The Lie Algebra of Homeomorphisms of the Circle

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We define and study an infinite-dimensional Lie algebra $\text{homeo}_+$ which is shown to be naturally associated to the topological Lie group $\text{Homeo}_+$ of all orientation-preserving homeomorphisms of the circle. Roughly, we rely on the universal decorated Teichmüller theory developed before as motivation to provide Fréchet coordinates on the homogeneous space given by $\text{Homeo}_+$ modulo the group of real fractional linear transformations, whose corresponding vector fields on the circle we then extend by the usual Lie algebra $\mathfrak{sl}_2$ of real traceless two-by-two matrices in order to define $\text{homeo}_+$. Surprisingly, $\text{homeo}_+$ turns out to be equal to the algebra of all vector fields on the circle which are “piecewise $\mathfrak{sl}_2$” in the obvious sense. It is evidently important to consider the relationship between our new Fréchet coordinates and the usual trigonometric functions on the circle, and we undertake here both natural infinitesimal calculations. We finally apply some further previous work in order to give sufficient conditions on the Fourier coefficients of a certain class of homeomorphisms of the circle which arises naturally in topology and number theory.

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INTRODUCTION

Perhaps the most naive attempt to build a Lie algebra for the group $\text{Homeo}_+$ of orientation-preserving homeomorphisms of the circle is to proceed topologically in analogy to the case (see [PS] for instance) of infinitely smooth orientation-preserving diffeomorphisms $\text{Diff}_+$ of the circle, namely, take the algebra of continuous vector fields on the circle and
hope for some manner of defining a bracket extending the well-known bracket for diffeomorphisms. Of course as is also well-known, this fails because “bracketing destroys one degree of smoothness,” so the bracket of continuous vector fields will fail to be continuous. Thus, something different from the obvious topological analogy with $\text{Diff}^+_+$ must be employed.

Another naive idea (that is not too far from our actual approach) is to begin with a collection of one-parameter families in $\text{Homeo}^+$ whose corresponding tangent vectors are seen from this or that point of view to be a topological basis for the desired Lie algebra. For instance, Thurston’s earthquakes (cf. [Th] or [Ke]) are “piecewise Möbius” on the circle (the “Möbius group” $\text{Möb}$ is the group of projective two-by-two matrices invertible over the reals) and therefore lead to continuous (but not differentiable) vector fields on the circle which are piecewise $\text{sl}_2$, where $\text{sl}_2$ denotes the usual algebra of real two-by-two traceless matrices. We might bracket such piecewise $\text{sl}_2$ vector fields on the circle in the obvious way (ignoring their values at finitely many convenient points) in an attempt to define a Lie algebra using earthquakes. Unfortunately, the expression for brackets of such vector fields on the circle as sums of such vector fields appears to be prohibitively complicated.

Here then is what we do. The universal decorated Teichmüller theory was developed in [P1] and provides a natural Fréchet manifold structure to the homogeneous space

$$\mathcal{Tess} = \text{Homeo}^+ / \text{Möb}$$

which is our model of a universal Teichmüller space (and is surveyed for our purposes here in Section 3 below). Indeed, there is furthermore a certain bundle $\mathcal{Tess}$ over $\mathcal{Tess}$ together with global affine coordinates called “lambda lengths”, and the corresponding coordinate deformations are again piecewise Möbius on the circle and lead to piecewise $\text{sl}_2$ vector fields called “elementary vector fields”. In this case, though, the elementary vector fields turn out to be once continuously differentiable on the circle.

Furthermore, the naive bracket described previously is tractable this time in the following sense. We take certain infinite linear combinations of elementary vector fields called “(left) fans” as well as certain infinite linear combinations of fans called “(left) hyperfans”. These vector fields have remarkable telescoping properties, and moreover, our main algebraic result here is that the vector space $\text{homeo}^+$ generated by $\text{sl}_2$ together with the set of all hyperfans “closes” under the naive bracket above. The facts that the elementary vector fields are once continuously differentiable and that the algebra of hyperfans closes together allow us to avoid the well-known
quantum mechanical paradoxes involved with bracketing point densities; indeed, our algebra contains certain Heaviside step functions but no Dirac delta functions.

In effect, our main algebraic result gives an algorithm for calculating brackets of left hyperfans with other left hyperfans as linear combinations of left hyperfans and elements of $\mathfrak{sl}_2$. This fact that $\text{homeo}_+$ closes together with simplicity of $\mathfrak{sl}_2$ turn out to imply that $\text{homeo}_+$ actually agrees with the algebra $\mathfrak{psl}_2$ of all piecewise $\mathfrak{sl}_2$ vector fields (where there are only finitely many pieces which are furthermore required to have rational endpoints). Thus, $\mathfrak{psl}_2$ admits the surprisingly simple basis consisting of the standard generators for $\mathfrak{sl}_2$ together with all hyperfans, and brackets in this basis may be calculated from our algorithm.

Just as earthquakes are the basic deformations of Thurston’s theory, here we discover other basic deformations, namely, hyperfans, which are suited to Lie theory; in effect, whereas earthquakes are “piecewise hyperbolic”, hyperfans are “piecewise parabolic”. Thus, both earthquakes and hyperfans arise naturally from considerations of hyperbolic geometry, but they are sufficiently disparate that it is not surprising that $\text{homeo}_+$ has not been discovered previously in the context of earthquakes.

In fact, it turns out that fans or hyperfans are actually indexed by elements of the “modular group” of projective two-by-two matrices invertible over the integers. Indeed, rather than rely upon the topological structure of the circle, our constructions depend instead upon a certain “rational” structure of the circle associated with the action of the modular group. Another fundamental aspect of our constructions is the canonical “regularization” of brackets we employ here arising from our “normalization” of homeomorphisms, which is again a manifestation of the rational structure of the circle.

Our investigations actually began with the explicit calculation in lambda lengths of certain group commutators in $\text{Homeo}_+$. Indeed, a finite but formidable collection of such calculations was originally undertaken symbolically on the computer using Mathematica [Wo] on a PowerMac 7100/66. The simplest such calculation is called the “model calculation”, and we reproduce this calculation here by hand (i.e., without the computer) for the insight it affords.

This paper is organized as follows. Section 1 establishes notation and gives the definition of $\text{homeo}_+$ as a vector space while the calculation of brackets which provides our main algebraic result is given in Section 2. This much of the paper is purely algebraic. We turn in Section 3 to a survey of [P1] for application in Section 4, where we give the model calculation. We undertake a study of the relationship between our new coordinates and the usual trigonometric functions on the circle in Section 5, and in particular (applying some previous joint work of the second-named author
with Dennis Sullivan), we discuss the harmonic analysis of a certain family of homeomorphisms of the circle arising in number theory (namely, the Nielsen extensions to the circle of the lift to the universal cover of any homeomorphism of a punctured surface which is uniformized by some finite-index subgroup of the modular group). Finally, Section 6 contains various speculative closing remarks.

Our algebra $\text{homeo}_+$ may or may not turn out to be a useful extension of harmonic analysis, but in any case, we believe that $\text{homeo}_+$ is important as an example of a naturally occurring infinite-dimensional Lie algebra which seems to share various attributes of $W$-algebras, the Virasoro algebra, and double loop algebras (cf. Section 6 below for a further discussion).

1. THE VECTOR SPACE $\text{homeo}_+$

Regard the Poincaré disk as the open unit disk $D$ in the complex plane in the usual way so that the unit circle $S^1$ is identified with the circle at infinity, and let $T$ denote the ideal hyperbolic triangle with vertices $+1$, 

![Figure 1](image-url)
Let $\Gamma$ denote the group generated by reflections in the sides of $T$, and define the Farey tessellation $\tau_*$ to be the full $\Gamma$-orbit of the frontier of $T$. We refer to geodesics in $\tau_*$ as edges of $\tau_*$ and think of $\tau_*$ itself as a set of edges. As is well-known (see for instance [P1]), the ideal vertices of the edges of $\tau_*$ are naturally identified with the set $\mathbb{Q} \cup \{\infty\}$ of rational numbers, where for instance $+1, -1, -\sqrt{-1} \in S^1$ correspond respectively to $\infty = \frac{1}{1}, 0 = \frac{0}{1}, 1 = \frac{1}{1}$ as illustrated in Fig. 1. Let $\mathbb{Q} \cup \{\infty\} \subseteq S^1$ denote the corresponding countable dense subset of $S^1$ which we refer to simply as the set of rational points $S^1$. Define the distinguished oriented edge or doe of the Farey tessellation to be the oriented edge from $0\ 1$ to $1\ 0$.

The modular group $\text{PSL}_2 = \text{PSL}_2(\mathbb{Z})$ of integral fractional linear transformations is just the subgroup of $\Gamma$ consisting of compositions of an even number of reflections, and $\text{PSL}_2$ acts simply transitively on the set of orientations on the edges of $\tau_*$. We shall take this to be a right action on $\tau_*$ itself, so

$$\tilde{e}_A = (\text{doe} \cdot A, \text{ for } A \in \text{PSL}_2),$$

establishes a bijection between $\text{PSL}_2$ and the set of oriented edges of $\tau_*$. In particular, the doe of $\tau_*$ is $\tilde{e}_I$, where $I = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \in \text{PSL}_2$.

We adopt the standard notation

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for certain elements of $\text{PSL}_2$, where $S$ is involutive and fixes the unoriented edge of $\tau_*$ underlying the doe while changing its orientation, and $U$ (respectively $T$) is the parabolic transformation with fixed point $\frac{1}{1}$ (respectively $\frac{1}{0}$) which cyclically permutes the incident edges of $\tau_*$ in the counter-clockwise sense about $\frac{1}{0}$ (respectively the clockwise sense about $\frac{1}{1}$). Of course, $U^{-1} = STS$, $T^{-1} = SUS$, and any two of $S$, $T$, $U$ generate $\text{PSL}_2$.

