^d \int_{B(b_i(\theta), \varepsilon)} e_\varepsilon(f_\varepsilon(\exp i\theta))$$

$$\leq W_{\phi}(b_1(\theta), \ldots, b_d(\theta)) + \pi d|\log \varepsilon| + C.$$
Set
\[ \beta = \sup_{\theta \in [0, 2\pi]} W_\theta(b_1(\theta), \ldots, b_d(\theta)) \]
so that:
\[ E_\varepsilon(f_\varepsilon(\exp i\theta)) \leq \pi d |\log \varepsilon| + \beta + C, \]
for some constant \( C \) depending only on \( \Omega, b_1, \ldots, b_d \). Hence \( f_\varepsilon \) is a continuous loop in \( E^a \), for
\[ a = \kappa_\alpha + \chi, \]
provided \( \chi \) is chosen sufficiently large (but independent of \( \varepsilon \)).

**Step 2.** - The loop \( f_\varepsilon \) is not contractible in \( E^a \). We argue by contradiction and we assume that there is some continuous map \( F : \bar{D}^2 \to E^a \) such that
\[ F|_{\partial D^2} = f_\varepsilon. \]
We are going to show that this leads to a contradiction. For that purpose consider the map \( \omega : \Omega' \to \Omega \) defined by:
\[ \omega(x) = \hat{\Phi}_\mu(F(\theta)) = \hat{\Phi}_\mu \circ F(\theta), \quad x \in \bar{D}^2. \]
Since \( F \) is continuous and since by Proposition V.1, \( \hat{\Phi}_\mu \) is \( \eta \)-almost continuous, for \( \eta \) given by
\[ \eta = C\left( \varepsilon^{\mu_{a+1}} + \varepsilon^{7|\log \varepsilon|} \right), \]
then \( \omega \) is also \( \eta \)-almost continuous (note that the composition of an \( \eta \)-almost continuous map with a continuous map is \( \eta \)-almost continuous for the same \( \eta \)). We have
\[ \eta \to 0 \quad \text{as} \quad \varepsilon \to 0, \]

since \( C \) is a constant which is independent of \( \varepsilon \). In view of Theorem 6, since \( \omega \) is a map in \( E^a \), with \( u \) satisfying (1.4), for any \( x \in \bar{D}^2 \), the vortices of \( \omega(x) \) have all degree +1, and there are exactly \( d \) of them: let us denote by \( \omega_1(x), \ldots, \omega_d(x) \) these vortices \( (\omega_i(x) \in \Omega, \text{for } i = 1, \ldots, d) \). By property (1.25) we have moreover
\[ |\omega_i(x) - \omega_j(x)| \geq \nu \quad \forall \ i \neq j \quad i, j \in \{1, \ldots, d\} \]
for some constant \( \nu > 0 \). It follows from Proposition IV.4 and (VII.4) that \( \omega \) may be considered as a \( \eta \)-almost continuous map from \( \bar{D}^2 \) to \( \Sigma_\nu \), where
\[ \Sigma_\nu = \{(x_1, \ldots, x_d) \in \Omega^d, \quad |x_i - x_j| \geq \nu, \quad \forall \ i \neq j \}, \]
provided $\varepsilon$ is sufficiently small. We deduce from Proposition IV.3 that, if $\varepsilon$ is sufficiently small, $\omega$ restricted to $\partial D^2$ is homotopic to a constant map (in the sense of Definition IV.1). On the other hand, we have

$$F|_{\partial D^2} = f_{\varepsilon}$$

and by the construction of $f_{\varepsilon}$, we have

$$\left| \Phi_n(f_{\varepsilon}(\exp i\theta)) - f(\exp i\theta) \right| \leq \eta$$

that is

$$(\text{VII.6}) \quad |\omega(\exp i\theta) - f(\exp i\theta)| \leq \eta \quad \forall \theta \in [0, 2\pi],$$

where $f$ is the loop in $\Omega^d \setminus \Delta$ constructed in Proposition I.1. By Proposition 1, $f$ is not contractible in $\Omega^d \setminus \Delta$, and therefore not homotopic to a constant map. Hence if $\varepsilon$ is sufficiently small, (VII.6) expresses the fact that $\omega$ restricted to $\partial D^2$ is not homotopic to a constant map: a contradiction, and the Theorem is proved.

