Hamiltonian structure of the Yang–Mills functional

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1. Introduction

The aim of this paper is to study a Hamiltonian structure of the Yang–Mills theory. Many authors have worked on a geometric formulation of Hamiltonian theory (cf. [2,3,8,9] and references therein). The approach of the calculus of variations on fibered manifolds which is based on Krupková’s concept of a Lepagean $(n+1)$-form generalizing Krupka’s concept of a Lepagean $n$-form is adopted ($n$ is the dimension of the base manifold of the fibered manifold). This approach opens a possibility to regularize a Lagrangian whose standard Hessian is singular, it is the case of the Yang–Mills Lagrangian too.

This informative paper is organized as follows. In Section 2, a survey of the general variational theory [1,5,6] is given and the results involving a Hamiltonian field theory [10,11] needed in the Yang–Mills theory are summarized. In Section 3, a Hamiltonian system associated with the Yang–Mills theory is presented and Legendre transformation after the regularization is performed.

2. Lagrangian and Hamiltonian theory on fibered manifolds

Recall our standard notation [5-9]. We have a fibered manifold $\pi: Y \to X$, and write $n = \dim X$, $n + m = \dim Y$. $J^r Y$ is the $r$-jet prolongation of $Y$, and $\pi^{r-\delta}: J^r Y \to J^\delta Y$, $\pi^r: J^r Y \to X$ are the canonical jet projections. The $r$-jet prolongation of a section $\gamma$ is defined to be the mapping $x \to J^r \gamma(x) = J^r_x \gamma$. For any set $W \subset Y$ we denote $W^r = (\pi^r,0)^{-1}(W)$. Any fibered chart $(V, \psi)$, $\psi = (x^i, y^\sigma)$, on $Y$ induces the associated charts on $X$ and $J^r Y$, denoted by $(U, \varphi)$, $\varphi = (x^i)$, and $(V', \psi')$, $\psi' = (x^i, y'^\sigma, y'^{j_1}, y'^{j_1j_2}, \ldots, y'^{j_1j_2...j_r})$, respectively; here $1 \leq i, j_1, \ldots, j_r \leq n$, $1 \leq \sigma \leq m$, and $V^r = (\pi^r,0)^{-1}(V)$, $U = \pi^r(V)$.

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We denote $\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$, and its contractions $\omega_i = \delta_i/\partial x^i \omega_0$, $\omega_{ij} = \delta_{ij}/\partial x^i \omega_0$. We define the formal derivative operator with respect to $x^i$ by $d_i = \partial_i/\partial x^i + y^\sigma_i \partial_i/\partial y^\sigma_i + \ldots + y^{\sigma_n}_i \partial_i/\partial y^{\sigma_n}_i$.

For any open set $W \subset Y$, let $\Omega^n_0 W$ be the ring of functions on $W$. The $\Omega^n_0 W$-module of differential $q$-forms on $W'$ is denoted by $\Omega^n_{q-1} W$, and the exterior algebra of forms on $W'$ is denoted by $\Omega^n W$. The module of $\pi^r$-horizontal $q$-forms is denoted by $\Omega^n_{q, r} W$. A form $\rho \in \Omega^n_0 W$ is called $\pi^{r,s}$-projectable if there exists a form $\rho_0 \in \Omega^n_0 W$ such that $\pi^{r,s} \rho_0 = \rho$. The fibered structure of $Y$ induces a morphism of exterior algebras $h : \Omega^n W \rightarrow \Omega^{n+1} W$, called the horizontalization. In a fibered chart $h$ is defined by $h^f = f \circ \pi^{r,s}, h dx^i = dx^i, h dy^\sigma = y^\sigma_i \partial_i y^\sigma_i + \ldots + y^{\sigma_n} \partial_i y^{\sigma_n}$, where $f$ is a real function on $W'$, and $0 \leq p \leq r$.