We shall also require the full Möbius group $\text{Möb} = \text{PSL}_2(\mathbb{R}) \supseteq \text{PSL}_2(\mathbb{Z}) = \text{PSL}_2$ consisting of all real fractional linear transformations.

Let $sl_2 = sl_2(\mathbb{R})$ denote the usual Lie algebra of $SL_2(\mathbb{R})$ consisting of traceless two-by-two real matrices with standard generators

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}.$$
and brackets $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. We may exponentiate an element of $sl_2$ to get a one-parameter subgroup in $SL_2(\mathbb{R})$ which we regard as acting smoothly on $D \cup S^1$. The induced one-parameter family of diffeomorphisms of $S^1$ gives rise to a vector field on $S^1$ in the usual way. A direct calculation (which we omit and which differs from [P1; Lemma 3.5] since we use the right action here) proves

**Lemma 1.** The vector field corresponding to $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in sl_2$ is given by

$$\{(b + c) \cos \theta + (a - d) \sin \theta + (c - b)\} \frac{\partial}{\partial \theta},$$

where $\theta$ is the usual angular coordinate on the circle.

Thus, we refer to a vector field $\mathcal{V}$ on the circle as a (global) $sl_2$ vector field if it arises as above from some element of $sl_2$, and we write $\mathcal{V} \in sl_2$ in this case.

More generally, we say that a vector field $\mathcal{V}$ on $S^1$ is a finite piecewise $sl_2$ vector field provided $S^1$ can be decomposed into finitely many open connected sets with pairwise disjoint interiors whose endpoints (if any) are rational so that $\mathcal{V}$ restricts on the interior of each such set to some global $sl_2$ vector field, and we write $\mathcal{V} \in psl_2$ in this case (where the “p” stands for “piecewise”). The endpoints of the (maximal such) intervals are called the breakpoints of $\mathcal{V}$ itself; in particular, $sl_2$ sits inside $psl_2$ as the collection of elements which have no breakpoints. At this stage of our discussion, there are no further restrictions on the behavior of $\mathcal{V}$ at its breakpoints, and indeed, $\mathcal{V} \in psl_2$ need not even be defined at its breakpoints.

A bracket $[\mathcal{V}_1, \mathcal{V}_2] \in psl_2$ of $\mathcal{V}_1, \mathcal{V}_2 \in psl_2$ is defined in the natural way, i.e., the resulting vector field has breakpoints given by the union of the breakpoints of $\mathcal{V}_1$ with those of $\mathcal{V}_2$, and on each complementary component of these breakpoints the bracket is given by the usual bracket on $sl_2$. This completes the definition on the Lie algebra $psl_2$.

Geometric considerations from [P1] which will be discussed further in Section 3 below lead directly to the extremely special element of $psl_2$ illustrated in Fig. 2, to be denoted $\mathcal{V} \in psl_2$, and defined as follows: $\mathcal{V}$ has breakpoints $\pm 1, \pm \sqrt{-1} \in S^1$ (so the complementary intervals lie in respective quadrants I–IV in the complex plane enumerated as usual in the counterclockwise sense starting from quadrant I where both coordinates are positive), and

$$\mathcal{V} = \begin{cases} 
 h + 2e, & \text{in quadrant I,} \\
 -h + 2f, & \text{in quadrant II,} \\
 -h - 2f, & \text{in quadrant III,} \\
 h - 2e, & \text{in quadrant IV.}
\end{cases}$$
One checks directly that $\mathcal{H}$ in fact vanishes at each of its breakpoints, and we shall find in Section 3 below that $\mathcal{H}$ is actually the tangent vector field of a certain geometrically natural one-parameter subgroup of once continuously differentiable homeomorphisms of the circle. $\mathcal{H}$ is itself therefore a once continuously differentiable vector field on the circle.

Notice that in general a breakpoint may or may not be a point of smoothness or even continuity for a given element of $\text{psl}_2$, and $\mathcal{H}$ above is actually once continuously differentiable at each of its breakpoints in light of our remarks in the previous paragraph. Furthermore, notice that in general if an element of $\text{psl}_2$ is actually twice continuously differentiable at a putative breakpoint $t \in S^1$, then $t$ is not in fact a breakpoint at all, i.e., if two Möbius transformations agree to second order at a point, then they actually coincide.

Having thus defined $\mathcal{H} \in \text{psl}_2$, we proceed to define

$$\mathcal{H}_A = A^{-1} \mathcal{H} A,$$

for $A \in \text{PSL}_2$.

Using the adjoint action on each piece. A short calculation shows that

$$\mathcal{H}_e = \mathcal{H},$$

and therefore if $A_+, A_- \in \text{PSL}_2$ correspond to the two different orientations on a common underlying edge of $\tau_e$, i.e., if $A_\pm = S A_\pm$, then $\mathcal{H}_{A_+} = \mathcal{H}_{A_-}$.

Thus, if $e \in \tau_e$ is an unoriented edge, then we may associate a well-defined $\mathcal{H}_e = \mathcal{H}_{A_+} \in \text{psl}_2$, where $A_+$ maps the unoriented edge underlying the doe to the edge $e$ of $\tau_e$; we call $\mathcal{H}_e$ the elementary vector field corresponding to $e \in \tau_e$.

Notice that an elementary vector field $\mathcal{H}_e \in \text{psl}_2$ is defined everywhere on $S^1$ (i.e., even at its breakpoints) since $\mathcal{H}$ itself has this property in light of remarks above. It thus makes sense to define the normalization $\mathcal{H}_e$ of $\mathcal{H}_e$ to
be $\mathfrak{g}_{\mathfrak{A}} - x \in \mathfrak{psl}_2$, where $x \in \mathfrak{sl}_2$ is chosen so that $\mathfrak{g}_x = x$ at the vertices of the triangle $T$ above, i.e., at the points $\pm 1, -\sqrt{-1} \in S^1$ (or at $\frac{2\pi}{3}, 1, \frac{1}{3}$ if you prefer the Farey enumeration). It is clear that $x$ is uniquely determined by $\mathfrak{g}_x$, so the normalization is well-defined.

For some examples, one calculates directly that

$$\tilde{\mathfrak{g}}_{U^N} = \mathfrak{g}_{U^N} + h - 2(n - 1)f,$$

and furthermore

$$\mathfrak{g} = \tilde{\mathfrak{g}}$$

is already normalized (indeed, it vanishes at $+\sqrt{-1}$ as well, i.e., at $-\frac{1}{3}$ in the Farey enumeration).

Now, for each oriented edge $\tilde{\mathfrak{e}}_A = \tilde{\mathfrak{e}}_I$ of $\tau_{\mathfrak{A}}$, we next define two vector fields

$$\phi_A = \sum_{n \geq 0} \tilde{\mathfrak{g}}_{U^N A},$$

$$\phi^*_A = \sum_{n \leq 0} \tilde{\mathfrak{g}}_{U^N A},$$

respectively called a left fan and right fan. The oriented edge $\tilde{\mathfrak{e}}_A$ is called the edge of the fan, and its initial point (in the given orientation) is called the pin of the fan in either case. Because of the normalization, the infinite sums $\phi_A$ and $\phi^*_A$ converge pointwise to vector fields on the circle except perhaps at the pin, and the convergence is uniform on each compactum not containing the pin. On the other hand, there is no a priori algebraic reason that these vector fields lie in $\mathfrak{psl}_2$, i.e., have only finitely many pieces (though there are actually geometric reasons for this, as described in Section 4 below), yet direct calculation (relying on the expression above for $\mathfrak{g}_{U^N}$) yields

**Lemma 2.** As illustrated in Fig. 3, we have

$$\phi_U = \begin{cases} -2e, & \text{on quadrant } I, \\ 2h - 2f, & \text{on quadrant } II, \\ 0, & \text{on quadrants } III \text{ and } IV. \end{cases}$$

$$\phi^*_{U^{-1}} = \begin{cases} -h, & \text{on quadrants } I \text{ and } II, \\ 2f + h, & \text{on quadrant } III, \\ 2e - h, & \text{on quadrant } IV. \end{cases}$$
Further direct calculation using Lemma 2 and the definition of $\mathcal{G}$ proves that

$$\phi_U + \phi_{U^{-1}} + \mathcal{G} = 0,$$

and it follows that any right fan is itself the linear combination of a left fan and an elementary vector field. In light of this, we make the choice here and hereafter to use left fans (instead of right fans) in our calculations.

Another direct calculation from the definitions proves that

$$\phi_I - \phi_U = \mathcal{G} = \mathcal{G},$$

and it follows that any elementary vector field is itself a difference of two left fans. Indeed, we also have

$$\phi_S - \phi_{US} = \mathcal{G} = \mathcal{G},$$

so each elementary vector field can actually be expressed as a difference of left fans in two distinct ways, i.e.,

$$\phi_A - \phi_{UA} = \phi_{SA} - \phi_{USA}, \quad \text{for} \ A \in \text{PSL}_2.$$

Summarizing some of these observations we have

**Lemma 3.** The span in $\text{psl}_2$ of the set of left fans $\{\phi_A : A \in \text{PSL}_2\}$ contains every right fan as well as every elementary vector field. There is furthermore a relation for each edge of $\tau_+$, namely, if $A \in \text{PSL}_2$, then we have

$$\phi_A - \phi_{UA} = \phi_{SA} - \phi_{USA}.$$
For application in the following section, we next record the expression as explicit elements of $\text{psl}_2$ of a particular fan collection.

