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Semiclassical orthogonal polynomials in two variables☆

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Abstract

In this work, semiclassical orthogonal polynomials in two variables are defined as the orthogonal polynomials associated with a quasi definite linear functional satisfying a matrix Pearson-type differential equation. Semiclassical functionals are characterized by means of the analogue of the structure relation in one variable. Moreover, non trivial examples of semiclassical orthogonal polynomials in two variables are given.

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1. Introduction

Classical orthogonal polynomials in one variable (Hermite, Laguerre, Jacobi and Bessel) can be defined as the only sequences of polynomials which are orthogonal with respect to a linear functional u satisfying the Pearson differential equation

$$D(\phi u) = \psi u, \tag{1}$$

where ϕ and ψ are fixed polynomials with $\deg \phi \leqslant 2$, and $\deg \psi = 1$.

In 1985, Hendriksen and van Rossum [4], extended these ideas introducing a new class of orthogonal polynomials. In fact, these authors studied orthogonal polynomials associated with linear functionals satisfying Eq. (1), with no restrictions in the degrees of the polynomials ϕ, and ψ.

Obviously, orthogonal polynomials defined as above, generalize in a natural way the classical ones. They were called *semiclassical* orthogonal polynomials. Later, Maroni in [13] showed that semiclassical functionals can be characterized as quasi definite functionals, with an orthogonal polynomial sequence $\{P_n\}_n$ satisfying

$$\phi P_n' = \sum_{j=n-s-1}^{n+p-1} c_{n,j} P_j,$$

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where ϕ is a fixed polynomial, s and p are integers (in fact, ϕ is the same polynomial as in Eq. (1)). This property is the so-called *Structure relation*.

Classical orthogonal polynomials in two variables constitute a very old subject in the literature. Usually, they are studied as the polynomial eigenfunctions of a partial differential operator. The classification of the classical orthogonal polynomials in two variables was started in a pioneering paper by Krall and Sheffer [11]. Later, several authors (Kim, et al. [6], Koornwinder [8], Littlejohn [12], and Suetin [14], among others), made some contributions in this direction.

Our point of view in [1,3] provides a wider perspective in the subject. We consider as *classical* every quasi definite linear moment functional u satisfying the matrix Pearson differential equation

$$\mathrm{div}(\Phi u) = \Psi^{t} u,$$

where Φ is a symmetric 2×2 polynomial matrix in two variables of total degree not greater than 2, and Ψ is a 2×1 polynomial vector in two variables of total degree 1. A restricted version of this kind of relation appears as a consequence of the partial differential equation in the above-mentioned papers.

Our matrix notation for the Pearson equation was inspired by the vector representation for orthogonal polynomials in several variables introduced by Kowalski in [9,10], and developed by Xu in [15].

In this work, using this matrix formalism, we introduce the concept of semiclassical linear functional in two variables, and we prove the bivariate analogues of the Structure relation. Of course, as the reader can check, all the results in this paper can be easily extended to the multivariate case.

The structure of the paper is as follows. Section 2 is devoted to collect definitions and basic results, essential for the rest of the paper. In the next section, the definition of the semiclassical linear functional is given. The Structure relation is proved in Section 4, and finally, in the last section, we analyze some nontrivial examples of semiclassical orthogonal polynomials. In particular, we prove that the *examples of two–variables analogues of the Jacobi polynomials* given by Koornwinder in [8] are semiclassical orthogonal polynomials in two variables, according to our definition.

In [8], those orthogonal polynomials are called classical since they are eigenfunctions of two commuting and algebraically independent partial differential operators. The definition of classical orthogonal polynomials in two variables considered in this paper provides a different perspective on the subject.

2. Preliminaries

Let $\mathscr{P}_n$, $n \geqslant 0$, denote the linear space of real polynomials in two variables of total degree not greater than n, and let $\mathscr{P} = \bigcup_{n\geqslant 0} \mathscr{P}_n$. Let $\mathscr{M}_{h\times k}(\mathbb{R})$ denote the linear space of $h \times k$ real matrices and $\mathscr{M}_h(\mathbb{R})$ the real space of square matrices. The linear spaces of polynomial matrices will be denoted by $\mathscr{M}_{h\times k}(\mathscr{P})$ and $\mathscr{M}_h(\mathscr{P})$. In addition, we denote by I_n the identity matrix of order n.

