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q-Cesáro matrix and q-statistical convergence

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ABSTRACT

It is obvious that a q-analog of C_α, the Cesáro matrix of order α, can be defined in different ways. In this paper we introduce a method to find q-analogs of C_α, where α is a positive integer. Using this method, we obtain the most natural q-analogs of C_α. We also prove that the strength of $C_1(q^k)$ does not depend on q, where $C_1(q^k)$ is the most natural q-analog of C_1. Finally, we define a density function and q-statistical convergence using $C_1(q^k)$.

1. Introduction

Let $A = (a_{nk})\ n, k = 0, 1, 2, \ldots$ be an infinite matrix and let $x = (x_j)_{j\in\mathbb{N}}$ be a sequence of real numbers. The A-transform of x is denoted by $A(x) = (Ax)_n$ and defined as;

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k$$

provided that the series converges for each $n \in \mathbb{N}^0$. The sequence space $c_A := \{x \in w : Ax \in c\}$ is called the convergence domain of A, where w and c are the spaces of all and convergent sequences respectively. The matrix A is said to be conservative if the convergence of the sequence x implies the convergence of $A(x)$, (or equivalently $c \subset c_A$). In addition, if $A(x)$ converges to the limit of x, for each convergent sequence x, then it is called regular. The following Theorem states the well known characterization of conservative matrices and can be found in any standard summability book [1].

Theorem 1. *An infinite matrix $A = (a_{nk})\ n, k = 0, 1, 2, \ldots$ is conservative if and only if*
(i) $\lim_{n\to\infty} a_{nk} = \lambda_k$, *for each* $k = 0, 1, \ldots$
(ii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \lambda$, *and*
(iii) $\sup_n \sum_{k=0}^{\infty} |a_{nk}| \le M < \infty$, *for some* $M > 0$.

Here, of course, the limits λ_k and λ are finite. If $\lambda_k = 0$, for all k and $\lambda = 1$ then the above theorem reduces to the well known theorem of Silverman and Toeblitz which provides necessary and sufficient conditions for regularity of the infinite matrix $A = (a_{nk})\ n, k = 0, 1, 2, \ldots$.

Let α be a real number with $-\alpha \notin \mathbb{N}$. Then, the regular matrices $C_\alpha := (c_{nk}^\alpha)$ defined by

$$c_{nk}^\alpha = \begin{cases} \dfrac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}} & \text{if } k \le n,\ \ n, k = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

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and the associated matrix summability methods, are called the Cesáro matrix and Cesáro summability method of order α respectively. The Cesáro mean (t_n) of a real or complex sequence $x = (x_j)$ is defined to be the sequence, $t_n = \frac{x_0+x_1+\cdots+x_n}{n+1}$, $n = 0, 1, \ldots$ and in the case of $\lim_{n\to\infty} t_n = t$, x is said to be Cesáro summable to t. The Cesáro methods have played a central role in connection with the applications of summability theory to different branches of mathematics.

An obvious generalization of C_1 is the weighted mean (or Riesz method). Let $p = (p_k)$ be a sequence of real numbers with $p_0 > 0$, $p_k \geq 0$, $k \in \mathbb{N}$ and $P_n = \sum_{k=0}^{n} p_k$, then the matrix method $R_p = (r_{nk})$ defined by

$$r_{nk} = \begin{cases} \dfrac{p_k}{P_n} & \text{if } k \leq n, \\ 0 & \text{otherwise} \end{cases} \qquad n, k = 0, 1, \ldots$$

is called a weighted mean associated with the sequence p. Obviously C_1 is a weighted mean associated with $e = (1, 1, \ldots)$.

On the other hand, the value $[r]_q$ denotes the q-integer of r, which is given by

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q}, & q \in \mathbb{R}^+ - \{1\} \\ r, & q = 1. \end{cases}$$

For the last thirty years, studies involving q-integers and their applications (for example, q-analogs of positive linear operators and their approximation properties) have become active research areas. During the same period a large number of research papers on q-analogs of existing theories, involving interesting results, have been published (see [2–4]). The motivation of the present paper is the following question "What kind of results can be achieved by considering q-analogs of regular matrices and the existing summability theory?" In this paper we will mainly focus on q-analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$ and its properties. The next section of this paper will introduce a method to find q-analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$.