**Lemma 4.** We have

$$\phi_T = \begin{cases} 
2e, & \text{on quadrant I,} \\
-2f, & \text{on quadrant II,} \\
0, & \text{on quadrants III and IV;} 
\end{cases}$$

$$\phi_{SU^{-1}} = \begin{cases} 
-h, & \text{on quadrants I and II,} \\
-h - 2f, & \text{on quadrant III,} \\
-h + 2e, & \text{on quadrant IV;} 
\end{cases}$$

$$\phi_{US} = \begin{cases} 
h, & \text{on quadrants I and II,} \\
h + 2f, & \text{on quadrant III,} \\
-h + 2e, & \text{on quadrant IV.} 
\end{cases}$$

Continuing now with the definitions, for each $A \in PSL_2$, consider the two vector fields

$$\psi_A = \sum_{n \geq 0} n\tilde{3}_A$$

$$\psi_A^* = \sum_{n < 0} n\tilde{3}_A$$

respectively called a *left hyperfan* and *right hyperfan*. Notice that hyperfans are to fans as fans are to elementary transformations in the sense that we may also evidently write

$$\psi_A = \sum_{n \geq 1} \phi_{U^nA}$$

$$\psi_A^* = \sum_{n \leq -1} \phi_{U^nA}.$$

Again, the oriented edge $\tilde{e}_A = \tilde{e}_T \cdot A$ is called the *edge of the hyperfan*, and its initial point is called the *pin of the hyperfan* in either case.

Again because of the normalization, the infinite sums $\psi_A$ and $\psi_A^*$ converge pointwise to vector fields on the circle except perhaps at the pin, and the convergence is uniform on each compactum not containing the pin. On the other hand, there is no *a priori* algebraic reason that these vector fields lie in $\text{psl}_2$ (and this time there are not even geometric reasons for it!), yet direct calculation again yields.
Lemma 5. As illustrated in Fig. 4, we have

\[ \psi_f = \begin{cases} -2e, & \text{on quadrants I and II}, \\ 0, & \text{on quadrants III and IV}. \end{cases} \]

\[ \psi_f^* = \begin{cases} -h, & \text{on quadrants I and II}, \\ 2e - h, & \text{on quadrants III and IV}. \end{cases} \]

Just as we have restricted our attention to left fans, we shall now restrict our attention to left hyperfans. Of course, there is an exactly parallel version of the subsequent theory using right fans and hyperfans rather than left ones as we do here.

Notices that any two left hyperfans are conjugate modulo an element of \( \text{sl}_2 \), i.e.,

\[ \psi_{AB} - B^{-1}\psi_A B \in \text{sl}_2, \quad \text{for } A, B \in \text{SL}_2. \]

The reason this difference does not vanish in general is that a hyperfan is a linear combination of normalized elementary vector fields and the latter, unlike elementary vector fields, are only conjugate modulo an element of \( \text{sl}_2 \). Of course, an analogous remark holds for left fans as well.

It follows from this remark together with Lemma 5 that the hyperfan \( \psi_A \), for \( A \in \text{PSL}_2 \), is described up to a global \( \text{sl}_2 \) vector field as follows. If the triangle to the right of the doe lies to the right of the oriented edge \( \tilde{e}_A \), then

\[ \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \]

FIGURE 4
vanishes on the right of \( \tilde{e}_A \), while on the left, \( \psi_A \) is the vector field which integrates to the parabolic transformation fixing the terminal point of \( \tilde{e}_A \) and rotating counter-clockwise around this terminal point in positive time; more precisely, \( \psi_A \) integrates to the square of the primitive such parabolic in \( \text{PSL}_2 \). On the other hand, if the doe lies to the left of \( \tilde{e}_A \), then \( \psi_A \) vanishes to the left of \( \tilde{e}_A \), while on the right, it is the negative of the vector field just described.

In our previous lemmas we have only asserted the existence of certain linear relations, and in contrast, the next result completely describes the linear dependences among elementary transformations, left fans, and left hyperfans in \( \text{PSL}_2 \).

**Proposition 6.** The span in \( \text{PSL}_2 \) of the set \( \{ \psi_A : A \in \text{PSL}_2(\mathbb{Z}) \} \) of left hyperfans contains every left fan as well as every elementary transformation. For each unoriented edge \( e \in \tau_A \) where \( A \in \text{PSL}_2 \) maps the doe to an orientation on \( e \), there is the corresponding relation

\[
R_e : (\psi_{U_1 A} - 2\psi_A + \psi_{U A}) = (\psi_{U_1 A} - 2\psi_{A} + \psi_{U A}),
\]

which is illustrated in Fig. 5. The relations \( \{ R_e : e \in \tau_A \} \) are linearly independent, and they span the space of (finite) linear dependences among left hyperfans.

**Proof.** That the span of left hyperfans contains each left fan is proved as in Lemma 3 by the analogy between fans and hyperfans, and it follows then from Lemma 3 that this span moreover contains all elementary transformations. Furthermore, the relation \( R_e \) is just the expression in left hyperfans of the relation in Lemma 3, and the main point requiring proof in Proposition 6 is thus the last sentence.
We next prove that the given relations span the linear dependences among left hyperfans. We shall consider ideal polygons in $\mathbb{D}$ whose frontier edges lie in $\tau_a$ and shall call them “sub-polygons” of $\tau_a$ itself. Of course, any such polygon inherits an ideal triangulation from $\tau_a$. Furthermore, an elementary inductive argument (which we omit) shows that for any triangulation of a (connected) polygon, there are at least two triangles that have only one frontier edge in the interior of the polygon.

Given any formal finite linear combination of left hyperfans, we say that a connected sub-polygon $P$ of $\tau_a$ carries the linear combination if the edge of each hyperfan occurring in the linear combination lies in $P$. Our proof is basically by descent on the number of sides of a minimal polygon carrying a putative relation which is independent of the relations $\{R_e : e \in \tau_a\}$.

To begin, let us study the possibility of relations among left hyperfans carried by a standard quadrilateral. It will be convenient and sufficient for us to work modulo global $sl_2$ vector fields, and we shall write $\mathcal{A} \equiv \mathcal{B}$ if $\mathcal{A} - \mathcal{B} \in sl_2$ for $\mathcal{A}, \mathcal{B} \in psl_2$. In fact in the interests of efficient presentation, we go a bit further and write, for instance,

$$\psi_I = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \text{ on I, II to abbreviate}$$

$$\psi_S = \begin{pmatrix} 0 \\ -2 & 0 \\ 0 \end{pmatrix} \text{ on I, II,}$$

$$\psi_{U^+} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \text{ on I, II,}$$

$$\psi_{S_U^+} = \begin{pmatrix} 0 \\ -2 & 0 \\ 0 \end{pmatrix} \text{ on II,}$$

$$\psi_{U^-} = \begin{pmatrix} -2 \\ 0 & 2 \\ -2 \\ 0 \end{pmatrix} \text{ on III,}$$

$$\psi_{S_U^-} = \begin{pmatrix} 0 \\ -2 & 0 \\ 0 \end{pmatrix} \text{ on III,}$$

$$\psi_{U^+S} = \begin{pmatrix} +2 \\ -2 & -2 \\ +2 \end{pmatrix} \text{ on IV,}$$

$$\psi_{S_U^+S} = \begin{pmatrix} 0 \\ +2 \end{pmatrix} \text{ on IV,}$$

$$\psi_{U^-S} = \begin{pmatrix} +2 \\ -2 & +2 \\ -2 \end{pmatrix} \text{ on I,}$$

$$\psi_{S_U^-S} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \text{ on I.}$$

Tactily taking the specified quadrants as the support of the vector field. In a hybrid of these notations and in light of remarks above, we find that
Armed with these results, direct and routine calculations (which we omit) establish the following two facts:

- There are no non-trivial linear combinations of left hyperfans carried by the triangle to the left of the doe which give a global $sl_2$ vector field.

- There is a unique non-trivial linear combination, namely, the relation above, among the left hyperfans carried by the quadrilateral comprised of the triangles on either side of the doe which gives a global $sl_2$ vector field.

It then follows from the fact that any two hyperfans are conjugate modulo an element of $sl_2$ that there are no relations carried by any sub-triangle of $\tau_\ast$, and furthermore, the only relation carried by any sub-quadrilateral of $\tau_\ast$ is the relation discussed above. These facts together constitute the basis step of our induction.

For the induction, suppose we have a non-trivial finite linear combination of left hyperfans which gives a global $sl_2$ vector field. Of course, since we take a finite linear combination, there must be some finite-sided sub-polygon of $\tau_\ast$ carrying the putative relation, and for each such linear relation, we can take a polygon $P$ with a minimal number of sides. Let us also choose a linear combination which minimizes the number of sides of such a minimal polygon $P$ and induct on the number $n$ of sides of $P$. The basis step $n = 3, 4$ has already been discussed.

Suppose then that $P$ is a minimal polygon with $n \geq 5$ sides carrying a linear relation independent of $\{R_e : e \in \tau_\ast\}$. As remarked above, in the induced triangulation of $P$, there are at least two triangles which have only one frontier edge interior to $P$, and we choose one such triangle, call it $T$. Again using that any two hyperfans are conjugate modulo an element of $sl_2$, we may assume that $T$ coincides with the triangle to the left of the doe and its two frontier edges in common with $P$ are the edges underlying the doe $e^\ast_I$ itself and $e^\ast_U$. Since $P$ is minimal, at least one of $\psi_U, \psi_{SU}, \psi_I, \psi_S$ must have a non-vanishing coefficient in the putative linear relation. Serial consideration of the various cases (which we omit) using the expressions above for these hyperfans shows that $-1 \in S^1$ (i.e., $\frac{1}{2}$ in the Farey enumeration) is necessarily a breakpoint (and this is in contradiction to the assumption that the putative linear combination is a global $sl_2$ vector field) unless the coefficients of $\psi_I$ and $\psi_U$ vanish and the coefficient of $\psi_{SU}$ equals the negative of the coefficient of $\psi_S$.