VIII. Proof of Theorem 1

Recall that $E_\varepsilon$ is a $C^1$-functional on $H^1_\#(\Omega; \mathbb{R}^2)$ which satisfies the Palais-Smale condition, that is for any sequence $u_n \in H^1_\#(\Omega; \mathbb{R}^2)$ such that $|E_\varepsilon(u_n)| \leq C$ and $dE_\varepsilon(u_n) \rightarrow 0$ in $H^{-1}(\Omega)$, up to a subsequence $u_n$ converges strongly in $H^1_\#(\Omega; \mathbb{R}^2)$ to some map $u$ which is consequently a solution to (1.1) and (1.2).

For a $C^1$-functional $E$ which satisfies the Palais-Smale condition a fundamental principle of the calculus of variation asserts that if $E$ does not have critical values in $[a, b]$, then $E_b$ retracts by deformation on $E_a$ and hence $E_b$ and $E_a$ have similar topology (see for instance Struwe [Str2]). We are going to use this principle for proving Theorem 1. We divide the proof in two steps.

Step 1. – $E_\varepsilon$ has a critical value larger than $\alpha$ (given in Theorem 2).

Assume by contradiction it were false. Then for any $b > a$, $E_b^a$ would retract by deformation on $E_a^a$. Since the space $H^1_\#(\Omega; \mathbb{R}^2)$ is affine, hence contractible, there is a continuous function $F : \bar{D}^2 \rightarrow H^1_\#(\Omega; \mathbb{R}^2)$ such that $F|_{\partial D^2} = f_{\varepsilon}$, the loop construction in Theorem 2. Set

$$b = \sup_{x \in \bar{D}^2} E_\varepsilon(F(x)) + 1 < +\infty$$

so that $F$ is a map from $\bar{D}^2$ to $E^b_\varepsilon$. Let $W : E_a^b \times [0, 1] \rightarrow E^b_\varepsilon$ be a (strong) retraction by deformation of $E^b_\varepsilon$ on $E_a^a$, that is $W$ is continuous and

\begin{align*}
(VII.1) & \quad W(u, 0) = u \quad \forall \ u \in E_a^b, \\
(VII.2) & \quad W(u, 1) \in E_a^a \quad \forall \ u \in E_a^b, \\
(VII.3) & \quad W(u, t) = u \quad \forall \ u \in E_a^a.
\end{align*}
Consider next the map $\tilde{F} : \tilde{D}^2 \to E_\varepsilon^b$ defined by

$$\tilde{F}(x) = W(F(x), 1), \quad \forall \, x \in \tilde{D}^2,$$

so that $\tilde{F}$ is continuous, $\tilde{F}$ takes values in $E_\varepsilon^a$ (by (VII.2)) and

$$\tilde{F}(\exp i\theta) = f_\varepsilon(\exp i\theta), \quad \forall \, \theta \in [0, 2\pi].$$

i.e. $\tilde{F}|_{\partial D^2} = f_\varepsilon$, where we have used (VII.3). Hence $f_\varepsilon$ would be contractible in $E_\varepsilon^a$, a contradiction with Theorem 2. Hence $E_\varepsilon$ has a critical value $c_1 > a$, and hence a second solution $u_1$ to (1.1) (1.2) such that $E_\varepsilon(u_1) = c_1 > a$.

**Step 2. — Construction of a third solution.** We assume by contradiction that we have only two solutions, $u_1$ the unique minimizer and $u_\varepsilon$ constructed in Step 1. We consider level set $E_\varepsilon^b$ for $\mathcal{K}_\varepsilon < b < a$ (for $a = \mathcal{K}_\varepsilon + \chi_1$). We claim that, for any $\delta > 0$, there exists $\delta_0 > 0$ such that:

(VIII.1) $E_\varepsilon^b \subset V_\delta = \{v \in H^1_0(\Omega; \mathbb{R}^2), \|u_\varepsilon - v\| < \delta\},$ if $\varepsilon$ is sufficiently small,

where $b = \chi_\delta + \mathcal{K}_\varepsilon$. Indeed if the claim were false, then we would have a sequence $u_n$, and $\delta_0 > 0$ such that

(VIII.2) $\|u_n - u_\varepsilon\| \geq \delta_0$

and $E_\varepsilon(u_n) \to \mathcal{K}_\varepsilon$. Hence up to a subsequence $u_n$ would converge strongly in $H^1_0(\Omega; \mathbb{R}^2)$ to a minimizer of $F_\varepsilon$, different from $u_\varepsilon$ by (VIII.2) : a contradiction. Next choose $\delta$ so small (depending on $\varepsilon$) such that

(VIII.2) $V_\delta \subset E_\varepsilon^a$

and assume that there are no critical values $c$ in $]\mathcal{K}_\varepsilon, a[$. Then we would construct a retraction of $E_\varepsilon^a$ in $E_\varepsilon^b \subset V_\delta$. Since $V_\delta$ is contractible the same kind of arguments as in Step 1 shows this is contradictory. Hence there is a third critical value $c_2$ in $]\mathcal{K}_\varepsilon, a[$ and the proof is complete.