We say that a form $\eta \in \Omega^n_{q, r} W$ is contact, if $h \eta = 0$. For any fibered chart the 1-forms $\eta^\sigma_{ij} = dy^\sigma_{ij} - y^\sigma_i \partial_j y^\sigma_j$, where $1 \leq i, j \leq r - 1$, are examples of contact 1-forms, defined on $W'$. Note that these forms define a basis of 1-forms on $\Omega^n_{q, r} W$, $(dx^i, \eta^\sigma_{ij}, dy^\sigma_{ij})$. For every $\eta \in \Omega^n_{q, r} W$ we have a unique canonical decomposition $\pi^{r,s} \eta = h \eta + p_1 \eta + \ldots + p_q \eta$ into a sum of a horizontal form $h \eta$ and $k$-contact forms $p_k \eta$, $1 \leq k \leq q$; in coordinates each term of a $k$-contact form with respect to a basis $(dx^i, \eta^\sigma_{ij} \partial_j y^\sigma_j, dy^\sigma_{ij} \partial_j y^\sigma_j)$, $1 \leq i, j \leq r$, contains exactly $k$ of the 1-contact forms $\eta^\sigma_{ij} \partial_j y^\sigma_j$.

A Lagrangian for $Y$ is a $\pi^{r,s}$-horizontal $n$-form on the $r$-jet prolongation $J^r Y$ of $Y$. The number $r$ is called the order of $\lambda$. In a fibered chart $(V, \psi) \rightarrow Y$, and the associated chart on $J^r Y$, a Lagrangian of order $r$ has an expression $\lambda = \mathcal{L} \omega_0$, where $\mathcal{L} : V' \rightarrow \mathbb{R}$ is the component of $\lambda$ with respect to $(V, \psi)$ (the Lagrange function associated with $(V, \psi)$). The Euler–Lagrange form of $\lambda$ is defined to be an $(n+1)$-form $E_\lambda$ on $J^r Y$, defined by

$$E_\lambda = E_\sigma (\mathcal{L}) \eta^\sigma \wedge \omega_0, \quad E_\sigma (\mathcal{L}) = \sum_{i=0}^r (-1)^i d_j d_1 \ldots \hat{d_j} \ldots d_p \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 j_2 \ldots j_l}}.$$ 

$E_\sigma (\mathcal{L})$ are the Euler–Lagrange expressions.

An $n$-form $\rho$ on $J^r Y$ is called a Lepagean $n$-form (of order $s$) if the $(n+1)$-form $p_1 d \rho$ is $\pi^{r+1,0}$-horizontal. If $h \rho = \lambda$ then we say that $\rho$ is a Lepagean equivalent of the Lagrangian $\lambda$. An $(n+1)$-form $E \in \Omega^n W$ is called a dynamical form if it is 1-contact and $\pi^{r,s}$-horizontal, i.e. in any fibered chart $E = E_\sigma (\mathcal{L}) \eta^\sigma \wedge \omega_0$, where $E_\sigma$ are functions on an open set in $J^r Y$. A closed $(n+1)$-form $\omega$ on $J^r Y$, $s \geq 1$, is called a Lepagean $(n+1)$-form if $p_1 \omega$ is a dynamical form. If $\lambda$ is a Lepagean $(n+1)$-form and $p_1 \omega = E$ then we say that $\omega$ is a Lepagean equivalent of $E$.

We say that Lepagean $(n+1)$-forms $\omega_1$ and $\omega_2$ are equivalent if (up to a projection) $p_1 \omega_1 = p_1 \omega_2$. The equivalence class $[\omega]$ of all Lepagean $(n+1)$-forms is called a Lagrangian system. Let $s \geq 0$ denote the dynamical order of the Lagrangian system $[\omega]$ defined as the minimum of the set of orders of the forms from $[\omega]$, then a section $\gamma : U \rightarrow Y$ defined on an open subset $U \subset X$ is an extremal of $E = p_1 \omega$, i.e. $E \circ J^r Y = 0$, iff for every $\pi^k$-vertical vector field $\xi$ on $Y$,

$$J^r Y \ast i_{\xi} \omega = 0,$$  

where $\omega$ is any representative of order $s$ of $[\omega]$. Eqs. (1) are called Euler–Lagrange equations corresponding to the Lagrangian system $[\omega]$.