Thus, letting $e$ denote the edge underlying $e^\ast_{U^{-1}S}$, we may apply the relation $R_e$ to modify the putative linear combination so as to be carried on the polygon $P - T$, and this finally contradicts the assumed minimality of $P$. 

To see that the relations $R_e$ are linearly independent for $e \in \tau_+$, again consider a minimal sub-polygon $P$ of $\tau_+$ carrying the edges $e$ in a putative relation among the $R_e$. As before, take one of the triangles $T$ with only one edge interior to $P$, and let $e, f$ denote the other two edges of $T$. By minimality, at least one coefficient of $R_e$ or $R_f$ must be non-zero; owing to the supports of $R_e$ and $R_f$, this is in any case impossible, establishing the independence of the relations. Q.E.D.

Finally, define the vector space $\text{homeo}_+ \subseteq \text{psl}_2$ to be the span of all left hyperfans together with the usual generators $e, f, h \in \text{sl}_2 \subseteq \text{psl}_2$. We shall prove the surprising algebraic and geometric fact in Section 2 below that $\text{homeo}_+$ “closes in $\text{psl}_2$” in the sense that if $L, M \in \text{homeo}_+$, then in fact $[L, M] \in \text{homeo}_+$, where $[L, M] \in \text{psl}_2$ denotes the bracket in $\text{psl}_2$; in fact, our proof gives an algorithm for the calculation of brackets in these hyperfan coordinates. As we shall see, it then follows easily from simplicity of $\text{sl}_2$ that in fact we have the still more surprising equality $\text{homeo}_+ = \text{psl}_2$ of Lie algebras.

As the notation suggests, we shall in Section 3 below explain the sense in which $\text{homeo}_+$ may be regarded as the Lie algebra of the topological Lie group $\text{Homeo}_+ (S^1)$ of orientation-preserving homeomorphisms of the circle using the “universal decorated Teichmüller theory” of [P1]. In Section 4 below, we sketch a “model calculation” in hyperbolic geometry which was in fact our starting point for the construction of $\text{homeo}_+$.

In the usual way (see for instance [PS]), the algebra of infinitely-smooth vector fields on the circle is regarded as the Lie algebra of the Lie group of infinitely-smooth orientation-preserving diffeomorphisms $\text{Diff}_+ (S^1)$ of the circle. An obvious goal is to relate the usual harmonic analysis on the circle to the deformation theory proposed here using the “lambda lengths” of [P1], and we undertake several such calculations in Section 5 below.

2. THE LIE ALGEBRA $\text{homeo}_+$

This entire section is dedicated to a proof of the following

**Theorem 7.** We have the equality $\text{homeo}_+ = \text{psl}_2$.

Because of its geometric roots, we shall continue to denote the algebra by $\text{homeo}_+$ in this paper. Of course, one interpretation of Theorem 7 is that $\text{psl}_2$ admits the surprisingly simple and elegant basis consisting of hyperfans together with $e, f, h$.

**Proof.** Since $\text{sl}_2$ is simple and the support of $\psi_f$ is a single interval $K \subseteq S^1$, for any $a \in \text{sl}_2$, there is some $b \in \text{sl}_2$ with $[\psi_f, b] = a$ on $K$ and
vanishing elsewhere. Using the definitions and conjugating by elements of \( PSL_2 \), we find that indeed \( psl_2 = [homeo_+, sl_2] \). Therefore, it remains to prove the following

**Claim.** The bracket of a hyperfan and an element of \( sl_2 \) is a linear combination of hyperfans and elements of \( sl_2 \).

To prove this claim, we first observe that \( PSL_2 = PSL_2(Z) \) operates on \( sl_2 \), and hence on \( psl_2 \), by conjugation. The space \( homeo_+ \) is preserved under this action. Indeed, by definition for any \( A \in PSL_2 \), we have

\[
\mathcal{G}_A = A^{-1} \mathcal{G} A,
\]

and this implies as before that any two left hyperfans are conjugate modulo an element of \( sl_2 \), i.e.,

\[
\psi_{AB} = B^{-1} \psi_A B \in sl_2.
\]

It follows that each hyperfan is conjugate to \( \psi_I \) by an element of \( sl_2 \). As conjugation is a Lie algebra automorphism, we are therefore reduced to proving

**Lemma 8.** For any \( x \in sl_2 \), we have \([x, \psi_I] \in homeo_+\).

In order to prove this lemma, we first compute brackets of the basis elements of \( sl_2 \) with normalized elementary vector fields as follows.

**Lemma 9.** We have the following identities

\[
[f, \mathcal{G}] = \phi_T + \phi_{SU^{-1}} - 2 \phi_{US} - \mathcal{G} + 2f + 2h,
\]

\[
[h, \mathcal{G}] = 2 \phi_T - 2 \phi_{SU^{-1}} - 2h.
\]

\[
[e, \mathcal{G}] = - \phi_T + 2 \phi_U - \phi_{SU^{-1}} + \mathcal{G} + 2e - 2h.
\]

Furthermore, if \( n \gg 0 \) (i.e., if \( n > 0 \) is sufficiently large), then we also have

\[
[f, \mathcal{G}_{V^n}] = \phi_{TV^n} + \phi_{SU^{-1}} - 2 \phi_{USV^n} - \mathcal{G}_{V^n},
\]

\[
[h, \mathcal{G}_{V^n}] = 2(n + 1) \phi_{TV^n} + 2(n - 1) \phi_{SU^{-1}} - 4n \phi_{USV^n} - 2n \mathcal{G}_{V^n},
\]

\[
[e, \mathcal{G}_{V^n}] = - (n + 1)^2 \phi_{TV^n} - (n - 1)^2 \phi_{SU^{-1}} + 2n^2 \phi_{USV^n}
\]

\[
+ 2 \phi_{TV^{n+1}} + (n^2 + 1) \mathcal{G}_{V^n}.
\]

**Proof.** Given the expressions for the left-hand and right-hand sides (see the definition of \( \mathcal{G} \) and Lemmas 2 and 4 in the previous section), one just directly checks the first three formulas.
(Though it is so easy to verify these formulas once they have been written, we cannot resist the temptation to mention that their discovery amounted to writing the left-hand sides as a linear combination of fans with unknown coefficients and subsequently solving a system of linear equations. In each such calculation, there were about twice as many equations as there were unknowns, and the existence of a unique solution, as expressed in the formulas above, was due to a series of miraculous cancellations.)

The last three formulas are the result of conjugating the first three by the parabolic \( U \). Observe that the disappearance of the elements of \( \mathfrak{sl}_2 \) (present in the first three formulas) from the last three is due to the vanishing of \( \delta_{1^+} \) and other normalized elementary vector fields on quadrants III and IV if \( n \) is sufficiently large. For a few small values of \( n \), these formulas must be corrected by suitable elements of \( \mathfrak{sl}_2 \). Q.E.D.

**Proof of Lemma 8.** By definition \( \psi_1 = \sum_{n=0}^\infty n\delta_{1^+} \), and we have already mentioned that the series \( \sum_{n=0}^\infty n\delta_{1^+} \) converges uniformly on each compactum not containing \(-1 \in S^1\); in fact, the sequence of partial sums of this series stabilizes on each such compactum. Term-wise bracketing is therefore legitimate, and we have

\[
[x, \psi_1] = \sum_{n=0}^\infty [x, n\delta_{1^+}].
\]

By Lemma 9, the bracket \([x, \delta_{1^+}]\) is a finite linear combination of fans, hence it is likewise a finite linear combination of hyperfans. Thus, it remains only to prove that \([x, \psi_1]\) is a finite linear combination of hyperfans, and this will be done presently by a direct (partial) calculation of the brackets \([x, \psi_2]\), \([x, \psi_3]\), \([h, \psi_1]\).

Consider the most interesting case \([h, \psi_1]\) (the other cases are analogous to this one but easier and are therefore left to the reader). The sum \([h, \psi_1] = \sum_{n=0}^\infty [h, n\delta_{1^+}]\) is a priori a linear combination of infinitely many fans, namely, the fans \( \phi_{US^n} \) for each \( n \) (with the pin at \( \frac{1}{2} \cdot U^n \in S^1 \)) and the infinitely many fans \( \phi_{U^{n+1}} \) (with pin at \( \frac{2}{3} \in S^1 \)). Using Lemma 9 to compute the coefficient of \( \phi_{US^n} \) in the answer for \( n \gg 0 \), we get

\[
2n(n-1) - 4n^2 + 2n(n+1) = 2n(n-1 - 2n + n + 1) = 0.
\]

Having thus observed that only a finite number of fans of the type \( \phi_{US^n} \) non-trivially contribute to the bracket \([h, \psi_1]\), we turn to computing the contribution of the fans \( \phi_{U^{n+1}} \). Equivalently, we must compute the coefficient with which the normalized elementary vector field \( \delta_{1^+} \) appears in
the bracket $[h, \psi_I]$. We again use Lemma 9 to find this coefficient for almost all $n$ as follows:

$$-2(n+1)^2 + 2(n+2)(n+1) = 2(n+1)(-n-1+n+2) = 2(n+1)$$

By definition

$$2\psi_I = \sum_{n=0}^{\infty} 2n\delta_U^n,$$

so the total contribution of all fans pinned at $\frac{\pi}{4}$ to the bracket $[h, \psi_I]$ equals $2\psi_I$ (at least modulo a finite linear combination of fans). This completes the proof of Lemma 8 and hence of Theorem 7. Q.E.D.

The Lie algebra $\text{homeo}^+_{\mathbb{R}}$ has been defined as a subalgebra of $\text{psl}_2$ and then found to be identical with $\text{psl}_2$. To close this section, we suggest another point of view on this algebra which is closer to the standard Lie theory.