**IX. On the vortices of the constructed solutions**

In view of Theorem 3, a natural question is to determine the nature of the vortices obtained, in the limit $\varepsilon \to 0$, for the solutions constructed above. In particular, their topological charge $d_i$ is of interest. For the solutions corresponding to the critical value $\mathcal{K}_\varepsilon < c_2 < a$, it is not difficult to verify that the vortices have all charges equal to $+1$, and that their number if consequently equal to $d$. 

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The nature of the vortices of the solution corresponding to the critical value \( c_1 > a \) is more subtle: at this stage, we do not have a definitive answer. The next discussion will however, we expect, clarify the mechanisms at play, and yield some plausible conjectures.

First, note that there is a minimax characterization of \( c_1 \), namely

\[
(1X.1) \quad c_1 = \inf_{F \in \Gamma} \left( \max_{x \in D^2} E_\varepsilon(F(x)) \right),
\]

where

\[
\Gamma = \{ F \in C^0(D^2, H^1_0(\Omega; \mathbb{R}^2)) \mid F|_{\partial D^2} = f_\varepsilon \}.
\]

In order to get some insight into the solutions corresponding to \( c_1 \), we have somehow to figure out what maps in \( \Gamma \) look like (at least those such that \( \max_{x \in \partial D^2} E_\varepsilon(F(x)) \) is not too large). Let us illustrate this in the simple case \( \Omega = D^2 \), and \( d = 2 \). In this case \( f_\varepsilon \) is given for instance by

\[
\hat{f}_\varepsilon(\exp i\theta) = \Phi(f_\varepsilon(\exp i\theta)) = \left\{ (0, +1), \left( \frac{1}{2}\exp i\theta, +1 \right) \right\}.
\]

[Here we represent the vortices of the map \( f_\varepsilon : \) the first coordinate stands for the position of the vortex, whereas the second stands for its charge: once the vortices are known the map \( f_\varepsilon : S^1 \to H^1_\varepsilon(\Omega; \mathbb{R}^2) \) is constructed as in Section VII, Step 1. We will describe two different maps \( F_1 \) and \( F_2 \) which will be described as \( f_\varepsilon \), in terms of their vortices \( \hat{F}_1, \hat{F}_2 \) (constructing these maps can then be performed as above, using the method of Section VII).

a) The deformation \( F_1 \). We describe \( \hat{F}_1 = \Phi \circ F_1 \) in polar coordinates \((r, \theta)\) on \( D^2 \) by:

\[
\hat{F}_1(r, \theta) = \left\{ (0, +1), \left( \frac{r}{2}\exp i\theta, +1 \right) \right\} \quad r \in [0, 1], \theta \in [0, 2\pi].
\]

As \( r \to 0 \), the second vortex comes closer to the one at the origin: at \( r = 0 \), the two vortices glue together to form a vortex of degree \( +2 \). Note that this deformation involves only vortices with positive charge.

b) The deformation \( F_2 \). This deformation involves also negatively charged particles. Let us describe briefly how it is constructed. First, we have to consider the coordinate \( r \) as a deformation parameter of \( f_\varepsilon \) to a constant map. Let \( b \in D^2 \setminus D^2(\frac{1}{2}) \) be an arbitrary (but fixed) point. As \( r = 1 \), we created some dipole at \( b \), that is a pair of a vortex of degree \( +1 \) and a vortex of degree \( -1 \). As \( r \) decreases to \( \frac{1}{2} \) the negatively charged vortex moves (on the line joining the origin to \( b \)) towards the origin and annihilates at \( r = \frac{1}{2} \) the vortex which is located there: during the same time, the vortex at \( b \) stays fixed. Hence at \( r = \frac{1}{2} \) we are left, for any \( \theta \in [0, 2\pi] \) with two positive vortices, which are not “linked”, as for \( \hat{f}_\varepsilon \). We then retract, as \( r \to 0 \), the configuration to a constant map. In analytical form, this writes:

\[
\hat{F}_2(r, \theta) = \left\{ (0, +1), \left( \frac{1}{2}\exp i\theta, +1 \right) \right\} \quad \text{for} \quad \frac{1}{2} \leq r \leq 1, \theta \in [0, 2\pi],
\]