A Hamiltonian system of order $s$ is given by a Lepagean $(n+1)$-form $\omega_0$ on $J^r Y$. A section of the fibered manifold $\pi^s$ is called a Hamilton extremal of $\omega_0$ if

$$\delta^s \omega_0 = 0,$$

where $\delta^s$ is any representative of order $s$ of $[\omega_0]$. Eqs. (2) are called Hamilton equations of $\omega_0$. If there exists an at most $k$-contact Lepagean $n$-form $\rho$, $1 \leq k \leq n$, in a neighborhood of every point in $J^r Y$ such that $\omega = d \rho$, we call Eqs. (2) Hamilton $p_k$-equations.

For a Lagrangian $\lambda = L \omega_0 \in \Omega^n_{1, r} W$ which is singular in the standard sense, i.e. the regularity condition $\det(\partial^2 \mathcal{L} / \partial y^\sigma_i \partial y^\sigma_j) \neq 0$ at each point of $W^1$ is not satisfied, we can consider its simple regularization $\rho \in \Omega^n_{1, r} W$, defined as a Lepagean equivalent of $\lambda$, such that $\rho$ is at most 2-contact, $p_2 \rho = p_2 \beta$ for a $\pi^{1,0}$-projectable form $\beta \in \Omega^n_1 W$, i.e. using a fibered chart

$$\rho = L \omega_0 + \frac{\partial \mathcal{L}}{\partial y^\sigma_i} y^\sigma_i \wedge \omega_i + g^i_{\sigma \nu} \eta^\sigma \wedge \eta^\nu \wedge \omega_i,$$

where $g^i_{\sigma \nu} = -g^i_{\nu \sigma}$, $g^i_{\sigma \nu}$ are functions on $W$ and $\rho$ satisfies the following regularity condition on $W^1$

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial y^\sigma_i \partial y^\sigma_j} - 4g^i_{\sigma \nu} \right) \neq 0,$$

where $(\sigma, i)$ labels rows and $(\nu, j)$ labels columns. The next statements were proved in [10,11].

**Theorem 2.1.** Let $\rho$ be the regularization of $\lambda$ as above (3) and $\alpha = d \rho$.

1. Then every Hamilton extremal $\delta = d \rho$ is of the form $\delta = J^1 Y$, where $\gamma$ is an extremal of $E_\lambda = p_1 d \rho$. 

2. If $\rho$ is a $\pi^{1,0}$-projectable form, then $\rho$ is regular at every point of $W^1$.

3. If $\rho$ is a $\pi^{1,0}$-projectable form, then $\rho$ is a $\pi^{1,0}$-horizontal form.
(2) Put
\[ p^i_\sigma = \frac{\partial \mathcal{L}}{\partial \dot{y}_i} - 4g^{ij}_i y_j. \] (4)

Then \((x^i, y^\sigma, y_j^\sigma) \rightarrow (x^i, y^\sigma, p^i_\sigma)\) is a coordinate transformation on \(W^1 \subset \mathcal{V}^1\).