If $M = (m_{i,j}(x, y))_{i,j=1}^{h,k} \in \mathscr{M}_{h\times k}(\mathscr{P})$ is a polynomial matrix, we define the *degree of* M as $\deg M = \max\{\deg m_{i,j}(x, y), 1 \leqslant i \leqslant h, 1 \leqslant j \leqslant k\} \geqslant 0$.

Let $\{\mu_{h,k}\}_{h,k\geqslant 0}$ be a doubly indexed sequence of real numbers and let u be the linear functional defined on $\mathscr{P}$ by

$$\langle u, x^h y^k \rangle = \mu_{h,k}, \quad h, k = 0, 1, \ldots,$$

then u is called a *moment functional* defined by $\{\mu_{h,k}\}_{h,k\geqslant 0}$ (see [2, p. 64]).

We will say that $f \in \mathscr{P}_n$ is an *orthogonal polynomial* with respect to u if

$$\langle u, fg \rangle = 0, \quad \forall g \in \mathscr{P}, \quad \deg g < \deg f.$$

In this way, we can define the linear space

$$\mathscr{V}_n = \{f \in \mathscr{P}_n : \langle u, fg \rangle = 0, \forall g \in \mathscr{P}_{n-1}\}.$$

A linear functional u will be said *quasi definite* if $\dim \mathscr{V}_n = n + 1, \quad \forall n \geqslant 0$.

The action of u over a polynomial matrix $M = (m_{i,j}(x, y))_{i,j=1}^{h,k} \in \mathscr{M}_{h\times k}(\mathscr{P})$, (see [7,15]) is defined in the following way,

$$\langle u, M \rangle = (\langle u, m_{i,j}(x, y) \rangle)_{i,j=1}^{h,k} \in \mathscr{M}_{h\times k}(\mathbb{R}),$$

and the *left multiplication* of u by the polynomial matrix M is given by

$$\langle Mu, f\rangle = \langle u, M^{\mathrm{t}} f\rangle, \quad \forall f \in \mathscr{P}.$$

Definition 1. A *polynomial system* (PS) is a sequence of vector polynomials $\{\mathbb{P}_n\}_{n\geqslant 0}$, defined by

$$\mathbb{P}_n = (P_{n,0}, P_{n-1,1}, \dots, P_{0,n})^{\mathrm{t}} \in \mathscr{M}_{(n+1)\times 1}(\mathscr{P}_n),$$

where $\{P_{n,0}, P_{n-1,1}, \dots, P_{0,n}\}$, are $n+1$ polynomials of total degree n, independent modulus $\mathscr{P}_{n-1}$, that is, for each $p \in \mathscr{P}_n$ there is a unique expansion

$$p = \sum_{j=0}^{n} c_j P_{n-j,j} + p_0 \quad \text{with } c_j \in \mathbb{R} \text{ and } p_0 \in \mathscr{P}_{n-1}.$$

Moreover, if a PS $\{\mathbb{P}_n\}_{n\geqslant 0}$ satisfies

$$\langle u, \mathbb{P}_n \mathbb{P}_m^{\mathrm{t}}\rangle = H_n \delta_{n,m}, \quad n, m \geqslant 0,$$

where $H_n \in \mathscr{M}_{n+1}(\mathbb{R})$ is a symmetric and non–singular matrix, then $\{\mathbb{P}_n\}_{n\geqslant 0}$ is called a *weak orthogonal polynomial system* (WOPS) associated with u.

In addition, $\{\mathbb{P}_n\}_{n\geqslant 0}$ is an *orthogonal polynomial system* (OPS) if H_n is diagonal, and an *orthonormal polynomial system* if $H_n = I_{n+1}$.

Observe that, a linear functional u is *quasi definite* if and only if there exists a WOPS with respect to u [6, Proposition 2.1].

If every polynomial $P_{h,k}(x, y)$, for $h + k = n$, contains only one higher degree term, that is,

$$P_{h,k}(x, y) = x^h y^k + R(x, y), \quad R(x, y) \in \mathscr{P}_{n-1},$$

then $\{\mathbb{P}_n\}_{n\geqslant 0}$ is called a *monic* WOPS.