2. q-Cesáro summability

Let $S := (s_{nk})$ be the summation matrix with $s_{nk} = 1$ for $k \leq n$ and $s_{nk} = 0$ otherwise and let I be the identity matrix. For any sequence $x = (x_k)$, define

$$B_n^0 x = I(x) = x_n, \tag{2.1}$$

$$B_n^1 x = S(x) = \sum_{\upsilon=0}^{n} x_\upsilon = \sum_{\nu=0}^{n} B_\nu^0(x) \tag{2.2}$$

and

$$B_n^\alpha(x) = S^\alpha(x) = \sum_{\nu=0}^{n} B_\nu^{\alpha-1} x. \tag{2.3}$$

$\alpha \in \mathbb{N}$, $\alpha \geq 2$. Recall that the entries s_{nk}^α of the matrix S^α can be determined in the following way. By (2.3) we have $\sum_{k=0}^{n} s_{nk}^\alpha x_k = B_n^\alpha(x)$. But,

$$(1-z)\sum_n B_n^\alpha(x) z^n = \sum_n (B_n^\alpha(x) - B_{n-1}^\alpha(x)) z^n = \sum_n B_n^{\alpha-1}(x) z^n$$

with $B_{-1}^\alpha(x) = 0$, therefore,

$$\sum_n B_n^\alpha(x) z^n = \frac{1}{(1-z)^\alpha} \sum_n B_n^0(x) z^n$$
$$= \sum_n \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} x_k z^n.$$

By comparing coefficients of z^n, we have

$$B_n^\alpha(x) = \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} x_k, \quad \text{with } n, k = 0, 1, \ldots, k \leq n,$$

or equivalently,

$$s_{nk}^\alpha = \binom{n-k+\alpha-1}{n-k}.$$

On the other hand, the sum of the nth row is;

$$\sum_{k=0}^{n} s_{nk}^\alpha = \sum_{k=0}^{n} \binom{n-k+\alpha-1}{n-k} = \binom{n+\alpha}{n}$$

and the matrix defined by

$$c_{nk}^{\alpha} := \frac{1}{\binom{n+\alpha}{n}} s_{nk}^{\alpha} \tag{2.4}$$

gives exactly the Cesáro matrix of order $\alpha \in \mathbb{N}$. Although the above calculations are not new and can be found in standard summability books, they can be modified to obtain q-analogs of Cesáro matrices of order $\alpha \in \mathbb{N}$. Before giving more details about this process, we need the following definition.

Definition 2. Let

$$S_q = \begin{cases} a_{nk}(q), & \text{if } k \leq n \\ 0, & \text{otherwise} \end{cases}$$

be the infinite, lower triangular matrix, satisfying

$$a_{nk}(1) = 1,$$

then S_q is called the q-analog of the summation matrix S generated by the sequence $a_{nk}(q)$.

Replacing S by its q-analog in the above process, we will obtain a q-analog of the Cesáro matrix of order $\alpha \in \mathbb{N}$ (or q-Cesáro matrix generated by $a_{nk}(q)$). In the following theorem, we introduce a general formula for Cesáro matrix of order one associated with $a_{nk}(q)$.

Theorem 3. *The q-analog of the Cesáro matrix of order one associated with $a_{nk}(q)$ is $C_1(a_{nk}(q)) = (c_{nk}^1(a_{nk}(q)))$ where*

$$c_{nk}^1(a_{nk}(q)) = \begin{cases} a_{nk}(q)\left(\sum_{k=0}^{n} a_{nk}(q)\right)^{-1}, & \text{if } k \leq n \\ 0, & \text{otherwise}, \end{cases} \qquad n, k = 0, 1, \ldots. \tag{2.5}$$

Proof. Let S_q be the q-analog of S associated with $a_{nk}(q)$. By applying the above process for $\alpha = 1$, Eqs. (2.1) and (2.2) become

$$B_n^0 x = I(x) = x_n,$$

$$B_n^1 x = \left(S_q(x)\right)_n = \sum_{\nu=0}^{n} a_{n\nu}(q) x_\nu,$$

respectively. The matrix multiplication yields that $S_q = (s_{nk}^1(a_{nk}(q)))$ where

$$s_{nk}^1(a_{nk}(q)) = \begin{cases} a_{nk}(q) & \text{if } k \leq n \\ 0 & \text{otherwise}, \end{cases} \qquad n, k = 0, 1, \ldots.$$

Now, the sum of the nth row will be $a_{n0}(q) + a_{n1}(q) + \cdots + a_{nn}(q) = \sum_{k=0}^{n} a_{nk}(q)$, therefore in a way parallel to (2.4), one can obtain the q-Cesáro matrix of order one given in (2.5). □

It is obvious that in the case $q = 1$, $C_1(a_{nk}(q))$ reduces to the ordinary Cesáro matrix C_1 given in (1.1) for $\alpha = 1$.

