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An Approximation Technique for Nonlinear Integral Operations*

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Approximation results for J. S. Mac Nerney's theory of nonlinear integral operations are established. For the nonlinear product integral ${}_x\prod^y (1 + V)P$, approximations of the form $\prod_{i=1}^n [1 + L_q(x_{i-1}, x_i)]P$ are considered, where $L_1(u, v)P = \int_u^v VP$ and $L_q(u, v)P = \int_u^v V(r, s)[1 + L_{q-1}(s, v)]P$ for $q = 2, 3,\ldots$. Error bounds are obtained for the difference between the product integral and the preceding product.

In a recent paper [3], the authors develop approximation techniques for Riemann product integrals. The purpose of this note is to present some related results for Mac Nerney's [13] theory of nonlinear integral operations.

The letters S, G, and H denote a nondegenerate, linearly ordered set, a normed complete Abelian group with zero element 0, and the set of functions from G to G that map 0 into 0, respectively. Further, $\mathcal{OA}$ denotes the set such that V is in $\mathcal{OA}$ if, and only if, V is an additive function from $S \times S$ to H and there exists an additive function α from $S \times S$ to the nonnegative numbers such that, if $\{x, y, P, Q\}$ is in $S \times S \times G \times G$, then

$$\| V(x, y) P - V(x, y) Q \| \leqslant \alpha(x, y) \| P - Q \| .$$

For convenience, α is referred to as the Lipschitz function for V. If V is in $\mathcal{OA}$ and $\{x, y, P\}$ is in $S \times S \times G$, then the product integral ${}_x\prod^y (1 + V) P$ exists and is equal to

$$P + \int_x^y V(r, s) \left[\prod_s^y (1 + V) P \right].$$

The statement that ${}_x\prod^y (1 + V) P$ exists means there exists an element L of G such that, if $\epsilon > 0$, then there exists a subdivision D of $\{x, y\}$ such that, if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left\| L - \prod_{i=1}^n [1 + V(x_{i-1}, x_i)] P \right\| < \epsilon.$$

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All definitions are the same as those used by Mac Nerney [13]. The reader is referred to this reference for a complete presentation of the definitions and notation used in this paper. Additional studies involving nonlinear operations are presented by Neuberger [7], Herod [4–7], Reneke [16–17], Webb [18–20], Lovelady [9–12], Martin [14], Gibson [1], Kay [8], and Helton [2].

In the following, a technique for approximating the product integral ${}_x\prod^y (1 + V) P$ is presented, where V is in $\mathcal{OA}$ with Lipschitz function α. With respect to such a function V, let

$$L_1(x, y) P = \int_x^y VP, \qquad L_q(x, y) P = \int_x^y V(r, s) [1 + L_{q-1}(s, y)] P,$$

$$I_1(x, y) = \int_x^y \alpha, \qquad \text{and} \qquad I_q(x, y) = \int_x^y \alpha(r, s) I_{q-1}(s, y)$$

for $\{x, y, P\}$ in $S \times S \times G$ and $q = 2, 3, \ldots$. The following approximation result is established. If $\{x, y, P\}$ is in $S \times S \times G$, q is a positive integer, and $\{x_i\}_{i=0}^n$ is a subdivision of $\{x, y\}$, then

$$\left\| \prod_x^y (1 + V) P - \prod_{i=1}^n [1 + L_q(x_{i-1}, x_i)] P \right\|$$

$$\leqslant \left\{ \prod_a^b (1 + \alpha) - \prod_{i=1}^n \left[1 + \sum_{j=1}^q I_j(x_{i-1}, x_i) \right] \right\} \| P \| .$$

Two other closely related results are also established.

In each of the three results, the product integral ${}_x\prod^y (1 + V) P$ is approximated by products involving the iterated integrals $L_q(x_{i-1}, x_i)$. These products converge to ${}_x\prod^y (1 + V) P$ more rapidly than do products involving only $V(x_{i-1}, x_i)$. Hence, by considering products of the form

$$\prod_{i=1}^n [1 + L_q(x_{i-1}, x_i)] P$$

rather than products of the form

$$\prod_{i=1}^n [1 + V(x_{i-1}, x_i)] P,$$

one is able to increase the rate at which the product under consideration converges to ${}_x\prod^y(1 + V) P$.

The approximation results are now presented. Five lemmas are needed.