An infinitesimal version of the group of diffeomorphisms of the circle is the space of infinitely smooth vector fields $f(\theta) \partial / \partial \theta$, the commutator being defined by

$$[f(\theta) \partial / \partial \theta, g(\theta) \partial / \partial \theta] = (fg'(\theta) - f'g(\theta)) \partial / \partial \theta.$$

Naively, one would expect the infinitesimal version of the passage from diffeomorphisms to homeomorphisms to be this or that relaxation of the smoothness condition, but the above formula for the commutator evidently becomes ill-defined. The way around this is suggested by the decorated Teichmüller theory, as follows.

Start with the linear span $V$ of normalized elementary vector fields. The normalized elementary vector fields are all once continuously differentiable and the commutator of any two such vector fields is well-defined. Of course, $[V, V]$ is larger than $V$, but one can show (and it follows from Theorem 7 above) that each element of $[V, V]$ is equal to a sum of elements of $V$. Therefore, although the standard Lie bracket no longer applies, one can compute the commutator of any two elements of $[V, V]$ by formally using bilinearity. One must introduce a suitable topology on $V$ with respect to which the bracket is continuous, and then consider the closure, call it $\tilde{V}$, of $V$ under this topology.

Thus, Theorem 7 simply says that $\text{homeo}^+_{\mathbb{R}}$ is the smallest subspace of this hypothetical $\tilde{V}$ containing $V$ and closed under the bracket. Furthermore, from this point of view, the meaning of Proposition 6 is that the set of normalized elementary vector fields is a topological basis of $\text{homeo}^+_{\mathbb{R}}$. 
3. GEOMETRY OF THE HOMOGENEOUS SPACE \( \text{Homeo}_+</\text{Mob} \)

It is the purpose of this section to establish notation and recall various constructions and results from [P1], and the reader is referred there for further details. We undertake this survey in order to explain the geometrical considerations that have led to the construction of \( \text{homeo}_+ \) described previously and to establish various basic facts for application in subsequent discussions.

Let \( \text{Homeo}_+ = \text{Homeo}_+ (S^1) \) denote the topological group of orientation-preserving homeomorphisms of the circle, and consider its subgroups

\[
\text{Mob} = \text{PSL}_2(\mathbb{R}),
\]

\[
\text{Homeo}_+ = \{ f \in \text{Homeo}_+ : f(t) = t \text{ for } t = +1, -1, -\sqrt{-1} \in S^1 \}.
\]

We can really only understand \( \text{Homeo}_+ \) itself in the weak sense of a topological group, but on the other hand, the homogeneous space of right cosets

\[
\text{Tess} = \text{Homeo}_+/\text{Mob}
\]

admits a natural Fréchet tangent space \( \text{tess} \) to be recalled below. \( \text{Tess} \) in fact comes equipped with natural affine coordinates plus further structure as well and is our model of a universal Teichmüller space described in [P1].

Of course, since a Möbius transformation is uniquely determined by its values at any three specified points, we have the (set) isomorphism \( \text{Homeo}_+ \approx \text{Homeo}_+/\text{Mob} \) and think of \( \text{tess} \times \text{sl}_2 \) as a sensible “tangent space” to \( \text{Homeo}_+ \) itself. (Various natural calculations involving the subgroup \( \text{Homeo}_+ \subseteq \text{Homeo}_+ \) are postponed until Section 5.)

As we shall see presently, the generators \( \beta_A \), for \( A \in \text{PSL}_2 \), of \( \text{homeo}_+ \) in \( \text{psl}_2 \) given before are actually continuous vector fields on the circle, each of which arises in the usual way (cf. Lemma 1) from a corresponding one-parameter subgroup \( \beta_A(s) : S^1 \to S^1 \) of once continuously differentiable piecewise Möbius homeomorphisms of the circle. These \( \beta_A(s) \subseteq \text{Homeo}_+ \) are the coordinate deformations of certain affine coordinates on a fiber space \( \text{Tess} \) over \( \text{Tess} \), as we shall briefly discuss later, and the formal combination of \( \text{sl}_2 \subseteq \text{psl}_2 \) with the vector fields associated to these coordinates on \( \text{Tess} \) explains the genesis of the algebra \( \text{homeo}_+ \) discussed in this paper as well as its relation to the constructions of [P1].

Let us begin with the transformation associated to the doe \( c_i \) itself and define the one-parameter family \( A(s) = A_i(s) \in \text{Homeo}_+ \), for \( s \in \mathbb{R} \), by setting

\[
A(s) = A_i(s) \in \text{Homeo}_+.
\]
\[ A(s) = \begin{cases} 
  \begin{pmatrix} s & s-s^{-1} \\ 0 & s^{-1} \end{pmatrix}, & \text{in quadrant I}, \\
  \begin{pmatrix} s^{-1} & 0 \\ s-s^{-1} & s \end{pmatrix}, & \text{in quadrant II}, \\
  \begin{pmatrix} s^{-1} & 0 \\ s^{-1}-s & s \end{pmatrix}, & \text{in quadrant III}, \\
  \begin{pmatrix} s & s^{-1}-s \\ 0 & s^{-1} \end{pmatrix}, & \text{in quadrant IV}, 
\end{cases} \]

as illustrated in Fig. 6.

Let us explicitly but parenthetically observe here that the given Möbius transformations on each fixed interval in \( S^1 - \{ \pm 1, \pm \sqrt{-1} \} \) are hyperbolic transformations in \( D \) with fixed points given by the end points of the interval. Indeed, \( A(s) \) is actually the composition of four “earthquakes” in the sense of Thurston (cf. [Th] or [Ke]), namely, two right and two left earthquakes, as illustrated in Fig. 6, which are carefully chosen so as to be once continuously differentiable on the circle. As we have seen, the basic deformations here are the hyperfans described before rather than the earthquakes of Thurston’s theory.

More generally, for any \( A \in PSL_2 \), define the conjugates

\[ A_A(s) = A^{-1}A(s)A. \]

Just as for vector fields on the circle, calculation shows that

\[ A_A(s) = A_A(s), \quad \text{for } A \in PSL_2, \]

and we define

\[ A_A(s) = A_A(s), \quad \text{for } e \in T^*_a, \]

if \( A \in PSL_2 \) maps the unoriented edge underlying the doe to \( e \). These one-parameter families \( A_A(s) \) in \( \text{Homeo}_+ \) were introduced and studied in [P1] (and called “log lambda deformations” there), and some of their properties mentioned above are formalized in

**Lemma 10.** For any \( A \in PSL_2 \), the one-parameter family \( A_A(s) \) is actually a one-parameter multiplicative subgroup of \( \text{Homeo}_+ \). Furthermore, each homeomorphism \( A_A(s) \) is actually once continuously differentiable on the circle with four fixed points given by \( (\pm 1) \cdot A, (\pm \sqrt{-1}) \cdot A \in S^1 \), and the \( psL_2 \) vector field on the circle associated to \( A_A \) is the elementary vector field \( g_A \).
Proof. One first explicitly and painlessly checks each of these assertions for \( A = A_i \) itself using Lemma 1, and the corresponding claims for the conjugate \( A_A \) then follow. Q.E.D.

Recalling some further geometry from [P1] (which explains the terminology), \( \mathcal{T} \) is identified with the “space of Möbius-orbits of tessellations with doe of \( D^* \)." We describe in [P1; Section 3] global coordinates on \( \mathcal{T} \), one such coordinate (namely, a suitable cross ratio of points in \( S^1 \)) for each edge of the Farey tessellation \( \tau_{\ast} \); that is, the coordinates identify \( \mathcal{T} \) with a path-connected open subset of the Fréchet space \( \mathbb{R}^* \), and there is a Fréchet tangent space \( \mathcal{T} \) to \( \mathcal{T} \) at (the class of) \( \tau_{\ast} \) itself (i.e., at the identity map on the circle) as defined by Nash–Moser theory [Ha] (namely, the tangent space is the space of derivatives of paths). Roughly, if \( f \in \text{Homeo}_{+} \), then there is a corresponding “image” tessellation with doe \( \tau = f(\tau_{\ast}) \) of the Farey tessellation \( \tau_{\ast} \) with doe defined in the obvious way. (Two points of \( S^1 \) span an edge of \( \tau \) if and only if their pre-images under \( f \) span an edge of \( \tau_{\ast} \).)

Turning to the fiber space mentioned above and continuing to follow [P1], define a “decoration” on a tessellation to be the “specification of one horocycle centered at each ideal point of the tessellation”; for instance, the Farey tessellation \( \tau_{\ast} \) admits a canonical decoration described as follows: there are unique horocycles centered at \( \pm 1, -\sqrt{-1} \in S^1 \) which are pairwise tangent, and we take their reflections (in the sides of the triangle \( T \) in the construction in Section 1 of \( \tau_{\ast} \)) to define the canonical decoration \( \tau_{\ast} \) on \( \tau_{\ast} \). Just as for \( \mathcal{T} \), define \( \mathcal{T} \) to be the “space of Möbius-orbits of decorated tessellations with doe of \( D^* \).” Again, we give in [P1; Section 3] global coordinates on \( \mathcal{T} \), one such coordinate (called a “lambda length”) for each edge of the Farey tessellation; that is, the coordinates identify \( \mathcal{T} \) with a path-connected open subset of the Fréchet space \( \mathbb{R}^* \), and there is again a Fréchet tangent space \( \mathcal{T} \) to \( \mathcal{T} \) at (the class of) \( \tau_{\ast} \) as defined by Nash–Moser theory.
More specifically and for completeness, define the lambda length $\lambda$ of an edge $e$ of a tesselation together with a pair $h_+, h_-$ of horocycles at its ideal end points as follows: let $\delta$ denote the (signed) hyperbolic distance between $h_+$ and $h_-$, (taken with a positive sign if and only if $h_+ \cap h_- = \emptyset$), and set $\lambda = \sqrt{2} \exp \delta$. For instance, the decorated Farey tesselation $\tilde{\tau}_*$ has coordinates given by setting all lambda lengths to be $\sqrt{2}$.