In this paper, we will use the standard differential operators in two variables, the *gradient operator*, ∇, and the *divergence operator*, div. These operators can be extended to matrices $M, M_1, M_2 \in \mathscr{M}_{h\times k}(\mathscr{P})$ as follows:

$$\nabla M = \begin{pmatrix} \partial_x M \\ \partial_y M \end{pmatrix} \in \mathscr{M}_{2h\times k}(\mathscr{P}), \quad \operatorname{div}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \partial_x M_1 + \partial_y M_2 \in \mathscr{M}_{h\times k}(\mathscr{P}).$$

We will use the same symbols to denote the dual of these operators and thus, we define the *distributional gradient and divergence operators* as follows:

$$\left\langle \nabla u, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle = -\left\langle u, \operatorname{div}\begin{pmatrix} f \\ g \end{pmatrix}\right\rangle = -\langle u, \partial_x f + \partial_y g\rangle, \quad \forall f, g \in \mathscr{P},$$

$$\left\langle \operatorname{div}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, f\right\rangle = -\left\langle \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \nabla f\right\rangle = -(\langle u_1, \partial_x f\rangle + \langle u_2, \partial_y f\rangle), \quad \forall f \in \mathscr{P}.$$

3. Semiclassical orthogonal polynomials in two variables

Definition 2. Let u be a quasi definite linear functional, and let

$$\Phi = \begin{pmatrix} \phi_{1,1} & \phi_{1,2} \\ \phi_{2,1} & \phi_{2,2} \end{pmatrix} \in \mathscr{M}_2(\mathscr{P}) \quad \text{and} \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathscr{M}_{2\times 1}(\mathscr{P}),$$

be polynomial matrices with Φ symmetric ($\phi_{1,2} = \phi_{2,1}$), and $\deg \Phi = p \geqslant 0$, $\deg \Psi = q \geqslant 1$. Define $s = \max\{p-2, q-1\}$.

We say that u is *s-Pearson* if it satisfies the matrix Pearson–type differential equation

$$\operatorname{div}(\Phi u) = \Psi^{t} u, \tag{2}$$

and $\det\langle u, \Phi\rangle \neq 0$.

The above definition means that, for all polynomial $f \in \mathscr{P}$, then

$$\langle \operatorname{div}(\Phi u), f\rangle = \langle \Psi^{t} u, f\rangle,$$

that is,

$$\langle u, \Phi\nabla f + \Psi f\rangle = 0 \iff \begin{cases} \langle u, \phi_{1,1}\partial_x f + \phi_{1,2}\partial_y f + \psi_1 f\rangle = 0, \\ \langle u, \phi_{2,1}\partial_x f + \phi_{2,2}\partial_y f + \psi_2 f\rangle = 0. \end{cases}$$

The natural extension of this property to matrices with h rows involves the Kronecker product $\otimes$ (see, for instance, [5]). Therefore, matrix equation (2) can be extended as follows:

$$\operatorname{div}((\Phi \otimes I_h)u) = (\Psi^{t} \otimes I_h)u, \quad h \geqslant 1. \tag{3}$$

Expression (3) means that, for all $M \in \mathscr{M}_{h\times k}(\mathscr{P})$, we get

$$\langle \operatorname{div}((\Phi \otimes I_h)u), M\rangle = \langle (\Psi^{t} \otimes I_h)u, M\rangle.$$

In particular, if we consider a monic PS $\{\mathbb{P}_n\}_{n\geqslant 0}$, the above relation can be written as follows:

$$\langle u, (\Phi \otimes I_{n+1})\nabla\mathbb{P}_n + (\Psi \otimes I_{n+1})\mathbb{P}_n\rangle = 0, \quad n \geqslant 0.$$

The expression $(\Phi \otimes I_{n+1})\nabla\mathbb{P}_n + (\Psi \otimes I_{n+1})\mathbb{P}_n$ is a $2(n+1)\times 1$ polynomial matrix of degree at most $n+s+1$. So, there exist matrices $\Omega_{n+s+1} \in \mathscr{M}_{2(n+1)\times(n+s+2)}(\mathbb{R})$ and $A_m^n \in \mathscr{M}_{2(n+1)\times(m+1)}(\mathbb{R})$ such that

$$(\Phi \otimes I_{n+1})\nabla\mathbb{P}_n + (\Psi \otimes I_{n+1})\mathbb{P}_n = \Omega_{n+s+1}\mathbb{P}_{n+s+1} + \sum_{m=0}^{n+s} A_m^n \mathbb{P}_m. \tag{4}$$