LEMMA 1. *Suppose* $\{A_i\}_{i=1}^n$ *is a sequence with values in* H *and* $\{a_i\}_{i=1}^n$ *is a sequence of nonnegative numbers such that, if* $\{P, Q\}$ *is in* $G \times G$ *and* $1 \leqslant i \leqslant n$, *then*

$$\| A_i P - A_i Q \| \leqslant a_i \| P - Q \| .$$

Then, in conclusion, if P *is in* G, *the following inequalities hold:*

$$\text{(i)} \qquad \left\| \prod_{i=1}^{n} (1 + A_i) P \right\| \leqslant \left[\prod_{i=1}^{n} (1 + a_i) \right] \| P \|$$

and

$$\text{(ii)} \qquad \left\| \left[\prod_{i=1}^{n} (1 + A_i) \right] P - \left[1 + \sum_{i=1}^{n} A_i \right] P \right\|$$
$$\leqslant \left[\prod_{i=1}^{n} (1 + a_i) - \left(1 + \sum_{i=1}^{n} a_i \right) \right] \| P \| .$$

Proof. This lemma can be established by induction. The proof of conclusion (i) is straightforward and Mac Nerney [13, Lemma 1.1(ii), p. 623] presents a proof for conclusion (ii).

LEMMA 2. *Suppose each of* $\{A_i\}_{i=1}^n$ *and* $\{B_i\}_{i=1}^n$ *is a sequence of functions in* H *and each of* $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$ *is a sequence of nonnegative numbers such that, if* $\{P, Q\}$ *is in* $G \times G$ *and* $1 \leqslant i \leqslant n$, *then*

$$\| A_i P - A_i Q \| \leqslant a_i \| P - Q \| , \qquad \| B_i P \| \leqslant b_i \| P \| ,$$

and

$$\| B_i P - A_i P \| \leqslant (b_i - a_i) \| P \| .$$

Then, in conclusion, if $\{P, Q\}$ *is in* $G \times G$, *the following inequality holds:*

$$\left\| \left(\prod_{i=1}^{n} B_i \right) P - \left(\prod_{i=1}^{n} A_i \right) Q \right\| \leqslant \left[\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i \right] \| P \| + \left[\prod_{i=1}^{n} a_i \right] \| P - Q \| .$$

Proof. This lemma is established by Mac Nerney [13, Lemma 1.2, p. 623]. Mac Nerney states the lemma with $P = Q$ but establishes it in the form given here.

LEMMA 3. *If* V *is in* $\mathcal{O}\mathcal{A}$ *with Lipschitz function* α, q *is a positive integer and* $\{x, y, P, Q\}$ *is in* $S \times S \times G \times G$, *then*

$$\| [1 + L_q(x, y)] P - [1 + L_q(x, y)] Q \| \leqslant \left[1 + \sum_{j=1}^{q} I_j(x, y) \right] \| P - Q \| .$$

Proof. This lemma is established by induction. It follows immediately that the desired inequality is true for $q = 1$. Now, the inequality is assumed to true for q and established for $q + 1$. Suppose $\{x, y, P, Q\}$ is in $S \times S \times G \times G$. Then,

$$\begin{aligned}
&\|[1 + L_{q+1}(x, y)]\, P - [1 + L_{q+1}(x, y)]\, Q\,\| \\
&\quad\leqslant \| P - Q \| + \left\| \int_x^y V(r, s)\, [1 + L_q(s, y)]\, P - \int_x^y V(r, s)\, [1 + L_q(s, y)]\, Q \right\| \\
&\quad\leqslant \| P - Q \| + \int_x^y \alpha(r, s)\, \|[1 + L_q(s, y)]\, P - [1 + L_q(s, y)]\, Q\,\| \\
&\quad\leqslant \| P - Q \| + \int_x^y \alpha(r, s) \left[1 + \sum_{j=1}^{q} I_j(s, y)\right] \| P - Q \|
\end{aligned}$$

[From induction hypothesis]

$$= \left[1 + \sum_{j=1}^{q+1} I_j(x, y)\right] \| P - Q \| .$$

Thus, the inequality is established for $q + 1$. This completes the proof of Lemma 3.

LEMMA 4. *If V is in $\mathcal{OA}$ with Lipschitz function α and $\{x, y, P\}$ is in $S \times S \times G$, then*

$$\left\| {}_x\prod^y (1 + V)\, P \right\| \leqslant \left[{}_x\prod^y (1 + \alpha) \right] \| P \| .$$

Proof. This lemma follows as a corollary to Lemma 1(i).