It follows immediately from this discussion that $\tilde{\tau}_*$ is spanned by $\theta_*$, for $e \in \tau_*$, where $\theta_e$ is the Fréchet tangent vector at the decorated Farey tesselation to the coordinate curve given by the lambda length of edge $e \in \tau_*$. Furthermore, the vector field on the circle corresponding to $\theta_e$ is none other than the elementary vector field $e \in \text{psl}_2$ by Lemma 10.

Since we shall require it below, recall that the forgetful mapping $\tilde{\tau}_* \to \tau_*$ is simply expressed in lambda lengths as follows. Consider a decorated quadrilateral with ideal vertices given counter-clockwise by $x, y, z, \zeta \in S^1$, and adopt notation $a, b, c$ for the lambda lengths of the frontier edges given counter-clockwise starting from $x$. According to [P1: Section 3], the cross ratio $CR$ of $x, y, z, \zeta$ is given in lambda lengths by

$$CR = \frac{\zeta - x \ y - z}{\zeta - z \ y - x} = \frac{bd}{ac},$$

and this describes the mapping $\tilde{\tau}_* \to \tau_*$ in coordinates.

Another point from [P1: Section 5] which we shall require peripherally is the existence of a canonical formal two-form $\omega$ expressed in lambda lengths on $\tilde{\tau}_*$ by

$$\omega = -\sum d\ln a \wedge d\ln b + d\ln b \wedge d\ln c + d\ln c \wedge d\ln a,$$

where the sum is over all triangles $T$ complementary to $\bigcup \tau_*$ in $D$ and the (counter-clockwise) frontier edges of $T$ have respective lambda lengths $a, b, c$. This two-form on $\tilde{\tau}_*$ generalizes a multiple of the Weil-Petersson Kähler two-forms on the classical Teichmüller spaces of punctured surfaces as well as the usual Kirillov-Kostant two-form on the coadjoint orbit $\text{Diff}_+ \text{Möb}$.

Our understanding of the tangent spaces $\text{tess}$ and $\tilde{\text{tess}}$ is only formal (i.e., they are defined by Nash–Moser theory) as a result of the fact that we do not know the equations determining $\tilde{\text{tess}} \subseteq \mathbb{R}^+$ in cross ratios or $\tilde{\text{tess}} \subseteq \mathbb{R}^+$ in lambda lengths. On the other hand, we recall sufficient conditions in either case as follows. Given $\lambda \in \mathbb{R}^+$, we say that “the lambda lengths are bounded” if there is $K \geq 1$ so that

$$K^{-1} \leq \lambda(e) \leq K, \text{ for each } e \in \tau_*.$$
Theorem 11 [P1; Theorems 6.3 and 6.4 (joint with Sullivan)]. If \( \lambda \in \mathbb{R}^+ \) has bounded lambda lengths, then there is a decorated tessellation whose lambda lengths are given by \( \lambda \). Furthermore, the corresponding homeomorphism of the circle is

(i) quasi-symmetric

(ii) differentiable at each rational point

(iii) with derivatives at the rational points uniformly near unity.

In particular, given any arithmetic punctured Riemann surface (i.e., the surface is uniformized by a finite-index subgroup \( \Gamma \) of \( \text{PSL}_2(\mathbb{Z}) \)) and the Nielsen extension \( f \in \text{Homeo}_+ \) to \( S^1 \) of the lift to \( \mathbb{D} \) of any homeomorphism of \( \mathbb{D}/\Gamma \), then \( f \) satisfies the three conditions.

The last assertion already follows from the discussion in the proof of [P1; Theorem 6.4] and is raised here just for emphasis. The seemingly bizarre class of functions satisfying the three conditions of the theorem thus occurs quite naturally. In fact, since the tessellation corresponding to a punctured surface homeomorphism automatically has bounded lambda lengths (since there are only finitely many values of lambda lengths), the last assertion actually follows from the previous one.

Having introduced \( \mathcal{C} \) and its basis \( \{ \theta_e; e \in \tau_+ \} \), suppose that \( A \in \text{PSL}_2 \) and define the left and right coordinate fans

\[
\Phi_A = \sum_{n \geq 0} \theta_{U^n A} \in \mathcal{C}, \\
\Phi^*_A = \sum_{n \leq 0} \theta_{U^n A} \in \mathcal{C},
\]

respectively.

As before for vector fields on the circle, we have the relations

\[
\Phi_A - \Phi_{U^A} = \theta_A = \Phi_{SA} - \Phi_{US A}, \quad \text{for} \quad A \in \text{PSL}_2,
\]

among left coordinate fans. In contrast to the previous discussion, though, for each \( A \in \text{PSL}_2 \), we define a boost

\[
\beta_A = \Phi_A + \Phi^*_A - \theta_A = \sum_{n = -\infty}^{\infty} \theta_{U^n A} \in \mathcal{C}.
\]

Notice that if \( \tilde{e}_1 \cdot A_1 \) and \( \tilde{e}_1 \cdot A_2 \) have the same initial point, then in fact \( \beta_{A_1} = \beta_{A_2} \), and we may thus define

\[
\beta_t = \beta_A, \quad \text{for} \quad t \in \mathbb{Q} \cup \{ \infty \} \subseteq S^1.
\]
where $A$ maps the initial point of the doe to the rational point $t$ in the circle.

The collection of all boosts evidently spans the vertical tangent space of the fiber space $\widetilde{\text{Fess}} \to \text{Fess}$, and we may thus identify $\text{tess}$ itself as the quotient of $\widetilde{\text{tess}}$ by the set of all boosts.

Notice that whereas the sum of left and right fans $\phi_A, \phi_A^*$ is an elementary vector field $\theta_A = \phi_A + \phi_A^*$ on the circle in the earlier discussion, in the current case of lambda length vector fields, the sum of left and right coordinate fans $\Phi_A, \Phi_A^*$ is given by $\Phi_A + \Phi_A^* = \theta_A + \beta_A$ in $\widetilde{\text{tess}}$, and we have the equality $\Phi_A + \Phi_A^* = \theta_A$ only “up to boosts”.

Finally, let us just observe that the expression of the vector field corresponding to an earthquake in the sense of Thurston along the edge of $\tau_s$ underlying the oriented edge $\hat{e}_A$, for $A \in \text{PSL}_2$, is simply given by $\Phi_A - \Phi_{S_A}$. On the other hand, one sees immediately that the expression of a hyperfan in terms of earthquakes is thus reasonably complicated.

4. THE MODEL CALCULATION

In this section, we identify $\text{Tess}$ with the space of tesselations having the distinguished triangle with vertices $-1, 1, -\sqrt{-1} \in S^1$ so that the doe has $-1, 1 \in S^1$ as its endpoints. The space of decorated tesselations $\text{Tess}_t$ will be understood in a similar way, so these identifications are compatible with the fibration $\widetilde{\text{Tess}} \to \text{Tess}$.

The group $\text{Homeo}_n$ naturally acts on $\text{Tess}$, and the assignment $\text{Homeo}_n \ni id \mapsto \tau_s$ determines an isomorphism $\text{Homeo}_n \approx \text{Tess}$ as topological $\text{Homeo}_n$-spaces. Our original motivation was to use this isomorphism and the global coordinates on $\text{Tess}$ to find this or that kind of Lie algebra structure associated with the group $\text{Homeo}_n$.

As it is easier to compute using lambda lengths, we define a decorated version $\text{Homeo}_{n,t}$ of $\text{Homeo}_n$ as follows. By definition, $\text{Homeo}_{n,t}$ is the set of all pairs $(f, \tilde{f})$ such that $f \in \text{Homeo}_n$ and $\tilde{f}$ is a homeomorphism of $\widetilde{\text{Tess}}$ covering $f$. The set $\text{Homeo}_{n,t}$ is endowed with a topological group structure in an obvious way. In particular, there is a topological group homomorphism

$$\text{Homeo}_{n,t} \to \text{Homeo}_n$$

$$(f, \tilde{f}) \mapsto f$$

In analogy to the isomorphism of $\text{Homeo}_n$-spaces $\text{Homeo}_n \approx \text{Tess}$, there is an isomorphism of $\text{Homeo}_{n,t}$-spaces $\text{Homeo}_{n,t} \approx \widetilde{\text{Tess}}$ determined by assigning
$\tau_*$ equipped with the canonical decoration to the pair $(id, id)$. In particular, under this isomorphism $\widetilde{\text{Tess}}$ becomes the tangent space to $\text{Homeo}_n \approx \text{Tess}$ at the identity.

Denote by $\text{Vect}(\widetilde{\text{Tess}})$ the space of vector fields on $\widetilde{\text{Tess}}$. There is the standard map

$$R: \tilde{\text{Tess}} \to \text{Vect}(\widetilde{\text{Tess}})$$

sending each $x \in \tilde{\text{Tess}}$ to the right-invariant vector field on $\widetilde{\text{Tess}}$ obtained by applying right translations by elements of $\text{Homeo}_n$ to $x$. (At this point we are using the fixed isomorphism $\text{Homeo}_n \approx \widetilde{\text{Tess}}$.)

Mimicking the finite-dimensional Lie theory, one may attempt to endow $\tilde{\text{Tess}}$ with a Lie algebra structure by

1. integrating $R(x)$, for $x \in \tilde{\text{Tess}}$, in order to get a one-parameter subgroup $X(t) \subseteq \text{Homeo}_n$,
2. defining for any $x, y \in \tilde{\text{Tess}}$

$$[x, y] = \frac{d^2}{ds dt} \bigg|_{s=t=0} \{X(s)Y(t)X(-s)Y(-t)\}.$$

There is not much hope of carrying out this plan for all $x, y \in \tilde{\text{Tess}}$. Especially suspicious is the step (ii) since it seems obvious that the group structure on $\text{Homeo}_n$ is not smooth enough to guarantee the existence of the derivative on the right hand side.