Remark 4. It should be mentioned that, under the conditions $a_{nk}(q) = a_k(q)$, for all n, with $a_0(q) > 0$, and $a_k(q) \geq 0$, $k \in \mathbb{N}$, $C_1(a_{nk}(q))$ is a Riesz method associated with $a_k(q)$.

Remark 5. The q-Cesáro matrix associated with $a_{nk}(q) = q^{-k}$ is the q-analog of the Cesáro matrix suggested by Bustoz and Gordillo [5].

Of course, there are many ways to define q-analogs of Cesáro matrices. In the following theorem, we suggest the most suitable q-analog of the Cesáro matrix of order $\alpha \in \mathbb{N}$.

Theorem 6. *$C_1(q^k) = (c_{nk}^1(q^k))$ with*

$$c_{nk}^1(q^k) = \begin{cases} \dfrac{q^k}{[n+1]_q} & \text{if } k \leq n \\ 0 & \text{otherwise}, \end{cases} \qquad n, k = 0, 1, \ldots \tag{2.6}$$

and $C_2(q^k) = (c_{nk}^2(q^k))$ with

$$c_{nk}^2(q^k) = \begin{cases} [n-k+1]\, q^{2k} \left(\sum_{k=0}^{n} q^{2k}\, [n-k+1]_q\right)^{-1} & \text{if } k \leq n \\ 0 & \text{otherwise}, \end{cases} \qquad n, k = 0, 1, \ldots$$

and more generally $C_\alpha(q^k) = \left(c^\alpha_{nk}(q^k)\right)$ *where*

$$c^\alpha_{nk}(q^k) = \begin{cases} \dfrac{q^{\alpha k} \sum\limits_{m_1=0}^{n-k} q^{m_1} \sum\limits_{m_2=0}^{m_1} q^{m_2} \cdots \sum\limits_{m_{\alpha-2}=0}^{m_{\alpha-1}} q^{m_{\alpha-2}} [m_{\alpha-2}+1]}{\sum\limits_{k=0}^{n} \left(q^{\alpha k} \sum\limits_{m_1=0}^{n-k} q^{m_1} \sum\limits_{m_2=0}^{m_1} q^{m_2} \cdots \sum\limits_{m_{\alpha-2}=0}^{m_{\alpha-1}} q^{m_{\alpha-2}} [m_{\alpha-2}+1] \right)} & \text{if } k \le n \\ 0 & \text{otherwise}, \end{cases} \qquad n, k = 0, 1, \ldots$$

with $\alpha > 2, \alpha \in \mathbb{N}$.

Proof. To find $C_1(q^k)$ given in (2.6), replace $a_{nk}(q)$ by q^k in Theorem 3. For $C_\alpha(q^k)$ take $a_{nk}(q) = q^k$ and apply the process described above. □

Recall that in the ordinary case the sum of the nth row of the summation matrix S was $n+1$, and the most natural q-analog of $n+1$ is $[n+1]_q$. To have the sum $[n+1]_q$ on the nth row of S_q the generating sequence can be selected as $a_{nk}(q) = q^k$. Therefore, it seems that the most suitable q-analog of the Cesáro matrix C_α is $C_\alpha(q^n)$.

In the rest of this paper, we will concentrate on the q-analog $C_1(q^n)$. As a direct consequence of Theorem 1, one can state the following lemma.

Lemma 7. (i) $C_1(q^n)$ *is conservative for each* $q \in \mathbb{R}$,
(ii) $C_1(q^n)$ *is regular for each* $q \ge 1$.

Remark 8. If $q_1 \ne q_2$, then $C_1(q_1^n) \ne C_1(q_2^n)$, moreover if $q_1 > 1$, then $C_1(q_1^n)$ is regular but $C_1(q_2^n)$ is not regular for $q_2 = q_1^{-1}$.

The following theorem will be used to compare $C_1(q_1^n)$ by $C_1(q_2^n)$ and C_1 by $C_1(q^n)$, as summability methods.

Theorem 9 (*[1], Theorem 3.2.8, p. 114*)**.** *Let* R_p *be a regular Riesz method with* $p_k > 0$ *for* $k = 0, 1, \ldots$ *and let* $A = (a_{nk})$ *be a conservative matrix method. Then, A is stronger than* R_p *if and only if the following conditions hold:*

(i) $\lim_{k\to\infty} \left(\frac{a_{nk}}{p_k}\right) = 0, n = 0, 1, \ldots$
(ii) $\sup_n \sum_k P_k \left| \frac{a_{nk}}{p_k} - \frac{a_{n(k+1)}}{p_{k+1}} \right| < \infty$.