LEMMA 5. *If V is in $\mathcal{OA}$ with Lipschitz function α, q is a positive integer, and $\{x, y, P\}$ is in $S \times S \times G$, then*

$$\begin{aligned}
&\left\| {}_x\prod^y (1 + V)\, P - [1 + L_q(x, y)]\, P \right\| \\
&\qquad \leqslant \left\{ {}_x\prod^y (1 + \alpha) - \left[1 + \sum_{j=1}^{q} I_j(x, y)\right] \right\} \| P \| .
\end{aligned}$$

Proof. This lemma is established by induction. The desired inequality can be established for $q = 1$ by using Lemma 1(ii). Now, the inequality is assumed to be true for q and established for $q+1$. Suppose $\{x, y, P\}$ is in $S \times S \times G$. Then,

$$\left\| {}_x\prod^y (1+V)\,P - [1 + L_{q+1}(x, y)]\,P \right\|$$

$$= \left\| \int_x^y V(r, s) \left[{}_s\prod^y (1+V)\,P \right] - \int_x^y V(r, s)\,[1 + L_q(s, y)]\,P \right\|$$

$$\leqslant \int_x^y \alpha(r, s) \left\| \left[{}_s\prod^y (1+V)\,P \right] - [1 + L_q(s, y)]\,P \right\|$$

$$\leqslant \int_x^y \alpha(r, s) \left\{ {}_s\prod^y (1+\alpha) - \left[1 + \sum_{j=1}^{q} I_j(s, y) \right] \right\} \| P \|$$

[From induction hypothesis]

$$= \left\{ {}_x\prod^y (1+\alpha) - \left[1 + \sum_{j=1}^{q+1} I_j(x, y) \right] \right\} \| P \| .$$

Thus, the inequality is established for $q+1$. This completes the proof of Lemma 5.

The approximation results now follow.

THEOREM 1. *If V is in $\mathcal{OA}$ with Lipschitz function α, q is a positive integer, $\{x, y, P\}$ is in $S \times S \times G$, and $\{x_i\}_{i=0}^n$ is a subdivision of $\{x, y\}$, then*

$$\left\| {}_x\prod^y (1+V)\,P - \prod_{i=1}^{n} [1 + L_q(x_{i-1}, x_i)]\,P \right\|$$

$$\leqslant \left\{ {}_x\prod^y (1+\alpha) - \prod_{i=1}^{n} \left[1 + \sum_{j=1}^{q} I_j(x_{i-1}, x_i) \right] \right\} \| P \| .$$

Proof. This theorem is established as a corollary to Lemma 2. For $i = 1, 2, \ldots, n$, let

$$A_i = 1 + L_q(x_{i-1}, x_i), \qquad B_i = {}_{x_{i-1}}\prod^{x_i} (1+V),$$

$$a_i = 1 + \sum_{j=1}^{q} I_j(x_{i-1}, x_i), \qquad \text{and} \qquad b_i = {}_{x_{i-1}}\prod^{x_i} (1+\alpha).$$

With the preceding definitions, it follows from Lemmas 3, 4, and 5 that the hypothesis of Lemma 2 is satisfied. Thus, Lemma 2 produces the desired inequality. This completes the proof of Theorem 1.

THEOREM 2. *If V is in $\mathcal{OA}$ with Lipschitz function α, q is a positive integer, $\{x, y, P\}$ is in $S \times S \times G$, and $\{x_i\}_{i=0}^n$ is a subdivision of $\{x, y\}$, then*

$$\left\| \prod_x^y (1 + V) P - \prod_{i=1}^n [1 + L_q(x_{i-1}, x_i)] P \right\|$$

$$\leqslant \sum_{i=1}^n \left\{ \prod_{k=1}^{i-1} \left[1 + \sum_{j=1}^q I_j(x_{k-1}, x_k) \right] \right\} \left\{ \sum_{j=q+1}^\infty I_j(x_{i-1}, x_i) \right\} \left\{ \prod_{x_i}^y (1 + \alpha) \right\} \| P \| .$$

Proof. If each of $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ is a sequence of numbers, then

$$\prod_{i=1}^n b_i - \prod_{i=1}^n a_i = \sum_{i=1}^n \left[\prod_{k=1}^{i-1} a_k \right] [b_i - a_i] \left[\prod_{k=i+1}^n b_k \right].$$

This identity can be established by induction. Further, if $\{r, s\}$ is in $S \times S$, then

$$\prod_r^s (1 + \alpha) = 1 + \sum_{j=1}^\infty I_j(r, s).$$

This identity is the Peano series representation for a product integral. The theorem now follows as a corollary to Theorem 1 by using the two preceding equalities. This completes the proof of Theorem 2.