Observe that we can split Ω_{n+s+1} as a block matrix in the form:

$$\Omega_{n+s+1} = \left(\frac{\Omega_{n+s+1}^{(1)}}{\Omega_{n+s+1}^{(2)}}\right),$$

where $\Omega_{n+s+1}^{(k)} = (w_{i,j}^{(k)})_{i,j=0}^{n,n+s+1} \in \mathscr{M}_{(n+1)\times(n+s+2)}(\mathbb{R})$, for $k = 1, 2$. Let us denote by $\phi_{i,j}^{(k,l)}$ and $\psi_{i,j}^{(k)}$ the coefficients in the polynomials

$$\phi_{k,l}(x, y) = \sum_{j=0}^{s+2}\sum_{i=0}^{s+2-j} \phi_{s+2-j-i,i}^{(k,l)} x^{s+2-j-i} y^i,$$

$$\psi_k(x, y) = \sum_{j=0}^{s+1}\sum_{i=0}^{s+1-j} \psi_{s+1-j-i,i}^{(k)} x^{s+1-j-i} y^i,$$

for $k, l = 1, 2$. Then, $\Omega_{n+s+1}^{(k)}$, $k = 1, 2$, are lower Hessenberg band matrices, that is, they are $(s+4)$–diagonal matrices whose entries are defined by

$$w_{i,i+j}^{(k)} = (n-i)\phi_{s+2-j,j}^{(k,1)} + i\phi_{s+1-j,j+1}^{(k,2)} + \psi_{s+1-j,j}^{(k)},$$

for $0 \leqslant i \leqslant n$, $-1 \leqslant j \leqslant s+2$, and $w_{i,j}^{(k)} = 0$, $k = 1, 2$, otherwise.

Definition 3. A quasi definite linear functional u is said to be *semiclassical* if it is s-Pearson and

$$\operatorname{rank}\Omega^{(k)}_{n+s+1} = n+1, \quad k=1,2, \qquad \operatorname{rank}\Omega_{n+s+1} \geqslant n+2, \quad n \geqslant s+1. \tag{5}$$

A WOPS with respect to u is called a *semiclassical WOPS*.

Remark. Observe that, for $s=0$, we recover the concept of classical WOPS in two variables introduced by the authors in [3]. And, of course, this definition includes the classical bivariate orthogonal polynomials given by Krall and Sheffer [11], and Suetin [14].

4. The structure relation

As in the univariate case, we are going to prove that the semiclassical character of a linear functional is equivalent to the existence of a short representation of the derivatives of the orthogonal polynomials in terms of the polynomials themselves. First, we need a technical Lemma

Lemma 4. (1) *For* $M \in \mathcal{M}_{h\times k}(\mathcal{P})$, *and* $N \in \mathcal{M}_{l\times m}(\mathcal{P})$, *we get*

$$(M \otimes I_l)(I_k \otimes N) = M \otimes N = (I_h \otimes N)(M \otimes I_m).$$

(2) *For* $M \in \mathcal{M}_{h\times k}(\mathcal{P})$, *and* $N \in \mathcal{M}_{k\times l}(\mathcal{P})$, *we have*

$$\nabla(MN) = (\nabla M)N + (I_2 \otimes M)\nabla N.$$

(3) *As a consequence*,

$$(I_2 \otimes \mathbb{P}_m)\Phi\nabla\mathbb{P}^{\mathrm{t}}_n = (\Phi \otimes I_{m+1})[\nabla(\mathbb{P}_m\mathbb{P}^{\mathrm{t}}_n) - (\nabla\mathbb{P}_m)\mathbb{P}^{\mathrm{t}}_n].$$

Theorem 5 (*Structure relation*). *Let* u *be a quasi definite linear functional and let* $\{\mathbb{P}_n\}_{n\geqslant 0}$ *be the monic WOPS associated with* u. *Then*, u *is semiclassical if and only if* $\{\mathbb{P}_n\}_{n\geqslant 0}$ *satisfy*

$$\Phi\nabla\mathbb{P}^{\mathrm{t}}_n = \sum_{j=n-s-1}^{n+p-1} (I_2 \otimes \mathbb{P}^{\mathrm{t}}_j)F^n_j \quad \text{for } n \geqslant s+1, \tag{6}$$

where $F^n_j = (F^n_{m,1}/F^n_{m,2}) \in \mathcal{M}_{2(j+1)\times(n+1)}(\mathbb{R})$, *such that*

$$\operatorname{rank} F^n_{n-s-1,i} = n-s, \quad i=1,2, \qquad \operatorname{rank} F^n_{n-s-1} \geqslant n-s+1.$$