It is natural to ask how the strength of $C_1(q^k)$ changes with q. The answer is given in the following theorem.

Theorem 10. $C_1(q_1^k)$ *is equivalent to* $C_1(q_2^k)$, *for* $1 < q_1 < q_2$.

Proof. Assume that $1 < q_1 < q_2$. Then,

$$\begin{aligned} \sup_n \sum_{k=0}^{n} [k+1]_{q_2} \left| \frac{q_1^k}{[n+1]_{q_1} q_2^k} - \frac{q_1^{k+1}}{[n+1]_{q_1} q_2^{k+1}} \right| &\le \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_2}}{q_2^k} \left| \frac{q_1^k}{[n+1]_{q_1}} - \left(\frac{q_1}{q_2}\right) \frac{q_1^k}{[n+1]_{q_1}} \right| \\ &\le \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_2}}{q_2^k} \left(\frac{q_1^k}{[n+1]_{q_1}} \right) \\ &\le \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_2}}{q_2^k} \le \sup_n \sum_{k=0}^{n} \left(\frac{1}{q_2}\right)^k \\ &\le \frac{q_2}{q_2-1}. \end{aligned}$$

Conversely,

$$\begin{aligned} \sup_n \sum_{k=0}^{n} [k+1]_{q_1} \left| \frac{q_2^k}{[n+1]_{q_2} q_1^k} - \frac{q_2^{k+1}}{[n+1]_{q2} q_1^{k+1}} \right| &\le \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left| \frac{q_2^k}{[n+1]_{q_2}} - \frac{q_2^{k+1}}{[n+1]_{q_2} q_1} \right| \\ &\le \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left| \frac{q_2^k}{[n+1]_{q_2}} - \frac{q_2}{q_1} \frac{q_2^k}{[n+1]_{q_2}} \right| \\ &\le \sup_n \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left(\frac{q_2}{q_1} \frac{q_2^k}{[n+1]_{q_2}} - \frac{q_2^k}{[n+1]_{q_2}} \right) \\ &\le \sup_n \frac{q_2}{q_1} \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k} \left(\frac{q_2^k}{[n+1]_{q_2}} \right) \end{aligned}$$

$$\leq \sup_n \frac{q_2}{q_1} \sum_{k=0}^{n} \frac{[k+1]_{q_1}}{q_1^k}$$
$$\leq \sup_n \frac{q_2}{q_1} \frac{q_1}{q_1 - 1} = \frac{q_2}{q_1 - 1}.$$

The proof is completed using Theorem 9 and the fact that $C_1(q_1^k)$ and $C_1(q_2^k)$ are both regular, row finite matrices. □

Theorem 11. *The summability method C_1 is stronger than $C_1(q^n)$ for $q \geq 1$.*

Proof. For $q = 1$, $C_1(q^n)$ reduces to C_1, therefore we may assume that $q > 1$. By using Theorem 9 and the fact that C_1 is a row finite regular method, it is enough to show that

$$\sup_n \frac{1}{n+1} \left(\frac{q-1}{q}\right) \sum_{k=0}^{n} \frac{[k+1]_q}{q^k} < \infty. \tag{2.7}$$

Using

$$\frac{[k+1]_q}{q^k} = \sum_{i=0}^{k} \frac{1}{q^i} \leq \frac{q}{q-1}$$

in (2.7), completes the proof. □

Theorem 12. *For $q \leq 1$, $c \subsetneq c_{C_1(q^n)}$.*

Proof. For any fixed $q \leq 1$, the divergent sequence $x = (x_k)$ with

$$x_k = \begin{cases} \dfrac{1}{q} & k = 0, 2, \ldots \\ -\dfrac{1}{q^2} & k = 1, 3, \ldots \end{cases}$$

is $C_1(q^n)$-summable to 0. □

3. *q*-statistical convergence

Freedman and Sember [6] showed that each non-negative regular matrix A can be associated by a density function

$$\delta_A(K) = \lim_{n\to\infty} \inf (A\chi_K)_n \tag{3.1}$$

where χ_K denotes the characteristic function of $K \subset \mathbb{N}$. Replacing A by C_1 and lim inf by an ordinary limit in (3.1), we obtain the well-known natural density function

$$\delta(K) = \delta_{C_1}(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\infty} \chi_K(k)$$

provided that a limit exists. A sequence $x := (x_j)$ is called statistically convergent to L and denoted by $st\text{-}\lim_{n\to\infty} x_n = L$, if for every $\varepsilon > 0$,

$$\delta \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} = 0$$

(see [7]). Using the regularity of C_1^q for $q \geq 1$, and replacing A by C_1^q in (2.1), we can define the following density functions $\delta_{C_1^q}$, between the subsets of natural numbers and the interval $[0, 1]$;

$$\delta_q(K) = \delta_{C_1^q}(K) = \lim_{n\to\infty} \inf \left(C_1^q \chi_K\right)_n, \quad q \geq 1. \tag{3.2}$$