THEOREM 3. *Suppose S is the set of real numbers, K is a positive constant, V is in $\mathcal{OA}$ with Lipschitz function α, q is a positive integer, and $\{x, y\}$ is an element of $S \times S$ such that, if $\{x, u, v, y\}$ is a subdivision of $\{x, y\}$, then*

$$\alpha(u, v) \leqslant K \mid v - u \mid .$$

Then, in conclusion, there exists a positive constant B such that, if P is in G, n is a positive integer, $h = (y - x)/n$ and $x_i = x + hi$ for $i = 0, 1, \ldots, n$, then

$$\left\| \prod_x^y (1 + V) P - \prod_{i=1}^n [1 + L_q(x_{i-1}, x_i)] P \right\| \leqslant | h |^q B \| P \| .$$

Proof. Let B_1 denote a number such that, if $\{x_i\}_{i=0}^n$ is a subdivision of $\{x, y\}$ and $1 \leqslant i \leqslant n$, then

$$\prod_{x_i}^y (1 + \alpha) < B_1 \qquad \text{and} \qquad \prod_{k=1}^{i-1} \left[1 + \sum_{j=1}^q I_j(x_{k-1}, x_k) \right] < B_1 .$$

Further, let B_2 denote a number such that, if $\{x_i\}_{i=0}^n$ is a subdivision of $\{x, y\}$, then

$$\sum_{i=1}^n \sum_{j=q+1}^\infty K^j \mid x_i - x_{i-1} \mid^{j-q} / j! < B_2 .$$

Let $B = B_1{}^2 B_2$.

Suppose P is in G, n is a positive integer, $h = (y - x)/n$ and $x_i = x + hi$ for $i = 0, 1, \ldots, n$. It can be established by induction that

$$I_j(x_{i-1}, x_i) \leqslant K^j \mid x_i - x_{i-1} \mid^j / j!$$

for $j = 1, 2, \ldots$. Hence,

$$\left\| {}_x\prod^y (1 + V) P - \prod_{i=1}^{n} [1 + L_q(x_{i-1}, x_i)] P \right\|$$

$$\leqslant \sum_{i=1}^{n} \left\{ \prod_{k=1}^{i-1} \left[1 + \sum_{j=1}^{q} I_j(x_{k-1}, x_k) \right] \right\} \left\{ \sum_{j=q+1}^{\infty} I_j(x_{i-1}, x_i) \right\} \left\{ {}_{x_i}\prod^y (1 + \alpha) \right\} \| P \|$$

[From Theorem 2]

$$\leqslant B_1^2 \left[\sum_{i=1}^{n} \sum_{j=q+1}^{\infty} K^j \mid x_i - x_{i-1} \mid^j / j! \right] \| P \|$$

$$= \mid h \mid^q B_1^2 \left[\sum_{i=1}^{n} \sum_{j=q+1}^{\infty} K^j \mid x_i - x_{i-1} \mid^{j-q} / j! \right] \| P \|$$

$$\leqslant \mid h \mid^q B_1^2 B_2 \| P \|$$

$$= \mid h \mid^q B \| P \| .$$

This completes the proof of Theorem 3.

Remark. In the preceding, we have restricted our study to interval functions in H. This restriction can be relaxed by considering a new class H^* of functions such that, if V is a function from $S \times S$ to the set of functions mapping G into G, then V is in H^* if, and only if, there exists an additive function γ from $S \times S$ to the nonnegative numbers such that, if $\{x, y\}$ is in $S \times S$, then

$$\| V(x, y)\, 0 \| \leqslant \gamma(x, y).$$

Then, our results can be extended to functions in H^* by suitably defining V on the direct product of G and the group of order 2. This technique is given by Mac Nerney [13, Remark, p. 637].

Two examples are now presented.