Proof. Since $\deg \Phi = p$, the entries in the matrix $\Phi\nabla\mathbb{P}^{\mathrm{t}}_n$ are polynomials of degree at most $n+p-1$. Then, we can write

$$\Phi\nabla\mathbb{P}^{\mathrm{t}}_n = \sum_{j=0}^{n+p-1} (I_2 \otimes \mathbb{P}^{\mathrm{t}}_j)F^n_j,$$

where $F^n_j \in \mathcal{M}_{2(j+1)\times(n+1)}(\mathbb{R})$. Using the orthogonality, we obtain

$$\begin{aligned}\langle u, (I_2 \otimes \mathbb{P}_m)(\Phi\nabla\mathbb{P}^{\mathrm{t}}_n)\rangle &= \langle u, (I_2 \otimes \mathbb{P}_m) \sum_{i=0}^{n+p-1} (I_2 \otimes \mathbb{P}^{\mathrm{t}}_i)F^n_i\rangle \\ &= (I_2 \otimes H_m)F^n_m, \quad 0 \leqslant m \leqslant n+p-1.\end{aligned} \tag{7}$$

On the other hand, using the above Lemma, and (3), we get

$$\begin{aligned}&\langle u, (I_2 \otimes \mathbb{P}_m)\Phi\nabla\mathbb{P}^{\mathrm{t}}_n\rangle \\ &\quad = \langle u, (\Phi \otimes I_{m+1})[\nabla(\mathbb{P}_m\mathbb{P}^{\mathrm{t}}_n) - (\nabla\mathbb{P}_m)\mathbb{P}^{\mathrm{t}}_n]\rangle \\ &\quad = -\langle(\Psi^{\mathrm{t}} \otimes I_{m+1})u, \mathbb{P}_m\mathbb{P}^{\mathrm{t}}_n\rangle - \langle u, (\Phi \otimes I_{m+1})(\nabla\mathbb{P}_m)\mathbb{P}^{\mathrm{t}}_n\rangle \\ &\quad = -\langle u, [(\Psi \otimes I_{m+1})\mathbb{P}_m + (\Phi \otimes I_{m+1})\nabla\mathbb{P}_m]\mathbb{P}^{\mathrm{t}}_n\rangle.\end{aligned} \tag{8}$$

Combining (7) and (8), we conclude

$$(I_2 \otimes H_m)F_m^n = -\langle u, [(\Psi \otimes I_{m+1})\mathbb{P}_m + (\Phi \otimes I_{m+1})\nabla \mathbb{P}_m]\mathbb{P}_n^t\rangle. \tag{9}$$

Observe that, $(\Psi \otimes I_{m+1})\mathbb{P}_m + (\Phi \otimes I_{m+1})\nabla \mathbb{P}_m$ is a polynomial matrix of degree at most $m+s+1$. So, from the orthogonality of the polynomials $\{\mathbb{P}_n\}_{n\geqslant 0}$, we deduce that $F_m^n = 0$, for $m+s+1<n$, and relation (6) holds.

Reciprocally, let us assume that (6) holds. Define

$$\Psi = -(I_2 \otimes H_0)\left(\sum_{i=0}^{s+1} F_0^i H_i^{-1}\mathbb{P}_i\right).$$