Lemma 13. (i) $\delta_q(2\mathbb{N}) = \delta_q(2\mathbb{N}+1) = \frac{1}{[2]}$
(ii) $\delta_q(a\mathbb{N}+b) = \frac{1}{[a]}$ *where a and b are positive integers.*

Proof. Using (3.2)

$$\delta_q(2\mathbb{N}) = \lim_{n\to\infty} \inf \sum_{k\in 2\mathbb{N}} \frac{q^{k-1}}{[n]}$$

where

$$\sum_{k\in 2\mathbb{N}} \frac{q^{k-1}}{[n]} = \begin{cases} \sum_{k=1}^{\frac{n}{2}} \frac{q^{2k-1}}{[n]} & \text{if } n \text{ is even} \\ \sum_{k=1}^{\frac{n-1}{2}} \frac{q^{2k-1}}{[n]} & \text{if } n \text{ is odd.} \end{cases}$$

(i) If n is even, then the nth partial sum

$$s_n = \frac{q}{[n]} + \frac{q^3}{[n]} + \cdots + \frac{q^{n-1}}{[n]} \tag{3.3}$$

and

$$q^2 s_n = \frac{q^3}{[n]} + \frac{q^5}{[n]} + \cdots + \frac{q^{n+1}}{[n]}. \tag{3.4}$$

Combining (3.3) and (3.4) we have

$$s_n = \frac{q(1-q^n)}{(1-q^2)\,[n]}$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{q(1-q^n)}{(1-q^2)\,[n]} = \frac{q}{1+q}.$$

If n is odd, then the nth partial sum

$$s_n = \frac{q}{[n]} + \frac{q^3}{[n]} + \cdots + \frac{q^{n-2}}{[n]} \tag{3.5}$$

and

$$q^2 s_n = \frac{q^3}{[n]} + \frac{q^5}{[n]} + \cdots + \frac{q^{n}}{[n]}. \tag{3.6}$$

Similarly combining (3.5) and (3.6) yields

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{q-q^n}{(1-q^2)\,[n]} = \frac{1}{1+q}.$$

Since $q \geq 1$, we have $\frac{1}{1+q} \leq \frac{q}{1+q}$ and

$$\delta_q(2\mathbb{N}) = \lim_{n\to\infty}\inf \sum_{k\in 2\mathbb{N}} \frac{q^{k-1}}{[n]} = \frac{1}{[2]}.$$

By using the above technique one can prove that $\delta_q(2\mathbb{N}+1) = \frac{1}{[2]}$.

(ii) Since $\{a\mathbb{N}+j : j = 0, 1, \ldots, a-1\}$ is a partition for $\mathbb{N}$ and using the method of (i)

$$\lim_{n\to\infty} \left(\sum_{k\in a\mathbb{N}+\mathbb{J}} \frac{q^{k-1}}{[n]} \right) = \frac{q^{a-1-j}}{[a]}$$

for fixed $j \in \{0, 1, \ldots, a-1\}$, we have

$$\delta_q(a\mathbb{N}+b) = \inf \left\{ \frac{q^{a-1-j}}{[a]} : j = 0, 1, \ldots, a-1. \right\} = \frac{1}{[a]}. \quad \square$$

Finally we will define a new type of convergence which is different from statistical convergence.

Definition 14. A number sequence $x = (x_k)$ is called q-statistically convergent to L, written $st^q\text{-}\lim x = L$, if for every $\varepsilon > 0$, $\delta_q(K_\varepsilon) = 0$, where $K_\varepsilon = \{k : |x_k - L| \geq \varepsilon\}$.

Example 15. Consider the sequence $x_k = \left(\underbrace{1}_{2^0}, \underbrace{0, 0}_{2^1}, \underbrace{1, 1, 1, 1}_{2^2}, \underbrace{0, 0, 0, \dots, 0}_{2^3}, 1, \dots, \right)$ and define the set $K = \{k \in \mathbb{N} : x_k = 1\}$ then $\delta(K)$ does not exist (see [8]) therefore x_k is not statistically convergent. On the other hand, since $\left[C_1^q \chi_K\right]_{2^{2n}-1} \to 0$, st_q- $\lim x_k = 0$.

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