\[
\hat{F}_2(r, \theta) = \left\{ \left( \frac{r}{2}\exp i\theta, +1 \right), (b, +1) \right\} \quad \text{for} \quad 0 \leq r \leq \frac{1}{2}, \theta \in [0, 2\pi].
\]
For both deformations $F_1$ and $F_2$, we verify that

$$\max_{x \in D^2} F_i(x) \leq 4\pi |\log \varepsilon| + C, \quad i = 1, 2,$$

where $C$ is some constant depending only on $g$. Actually we conjecture that, in this case ($d = 2$)

$$|c_1 - 4\pi |\log \varepsilon|| \leq C, \quad 0 < \varepsilon < 1$$

and for arbitrary $d \geq 2$,

$$|c_1 - \pi (d + 2)|\log \varepsilon|| \leq C, \quad 0 < \varepsilon < 1,$$

where $C$ is some constant depending only on $g$.

For $F_1$, one verifies that the maximum value $\beta_1 = \max_{x \in D^2} E_{\varepsilon}(F_1(x))$ is achieved at $r = 0$, a point where $F_1$ has a vortex at the origin of degree $+2$. For $F_2$ the situation is quite different and the maximum value $\beta_2 = \max_{x \in D^2} E_{\varepsilon}(F_2(x))$ is achieved at a point where $F$ has three vortices of charge $+1$ and a vortex of charge $-1$. Hence we are led to conjecture:

**Conjecture.** – For solutions $v_{\varepsilon}$ corresponding to $c_1$ one of the two possibilities occur

a) $v_{\varepsilon}$ has a vortex of degree $+2$,

b) $v_{\varepsilon}$ has three vortices of degree $+1$, and one vortex of degree $-1$.

At this point, we have not a clear idea of which of the two possibilities will actually occur. It is also possible that solutions corresponding to each of them exist. In the special case $g = \exp 2i\theta$, we think however that the solution corresponding to $c_1$ is the radially symmetric one, i.e. of the form $v_{\varepsilon} = f(r)\exp 2i\theta$, and hence has a vortex of degree $+2$.

To conclude this section, let us finally show how similar methods should yield solutions of higher energies (and higher Morse Index). Let $\Omega = D^2(2)$, and for instance $d = 3$. We consider the sphere $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3, \ |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$. Let $f_{\varepsilon} : S^5 \to H^1_g(\Omega; \mathbb{R}^2)$ be a map whose vortices are given by:

$$f_{\varepsilon}(z_1, z_2, z_3) = \{(z_1, +1), (z_2, +1), (z_3, +1)\}, \quad \forall (z_1, z_2, z_3) \in S^5,$$

and

$$\Gamma = \{F \in C^0(B^6; H^1_g(\Omega; \mathbb{R}^2)), \quad F|_{\partial B^6} = f_{\varepsilon}\}.$$

Consider

$$c_3 = \inf_{F \in \Gamma} \left(\max_{x \in B^6} E_{\varepsilon}(F(x))\right).$$

We believe that $c_3$ is a critical value of $E_{\varepsilon}$ and using similar constructions as above that the corresponding solution has a vortex of degree $+2$, two vortices of degree $+1$, and one vortex of degree $-1$. 

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X. Final remarks and comments

The argument of the proof of Theorem 1 was based on the close relationship between the level of \( E_e \) and the level sets of \( W_0 \) on \( W \). Actually here we only have considered level sets of small energy (i.e. a satisfying (1.4)). We expect to find more critical values for \( E_e \) for higher energy values. This requires to have a better understanding of suitable subsets in \( W \) (note that \( W \) itself has a trivial topology, since \( \Omega \) is assumed to be simply connected): one important difficulty, in our opinion, is that we have to deal with positive and negative particles and the possibility of cancellation: this difficulty was removed here thanks to Theorem 6. For solutions with higher energy, we however think (as shown in Section IX) that configurations with particles of opposite signs are central to the problem, raising interesting questions, both of topological and analytical nature.

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TOPOLOGICAL METHODS FOR THE GINZBURG-LANDAU EQUATIONS


(Manuscript received February, 1996.)

L. ALMEIDA
CMLA, ENS de Cachan, 61 avenue du Président Wilson, URA CNRS 1611, 94235 Cachan, France

F. BETHUEL
Université de Paris-Sud, Analyse Numérique et EDP, URA CNRS 760, Bâtiment 425, 91405 Orsay, France

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