In order to prove the semiclassical character of u, we want to check that $\langle \mathrm{div}(\Phi u), \mathbb{P}_n^t\rangle = \langle \Psi^t u, \mathbb{P}_n^t\rangle$, for $n\geqslant 0$. In the case $0\leqslant n\leqslant s+1$, we get

$$\begin{aligned}\langle \mathrm{div}(\Phi u), \mathbb{P}_n^t\rangle &= -\langle u, \Phi\nabla\mathbb{P}_n^t\rangle = -\left\langle u, \sum_{j=0}^{n+p-1}(I_2\otimes \mathbb{P}_j^t)F_j^n\right\rangle \\ &= -\left\langle u, (I_2\otimes \mathbb{P}_0)\sum_{j=0}^{n+p-1}(I_2\otimes \mathbb{P}_j^t)F_j^n\right\rangle = -(I_2\otimes H_0)F_0^n,\end{aligned}$$

and

$$\begin{aligned}\langle \Psi^t u, \mathbb{P}_n^t\rangle = \langle u, \Psi\mathbb{P}_n^t\rangle &= \left\langle u, -(I_2\otimes H_0)\left(\sum_{i=0}^{s+1}F_0^i H_i^{-1}\mathbb{P}_i\right)\mathbb{P}_n^t\right\rangle \\ &= -(I_2\otimes H_0)F_0^n H_n^{-1}\langle u, \mathbb{P}_n\mathbb{P}_n^t\rangle = -(I_2\otimes H_0)F_0^n.\end{aligned}$$

In the same way, if $n\geqslant s+2$,

$$\begin{aligned}\langle \mathrm{div}(\Phi u), \mathbb{P}_n^t\rangle &= -\langle u, \Phi\nabla\mathbb{P}_n^t\rangle = -\left\langle u, \sum_{j=n-s-1}^{n+p-1}(I_2\otimes \mathbb{P}_j^t)F_j^n\right\rangle \\ &= -\left\langle u, (I_2\otimes \mathbb{P}_0)\sum_{j=n-s-1}^{n+p-1}(I_2\otimes \mathbb{P}_j^t)F_j^n\right\rangle = 0,\end{aligned}$$

and $\langle \Psi^t u, \mathbb{P}_n^t\rangle = \langle u, \Psi\mathbb{P}_n^t\rangle = 0$, using that $\deg\Psi\leqslant s+1<n$.

Finally, the rank condition can be deduced from (9), taking $m+s+1=n$, and using (4). In fact, we get

$$(I_2\otimes H_{n-s-1})F_{n-s-1}^n = -\left\langle u, \left(\Omega_n\mathbb{P}_n + \sum_{m=0}^{n-1}A_m^{n-s-1}\mathbb{P}_m\right)\mathbb{P}_n^t\right\rangle = -\Omega_n H_n,$$

and thus, $\mathrm{rank}\, F_{n-s-1}^n = \mathrm{rank}\,\Omega_n$. □

5. Examples

In this section, some examples of semiclassical orthogonal polynomials in two variables are given. Naturally, classical orthogonal polynomials in two variables [3,11,14] are semiclassical with $s=0$. Moreover, it is very easy to check that tensor product of semiclassical orthogonal polynomials in one variable are semiclassical polynomials in two variables.

In 1975, Koornwinder studied in [8] *examples of two–variables analogues of the Jacobi polynomials*, introducing seven classes of such polynomials in two variables. Class I, II, IV, and V are well known classical orthogonal polynomials in two variables. Classes III, VI, and VII constitute a nontrivial example of *semiclassical orthogonal polynomials in two variables*, according to our definition.

Class III : For $\alpha, \beta > -1$, the polynomials

$$P_{n,k}^{(\alpha,\beta)}(x,y) = P_{n-k}^{(\alpha,\beta+k+1/2)}(2x-1)x^{(1/2)k}P_k^{(\beta,\beta)}(x^{-1/2}y), \quad n \geqslant k \geqslant 0,$$

are orthogonal with respect to the weight function $w(x,y) = (1-x)^\alpha(x-y^2)^\beta$, on the region $\{(x,y)/y^2 < x < 1\}$, which is bounded by a straight line and a parabola. Defining

$$\Phi = (1-x)(x-y^2)I_2 \quad \text{and} \quad \Psi = \begin{pmatrix} -(\alpha+1)(x-y^2)+(\beta+1)(1-x) \\ -2(\beta+1)(1-x)y \end{pmatrix},$$

these polynomials are semiclassical, since the weight function $w(x,y)$ satisfies (2), and the matrix

$$\Omega_{n+s+1} = \Omega_{n+2} = \left(\begin{array}{cccccc} 0 & 0 & n+\beta+1 & & & \bigcirc \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & \bigcirc & & & \beta+1 \\ \hline 0 & 2\alpha+2 & & & \bigcirc & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \bigcirc & & & n+2\alpha+2 & 0 \end{array}\right) \in \mathcal{M}_{2(n+1)\times(n+3)},$$

has rank $\Omega_{n+2} = n+2$ and rank $\Omega_{n+2}^{(1)} = \text{rank } \Omega_{n+2}^{(2)} = n+1$, so condition (5) holds. In this case, deg $\Phi = p = 3$, and deg $\Psi = q = 2$, so $s = \max\{p-2, q-1\} = 1$, and the structure relation has, at most, $p+s+1 = 5$ terms.