There is, however, a choice of $x, y$ such that both (i) and (ii) do indeed hold. Namely, choose $x = \theta_{U^{-1}}$ and $y = \theta_{T^{-1}}$. These naturally determine right-invariant vector fields on $\text{Homeo}_n$. (To see this, use the map $R$ above and then forget the decoration.) These vector fields can easily be integrated to give one-parameter subgroups of $\text{Homeo}_n$; indeed, we have seen (in Lemma 10 above) that the one-parameter subgroup associated to $\theta_V$ is none other than $A_V(s)$, where $V$ stands for either $U^{-1}$ or $T^{-1}$.

Furthermore, as explained in [P1], $A_V(s)$ naturally lifts to a one-parameter subgroup $\widetilde{A}_V(s)$ of $\text{Homeo}_n$. (Briefly, the arguments leading to this are as follows. Observe that for each $s$, $A_V(s)$ is a homeomorphism of the unit circle which is piecewise an element of the Möbius group $\text{Mob} = \text{PSL}_2(\mathbb{R})$. Realize the Poincaré disk as the upper sheet of the unit hyperboloid in Minkowski three-space. A horocycle is then identified with a point on the open positive light-cone. Any homeomorphism of the circle determined by an element of $\text{Mob}$ comes from a global transformation of Minkowski space preserving both the upper sheet of the hyperboloid and
the positive light-cone. Thus, \( \text{M"ob} \) operates on horocycles and therefore on decorated tessellations. We apply this construction to each of the four pieces of \( A_{V}(s) \) to get \( \tilde{A}_{V}^{(i)}(W_{s}) \), \( i = 1, \ldots, 4 \). Explicit formulas for \( A_{V}(s) \) (or, more generally, differentiability at the break-points) then show that \( \tilde{A}_{V}^{(i)}(s) \), \( i = 1, \ldots, 4 \) combine to give an element of \( \text{Homeo}_{\Lambda} \).

We see that \( R(\theta_{V}) = \tilde{A}_{V}(s) \), and this accomplishes step (i) of our program. As to step (ii), we have at least some reason to be optimistic, for the corresponding vector fields on the circle \( \partial_{U^{-1}} \) and \( \partial_{T^{-1}} \) are once differentiable and hence their Lie bracket is well-defined. Indeed, we get the following.

**Lemma 12.** We have

\[
[\theta_{U^{-1}}, \theta_{T^{-1}}] = 2(\Phi_{USUTU^{-1}} - \Phi_{TU^{-1}}),
\]

where the bracket is defined by means derivatives of group commutators as in (ii) above.

**Sketch of Proof.** The argument consists of a straightforward and perhaps tedious—yet at points surprising—calculation of the lambda length evolution under the group commutator

\[
A_{U^{-1}v(s)} A_{T^{-1}v(t)} A_{U^{-1}v(s)}^{-1} A_{T^{-1}v(t)}^{-1}.
\]

We shall record here the main steps of this calculation.

First of all, let us simplify notation by writing \( A_{1} \) instead of \( A_{V}(s) \) and \( A_{2} \) instead of \( A_{T^{-1}}(t) \). Of course, both \( A_{1} \) and \( A_{2} \) are piecewise \( \text{M"ob} \) = \( \text{PSL}_{2}(\mathbb{R}) \) with four pieces. To be more specific, let \( A, B, C, E \in S^{1} \) (and \( A, C, D, E \subseteq S^{1} \) respectively) denote the breakpoints of \( A_{1} \) (and \( A_{2} \)) taken in counter clockwise order starting from \( A = -1 \in S^{1} \) as indicated in Fig. 7. As a matter of notation and also as in Fig. 7, denote by \( R_{i} \) the restrictions of \( A_{i} \) to the intervals. Each \( R_{i} \) is easy to compute by conjugating the suitable element in the definition of \( A_{i}(s) \) by \( U^{-1} \) if \( i = 1 \) or \( T^{-1} \) if \( i = 2 \). In particular, \( R_{i} \) is a hyperbolic fixing the end points of the corresponding interval.

The group commutator \( A_{1} A_{2} A_{1}^{-1} A_{2}^{-1} \) is also piecewise \( \text{M"ob} \). In order to find its breakpoints, we denote by \( \Pi(f) \) the set of breakpoints of an arbitrary piecewise \( \text{M"ob} \) homeomorphism of the circle. One easily checks the following relation on breakpoints

\[
\Pi(f \circ g) = \Pi(f) \cup \Pi(g) f^{-1}.
\]
Figure 7

Recall that we always use the right action. The repeated application of this relation along with the known information on $R_j^i$ allows us to compute $\Pi(A_1 A_2 A_1^{-1} A_2^{-1})$ directly, and the answer is as follows:

$$\Pi(A_1 A_2 A_1^{-1} A_2^{-1}) = \{A, B, B_{+s}, C, D_{+t}, D, D_{+t}, E\}.$$ 

Here the points are listed in the order they appear when going from $A$ to $E$ in the counter-clockwise sense, and the subindex $\pm \epsilon$ means that the point is close (depending on $s, t$, $\approx 0$) to the indicated one as in Fig. 8. Furthermore, the commutator itself is given explicitly as a piecewise Möbius homeomorphism in Fig. 8.

An important consequence of the formulas is that the “combinatorics” of the transformation $A_1 A_2 A_1^{-1} A_2^{-1}$ is independent of the parameters $s, t$, provided these parameters are small enough. This is not what one would expect of the group commutator of two arbitrary one-parameter subgroups of $\text{Homeo}^+$ and is a “special” attribute of the particular commutator we compute here.

As lamba lengths are $\text{Möb}$-invariant, only those lambda lengths may evolve non-trivially under $A_1 A_2 A_1^{-1} A_2^{-1}$ whose underlying edges in the Farey tessellation have their endpoints in different complementary intervals among $S^1 = \Pi(A_1 A_2 A_1^{-1} A_2^{-1})$.

Inspection of the representation of $A_1 A_2 A_1^{-1} A_2^{-1}$ displayed above shows that only the following three groups of lambda lengths $\lambda(e)$ can possibly evolve:
(i) an edge obtained by parabolic rotation of $BC$ in the clockwise direction about $B$;

(ii) an edge obtained by parabolic rotation of $CD$ in the counterclockwise direction about $D$;

(iii) an edge obtained by parabolic rotation of $DE$ in the clockwise direction about $D$.

Now, observe that our calculation affords the flip about the imaginary axis as a symmetry; it follows at once that the third group (iii) of lambda lengths are actually constant. Furthermore, it is the vectors tangent to the coordinates from the first two groups (i) and (ii) which occur in Lemma 12.

To calculate the infinitesimal evolution of the lambda lengths from group (i) and (ii), one must use the explicit formulas for the action of $\text{Mo}_b$ on horocycles (cf. [P1]), and then differentiate. This is a lengthy but rewarding calculation mostly consisting of numerous operations on $2 \times 2$-matrices. The result is recorded in Lemma 12 above.

What has so far been computed is only the value of the bracket of two right-invariant vector fields at the identity. Since the bracket of right-invariant vector fields is right-invariant and a right-invariant vector field is uniquely determined by its value at the identity, this is sufficient for the proof.

Q.E.D.
Two remarks are in order:

Fans actually first appeared in the present work in this calculation of the bracket of the two elementary vector fields \( \theta_U^{-1} \) and \( \theta_T^{-1} \) (but see also [P1; Section 6.1]), and it was surprising and encouraging that the “structure constants on fans are constant” in the sense of Lemma 12.

At the level of vector fields on circle, Lemma 12 means that

\[
[\theta_U^{-1}, \theta_T^{-1}] = 2(\phi_{USUTU^{-1}} + \phi_{TU^{-1}})
\]

and this latter equality is easy to check using formulas from Section 1.

5. HARMONIC ANALYSIS OF HOMEOMORPHISMS

Recall that \( \text{Diff}_+ = \text{Diff}_+(S^1) \subseteq \text{Homeo}_+(S^1) \) denotes the subgroup of all infinitely differentiable orientation-preserving homeomorphisms of the circle, whose corresponding Lie algebra \( \text{diff}_+ \) is comprised of all infinitely differentiable real vector fields \( f(\theta) \frac{\partial}{\partial \theta} \) on the circle (where \( \theta \) is the usual angular coordinate on the circle and \( f(\theta) \in \mathbb{R}; \) one usually complexifies and considers \( f(\theta) \in \mathbb{C} \), but we shall stick with real vector fields for the calculations below). Thus, we expand

\[
f(\theta) \sim \sum_{n \geq 0} a_n \cos n\theta + b_n \sin n\theta
\]

\[
\sim \sum_{n \in \mathbb{Z}} c_n e^{in\theta}, \quad \text{for} \quad c_{-n} = c_n \quad \text{and} \quad i = \sqrt{-1}.
\]

Extending the previous notation, let \( \text{Diff}_n \subseteq \text{Diff}_+ \) denote the subgroup of normalized infinitely differentiable orientation-preserving homeomorphisms of the circle, and let \( \text{diff}_n \) denote its Lie algebra consisting of all infinitely differentiable real vector fields \( f(\theta) \frac{\partial}{\partial \theta} \) which vanish for \( \theta = 0, \pi, 3\pi/2 \) (namely, at the three distinguished points \(-1, +1, -\sqrt{-1} \in S^1\)).