(3) Put
\[ \mathcal{H} = -\mathcal{L} + p^i_\sigma y_i^\sigma + 2g^{ij}_i y_i^\sigma y_j^\sigma. \] (5)

Then Hamilton \(p_2\)-equations of \(\alpha\) take in the coordinates \((x^i, y^\sigma, p^i_\sigma)\) the following form
\[
\frac{\partial y^\sigma}{\partial x^i} = \frac{\partial \mathcal{H}}{\partial p^i_\sigma}, \quad \frac{\partial p^i_\sigma}{\partial x^i} = -\frac{\partial \mathcal{H}}{\partial y^\sigma}.
\]

If the n-form \(g^{ij}_i\) dy^\sigma \wedge dy^\nu \wedge \omega_j\) is closed then the Hamilton \(p_2\)-equations take the standard form
\[
\frac{\partial y^\sigma}{\partial x^i} = \frac{\partial \mathcal{H}}{\partial p^i_\sigma}, \quad \frac{\partial p^i_\sigma}{\partial x^i} = -\frac{\partial \mathcal{H}}{\partial y^\sigma}.
\]

We shall call \(p^i_\sigma\) in (4) momenta and \(\mathcal{H}\) in (5) the Hamiltonian of \(\alpha\), and the coordinates \((x^i, y^\sigma, p^i_\sigma)\) are called Legendre coordinates and the corresponding coordinate transformation Legendre transformation. From the first part of Theorem 2.1 we see that there is a bijection between Hamilton extremals of \(\alpha\) and extremals of \(E_\lambda\) and we say that Hamilton and Euler-Lagrange equations are equivalent.

3. Hamiltonian Yang–Mills theory

We shall analyze a Hamiltonian system for the Yang–Mills theory. Let \(C\) be the bundle of principal connections on the principal fiber bundle \((P, p, X, G)\) over a spacetime \((X, g)\). The Yang–Mills Lagrangian \(\lambda\) on \(J^1C\) is in fibered coordinates \((x^i, \Gamma^i_j, \Gamma^i)\) on \(C\) given by
\[ \lambda = \mathcal{L}_{YM} \omega_0 = -\frac{1}{4} R^i_{jkl} y^k y^l - \frac{1}{4} \pi^i_{jkl} \sqrt{g} \omega_0, \]
where \(g_{ij}\) are the components of a Lorentzian metric \(g\), we denote \(g = |\det g_{ij}|\) for simplicity, \(R^i_{jkl} = \Gamma^i_{j(k,l)} + c^j_{QR} \Gamma^j_Q \Gamma^i_{Rl}\) is the curvature (field strength) of the principal connection (Yang–Mills field) \(\Gamma^i_{j}p\) and \(h\) denotes an Ad-invariant form on the Lie algebra \(g\), i.e. invariant with respect to the adjoint representation \(Ad\) of the Lie group \(G\) in the sense that for all \(g \in G, X, Y \in g\) the relation \(h(Ad_g(X), Ad_g(Y)) = h(X, Y)\) holds, \(h_{RQ}\) are the components of an Ad-invariant form on the Lie algebra \(g\). In particular, we suppose the Lorentzian metric \(g\) to be fixed from the beginning (compare with [12,13]).

Since the Yang–Mills Lagrangian is singular in the standard sense for any Ad-invariant form \(h\) and any spacetime, we try to regularize it using the Lepagean equivalent (3) with \(g^i_j\)’s given by
\[ g^{ij}_i = \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} - \frac{\partial^2 \mathcal{L}}{\partial y^j \partial y^i}. \] (6)

Let us introduce the following notation:
\[ f^{ijkl}_{ij} = \sqrt{g} h_{RQ} (4g^{ij}_i g^{jk} - g^{il}_i g^{jk} - g^{ik}_i g^{jl}_j) \] and its inverse \(f^{-1}_{ijkl} = f^{-1}_{ij} Q_{OP}\), i.e.
\[ f^{-1}_{ijkl} = \sqrt{g} h_{RQ} (g^{il}_i g^{jk} - g^{ik}_i g^{jl}_j), \]
\[ a^{ij}_{CD} = \frac{1}{4} \sqrt{g} h_{RQ} g^{ik}_i (f^{-1}_{jCD} - f^{-1}_{ij} d \xi^D) (f^{-1}_{iCD} - f^{-1}_{kD} d \xi^D) + 2g^{ijkl} f^{-1}_{ij} f^{-1}_{kD} d \xi^D. \]