EXAMPLE 1. Let R^n denote the set of n-dimensional real vectors. Further, suppose $[a, b]$ is a number interval and $f(x, z)$ is a function from $[a, b] \times R^n$ to R^n which is continuous in x and satisfies the Lipschitz condition

$$\| f(x, z) - f(x, w) \| \leqslant B \| z - w \|$$

for x in $[a, b]$ and z, w in R^n. If V is defined by

$$V(u, v)\, z = \int_u^v -f(x, z)\, dx$$

for u, v in $[a, b]$ and z in R^n, then V is in $\mathcal{OA}$ on $[a, b]$ with respect to R^n. The Lipschitz function α for V is defined by $\alpha(u, v) = B \mid v - u \mid$. If V does not map 0 into 0, then it is necessary to use the embedding technique referred to in the remark.

The initial value problem

$$y(a) = c, \qquad dy/dx = f(x, y) \qquad \text{for } a \leqslant x \leqslant b \tag{1}$$

can be reformulated as

$$\begin{aligned} y(x) &= c + \int_a^x f[t, y(t)]\, dt \\ &= c + \int_x^a -f[t, y(t)]\, dt \\ &= c + \int_x^a V(u, v)\, y(v). \end{aligned}$$

Thus, $y(x) = {}_x\prod^a (1 + V)\, c$ is the solution of the initial value problem in (1). Hence, the results presented in this paper provide a means of approximating the solutions of initial value problems.

For this example, $L_q(u, v)$ is a function from R^n to R^n such that $[L_q(u, v)]\,(z)$ is an iterated integral of the function $f(x, z)$. In the case $q = 3$, this integral is given by

$$\begin{aligned} [L_3(u, v)]\,(z) &= \left[\int_v^u f \left\{t, \_ + \int_v^t f \left[s, \_ + \int_v^s f(r, \_)\, dr\right] ds\right\} dt\right](z) \\ &= \int_v^u f \left\{t, z + \int_v^t f \left[s, z + \int_v^s f(r, z)\, dr\right] ds\right\} dt \end{aligned}$$

for u, v in $[a, b]$ and z in R^n.

EXAMPLE 2. Let R^n and $[a, b]$ be defined as in Example 1. Further, suppose $f(x, t, z)$ is a function from $[a, b] \times [a, b] \times R^n$ to R^n which is continuous in each of x and t and satisfies the Lipschitz condition

$$\| f(x, t, z) - f(x, t, w) \| \leqslant B \| z - w \|$$

for x, t in $[a, b]$ and z, w in R^n. Let G denote the set of all continuous functions from $[a, b]$ to R^n and, for g in G, let

$$\| g \|_G = lub \| g(x) \|, \qquad a \leqslant x \leqslant b.$$

If V is defined by

$$[V(u, v)\, g]\,(x) = \left[\int_u^v -f\{\_\,, t, g(t)\}\, dt\right](x)$$
$$= \int_u^v -f\{x, t, g(t)\}\, dt$$

for u, v in $[a, b]$ and g in G, then V is in $\mathcal{O}\mathscr{A}$ on $[a, b]$ with respect to G. That is, $V(u, v)$ maps a continuous function into a continuous function. The Lipschitz function α for V is defined by $\alpha(u, v) = B \mid v - u \mid$. If V does not map 0 into 0, then it is necessary to use the embedding technique referred to in the remark.

The solution of the Volterra integral equation

$$y(x) = g(x) + \int_a^x f\{x, t, y(t)\}\, dt \qquad \text{for } a \leqslant x \leqslant b \tag{2}$$

can be obtained from an integral equation in which the unknown is a function from $[a, b]$ to G. This equation is

$$h(x) = g + \int_a^x f\{\_\,, t, [h(t)]\,(t)\}\, dt$$
$$= g + \int_x^a -f\{\_\,, t, [h(t)]\,(t)\}\, dt$$
$$= g + \int_x^a V(u, v)\, h(v).$$

The preceding integral equation has

$$h(x) = \prod_x^a (1 + V)\, g$$

as its solution. The function h maps $[a, b]$ into G, and the function y defined by

$$y(x) = [h(x)]\,(x)$$
$$= \left[\prod_x^a (1 + V)\, g\right](x)$$
$$= \left[g + \int_a^x f\{\_\,, t, [h(t)]\,(t)\}\, dt\right](x)$$
$$= g(x) + \int_a^x f\{x, t, y(t)\}\, dt$$

maps $[a, b]$ into R^n and is the solution of the integral equation in (2). Hence, the results presented in this paper provide a means of approximating the solutions of Volterra integral equations. This example is due to Reneke [16].

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