In the following examples, rank condition (5) is satisfied. The details are omitted for reasons of space.

Class VI : Let $\alpha, \beta, \gamma > -1$, $\alpha+\gamma+3/2 > 0$, and $\beta+\gamma+3/2 > 0$, and consider the weight function

$$w(u,v) = (1-u+v)^\alpha(1+u+v)^\beta(u^2-4v)^\gamma,$$

defined on the region $\{(u,v)/|u| < v+1, u^2-4v > 0\}$, which is bounded by two straight lines and a parabola touching these lines.

If we take $\Phi = (1-u+v)(1+u+v)(u^2-4v)I_2$, and

$$\begin{aligned}\psi_1 &= [-(\alpha+1)(1+u+v)+(\beta+1)(1-u+v)](u^2-4v) \\ &\quad + 2(\gamma+1)u(1-u+v)(1+u+v),\end{aligned}$$

$$\begin{aligned}\psi_2 &= [(\alpha+1)(1+u+v)+(\beta+1)(1-u+v)](u^2-4v) \\ &\quad - 4(\gamma+1)(1-u+v)(1+u+v),\end{aligned}$$

then conditions (2) and (5) are satisfied, and orthogonal polynomials associated with $w(u,v)$ are semiclassical. Therefore, deg $\Phi = p = 4$, and deg $\Psi = q = 3$, so $s = \max\{p-2, q-1\} = 2$, and the Structure relation has, at most, $p+s+1 = 7$ terms.

Class VII : Consider the weight function

$$w(x,y) = [-(x^2+y^2+9)^2 + 8(x^3-3xy^2)+108]^\alpha,$$

for $\alpha > -\frac{5}{6}$, defined on the region bounded by the three–cusped deltoid (or Steiner's hypocycloid) $-(x^2+y^2+9)^2 + 8(x^3-3xy^2)+108 = 0$. Orthogonal polynomials associated with $w(x,y)$ are semiclassical again. In fact, we can choose $\Phi = [-(x^2+y^2+9)^2+8(x^3-3xy^2)+108]I_2$, and

$$\Psi = \begin{pmatrix} (\alpha+1)[-4x(x^2+y^2+9)+24(x^2-y^2)] \\ (\alpha+1)[-4y(x^2+y^2+9)-48xy] \end{pmatrix}.$$

In this case, the rank condition (5) holds again, and deg $\Phi = p = 4$, and deg $\Psi = q = 3$, so $s = \max\{p-2, q-1\} = 2$, and the Structure relation has, at most, $p+s+1 = 7$ terms as in Class VI.

Finally, using Koornwinder's tools, we present an example of a semiclassical weight function with unbounded support. For $\alpha, \beta > -1$, the polynomials

$$P_{n,k}^{(\alpha,\beta)}(x,y) = L_{n-k}^{(\alpha+2k+1)}(x)x^k P_k^{(\beta,0)}(x^{-1}y), \quad n \geqslant k \geqslant 0,$$

are orthogonal with respect to the weight function

$$w(x, y) = x^{\alpha} e^{-x} (1 - x^{-1} y)^{\beta},$$

on the region $\{(x, y)/ - x < y < x, x > 0\}$. Defining

$$\Phi = \begin{pmatrix} x(x-y) & 0 \\ 0 & x^2(x-y) \end{pmatrix},$$

$$\Psi = \begin{pmatrix} -x^2 + xy + (\alpha+2)x + (\beta-\alpha-1)y \\ -(\beta+1)x^2 \end{pmatrix},$$

these polynomials are semiclassical, since the weight function $w(x, y)$ satisfies (2), and rank condition (5) holds. In this case, deg $\Phi = p = 3$, and deg $\Psi = q = 2$, so $s = \max\{p-2, q-1\} = 1$, and the Structure relation has, at most, $p + s + 1 = 5$ terms.

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