We denote the contact forms by \(\eta^i = d\Gamma^i_{j} - \Gamma^i_{j} d x^j\). Furthermore, we denote by
\[ \nabla \Gamma^i_{j} = d\Gamma^i_{j} + c\Gamma^i_{j} \Gamma^j_{R} R^j_{R} + I_{ij} R^i_{j}. \]

the components of the covariant derivative (with respect to a prolongation [4] of the principal connection with respect to the Levi-Civita connection), the Christoffel symbols are given by \(\Gamma^i_{jk} = (1/2) g^{ij}_i (g_{ij,k} + g_{sk,j} - g_{jk,s})\). The next result follows from Theorem 2.1.
Theorem 3.1. The Lepagean equivalent ρ given by (3) and (6) of the Yang–Mills Lagrangian with a symmetric regular Ad-invariant form h has a chart expression:

\[ ρ = -\frac{1}{4} R^i_{jk} R^j_{kl} \sqrt{g} \omega_0 + R^i_{jk} \sqrt{g} \eta^Q_i \wedge ω_j + \sqrt{g} h_{PQ} (g^{ij} g^{jk} - g^{ij} g^{jk}) \eta^P_i \wedge \eta^Q_k \wedge ω_{ji} \]

and it is its regularization. The momenta are given by

\[ p^i_{ij} = f^i_{PQ} R^Q_{ij} + \sqrt{g} h_{PQ} g^{ik} g^{jl} \eta^Q_k \wedge \eta^Q_l \wedge ω_{ji} \]

The Hamiltonian in the Legendre coordinates \((x^i, \Gamma^P_i, p^i_{ij})\) is given by

\[ H = (a^{OA}_{cdef} + f^{-1} A^{OA}) P^d P^e f_{ij} h_{AW} c_{ij} W (a^{OA}_{cdef} + a^{AO}_{efcd} + f^{-1} A^{OA} + f^{-1} A^{OA}_{efcd}) P^d P^e \Gamma^P_i \Gamma^P_j \]

\[ + \left( (a^{OA}_{cdef} + f^{-1} A^{OA}) g h_{AW} h_{OT} g^{ij} g^{kl} a^{AB} g^{cd} + \frac{1}{4} \sqrt{g} h_{TW} g^{ag} g^{bh} \right) c_{ij} W (a^{OA}_{cdef} + a^{AO}_{efcd} + f^{-1} A^{OA}_{efcd}) P^d P^e \Gamma^P_i \Gamma^P_j \]

The Yang–Mills equations \(\nabla_j R^i_{jk} = 0\) and the Hamilton p2-equations of \(d \rho\)

\[ \frac{\partial \Gamma^P_i}{\partial x^j} = \frac{\partial H}{\partial p^i_{ij}}, \quad \frac{\partial p^i_{ij}}{\partial x^j} = \frac{\partial H}{\partial \Gamma^P_i} + 4 \frac{\partial g^{ijkl}}{\partial x^k} \frac{\partial H}{\partial p^l_{ij}} \]

are equivalent.

A direct calculation shows that the regularization ρ is invariant under changes of fibered coordinates on C. We also note that for electromagnetism \((G = U(1))\) the momenta and the Hamiltonian simplifies to \(p^i_{ij} = f^i_{ij} \Gamma^P_i\). \(H = (a^{OA}_{cdef} + f^{-1} A^{OA}) P^d P^e f_{ij} \), which agrees with [11]. For the Minkowski spacetime the Yang–Mills equations \(d_j R^i_{jk} + R^i_{jk} \eta^P_k \wedge \eta^P_j = 0\) are equivalent to the Hamilton equations

\[ \frac{\partial \Gamma^P_i}{\partial x^j} = \frac{\partial H}{\partial p^i_{ij}}, \quad \frac{\partial p^i_{ij}}{\partial x^j} = \frac{\partial H}{\partial \Gamma^P_i} \]

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References