Since we shall need it later, we take a moment to describe the natural basis of \( \text{diff}_n \), namely, given a vector field \( \theta = f(\theta) \frac{\partial}{\partial \theta} \), we define its normalization to be the vector field

\[
\tilde{\theta} = \frac{\partial}{\partial \theta} \{ f(\theta) + \alpha \cos \theta + \beta \sin \theta + \gamma \} \frac{\partial}{\partial \theta}.
\]
where \( \alpha, \beta, \gamma \in \mathbb{R} \) are chosen so that \( f(\theta) \) vanishes for \( \theta = 0, \pi, \frac{3\pi}{2} \). For instance, setting \( z = e^i\theta \), a short calculation gives the following normalized trigonometric fields:

\[
\cos n\theta \frac{\partial}{\partial \theta} = \left\{ i \frac{z^n + z^{-n}}{2} + x i \frac{z^n + z^{-1}}{2} + \beta \frac{z^n - z^{-1}}{2} + \gamma i \right\} \frac{d}{dz},
\]

for

\[
\alpha = \frac{(-1)^n - 1}{2},
\]

\[
\beta = \frac{-1 - (-1)^n}{2} - \frac{(-1)^n + i^n + i^{-n}}{2},
\]

\[
\gamma = \frac{-1 - (-1)^n}{2},
\]

and

\[
\sin n\theta \frac{\partial}{\partial \theta} = \left\{ i \frac{z^n - z^{-n}}{2} + x i \frac{z^n + z^{-1}}{2} + \beta \frac{z^n - z^{-1}}{2} + \gamma i \right\} \frac{d}{dz},
\]

for

\[
\alpha = 0 = \gamma, \quad \beta = \frac{i(-1)^{n+1} + i^{-n}}{2}.
\]

These normalized trigonometric vector fields \( \cos n\theta \partial/\partial \theta \), \( \sin n\theta \partial/\partial \theta \) for \( n \geq 2 \) constitute the natural basis for the Lie algebra \( \text{diff}_n \), and we shall subsequently require them.

**Theorem 13.** If \( A = (a \ b \\ c \ d) \in \text{PSL}_2 \), then the normalized elementary vector field \( \mathcal{B}_A \in \text{diff}_n \) admits the Fourier expansion \( \mathcal{B}_A(\theta) \sim \sum_{n \in \mathbb{Z}} \pi i n^3 - n^2 \ c_n e^{in\theta} (\partial / \partial \theta) \), where for \( n^2 > 1 \), we have

\[
\pi i (n^3 - n) c_n = - \left[ (c - a)^2 + (b - d)^2 \right] \left[ \frac{(b - d) - i(a - c)}{(b - d) + i(a - c)} \right]^n \\
+ 2(c^2 + d^2) \left[ \frac{d - ic}{d + ic} \right]^n + 2(a^2 + b^2) \left[ \frac{b - ia}{b + ia} \right]^n \\
- \left[ (c + a)^2 + (b + d)^2 \right] \left[ \frac{(b + d) - i(a + c)}{(b + d) + i(a + c)} \right]^n,
\]
and the modes $c_0, c_1, c_{-1}$ are chosen to guarantee that $\tilde{\mathcal{G}}_A$ vanishes at $-1, +1, -\sqrt{-1} \in S^1$. In particular, for the basic elementary vector field $\mathcal{G} = \tilde{\mathcal{G}}_I$ (i.e., taking $A = I$) we find

$$\mathcal{G}(\theta) \sim \frac{8}{\pi i} \sum_{n \equiv 2 \pmod{4}} \frac{1}{n^3} e^{\pi i n \theta} \frac{\partial}{\partial \theta}.$$ 

**Sketch of Proof.** We shall rely on the fact that the elementary vector field $\mathcal{G}_A$ is once-continuously differentiable on the circle to twice integrate by parts the usual expression for Fourier coefficients (as in [P1; Proposition 3.6]) and conclude that

$$c_n = \frac{1}{2\pi i n^3} \sum_j z_j e^{-i n \theta_j}, \quad \text{for} \quad n^2 > 1,$$

where $\{I_j\}$ denotes the set of four intervals in the $psl_2$ structure of $\mathcal{G}_A$ or $\tilde{\mathcal{G}}_A$ and

$$\tilde{\mathcal{G}}_{\tilde{A}}(\theta) = \{x_j \cos \theta + y_j \sin \theta + z_j\} \frac{\partial}{\partial \theta} \quad \text{on} \ I_j.$$ 

Direct calculation from the definitions (conjugating the pieces of $\mathcal{G}$ by $A$ and acting by similarity on the circle itself) yields un-normalized $\tilde{\mathcal{G}}_A$ as given in Fig. 9, where we indicate points in the unit circle as complex numbers and in addition the various $sl_2$ vector fields in the $psl_2$ structure on the complementary intervals.

In our depiction in Fig. 9, we have of course tacitly assumed that “the image under $A$ of the third quadrant contains the doe”, and there are evidently four such cases we must actually consider when normalizing. Our argument here will be entirely naive in that we shall serially calculate (or leave for the faithful reader to do so!) in these four cases and find the answer as given by Theorem 13 independently of the case; this of course expresses a four-fold symmetry of the formulas.

Continuing then from Fig. 9, we normalize by subtracting the appropriate element of $sl_2$ to arrange that $\tilde{\mathcal{G}}_A$ vanishes at $-1, +1, -\sqrt{-1} \in S^1$. Direct application of the formula above for Fourier modes (using Lemma 1 to find the modes $z_j$ above from Fig. 9) together with a bit of calculation leads to the asserted expression.

As we have remarked, there are three further cases (corresponding to the various possible normalizations) whose serial consideration lead to exactly the same expression. Q.E.D.

Theorem 13 gives a satisfactory description of the harmonic analysis of $homeo_+$, and we turn our attention now to the reverse inclusion $diff_n \subseteq homeo_+$. To this end and in analogy to Section 4, define the decorated
version $\widetilde{\text{Diff}}_n$ of $\text{Diff}_n$ to be the set of all pairs $(f, \tilde{f})$, where $f \in \text{Diff}_n$ and $\tilde{f}$ is a homeomorphism of $\text{Tess}$ covering $f$. As before, $\text{Diff}_n$ becomes a group in the natural way, and
\[
\text{Diff}_n \to \text{Diff}_n
\]
\[
(f, \tilde{f}) \mapsto f
\]
is a group homomorphism.

Now, motivated by the calculations in the appendix, let us define a canonical section
\[
\sigma : \text{Diff}_n \to \widetilde{\text{Diff}}_n
\]
which is a group homomorphism as follows. Given $f \in \text{Diff}_n$, we may conjugate by the Cayley transform
\[
C : \mathbb{U} = \mathbb{R} \times \mathbb{R} \to \mathbb{D}
\]
\[
z \mapsto \frac{z - i}{z + i}
\]
to produce a homeomorphism
\[ f^C = C^{-1} \circ f : C \to R. \]

In fact, \( C^{-1} \) maps the distinguished points \(-1, +1, \) and \(-\sqrt{-1} \in S^1\), respectively, to \(0 \in R\), the point at infinity in \( C \supset \mathbb{U} \), and \( 1 \in R\), so \( f^C \) is indeed well-defined and furthermore fixes each of \( 0 \in R \) and \( 1 \in R \).

A horocycle in upper half space \( \mathbb{U} \) tangent to \( x \in R \) is determined by its Euclidean diameter \( d \), and (motivated by Lemma A.2), we define the Euclidean diameter of the horocycle for \( \sigma(f) \) at \( f^C(x) \) to be
\[ \left| \frac{df^C}{dx}(x) \right| d, \]
so the Euclidean diameters scale by the derivatives of \( f^C \). Likewise, we define the evolution under \( f^C \) of the horocycle centered at infinity so that its Euclidean height scales by the reciprocal of the derivative of \( f^C \) at infinity.

This gives a natural action of \( \Diff_n \) on decorations and hence the required section \( \sigma \). The proof that \( \sigma \) is a group homomorphism is then just an expression of the Chain Rule of Differential Calculus. Armed with our section \( \sigma \), we may regard
\[ \Diff_n \simeq \sigma(\Diff_n) \subseteq \Diff_n \subseteq \Homeo_n \]
in order to calculate lambda lengths. Actually, we shall undertake this calculation only infinitesimally near the identity, i.e., we calculate \( \diff_n \subseteq \homeo_\ast \) using the normalized trigonometric fields as follows.

Let \( w(z, t) \) be the one-parameter family of diffeomorphisms of the circle gotten by integrating the normalized trigonometric fields \( \cos n\theta \partial / \partial \theta \) or \( \sin n\theta \partial / \partial \theta \), which were computed above. In particular, we have
\[ \left. \frac{\partial w}{\partial z} \right|_{t=0} = 1. \]

In order to calculate \( \sigma \), we set
\[ z = C(s) = \frac{s - i}{s + i}, \quad \text{for } s \in R, \]
\[ s = C^{-1}(z) = \frac{1 + z}{1 - z}, \quad \text{for } z \in S^1, \]
and conjugate to define
\[ W(s, t) = C^{-1} \cdot w(C(s), t) \]
\[ = \frac{1 + w \left( \frac{s - i}{s + i}, t \right)}{1 - w \left( \frac{s - i}{s + i}, t \right)} : \mathbb{R} \to \mathbb{R}, \]
so in particular, we again find
\[ \frac{\partial W}{\partial s} \bigg|_{s=0} = 1 \]

Now, by Lemma A.1, the lambda lengths are given by
\[ \lambda(x, y) = |y - x| \frac{2}{\sqrt{d_x d_y}} \]
if the horocycles at \( x, y \in \mathbb{R} \) have respective diameters \( d_x, d_y \). By definition of our section \( \sigma \), the lambda length \( \lambda(x, y) \) evolves under \( W(\cdot, t) \) to
\[ \lambda_W(x, y, t) = |W(y, t) - W(x, t)| \frac{2}{\sqrt{(\partial W/\partial s)_x d_x | \partial W/\partial s|_y d_y}}. \]

Thus, we seek
\[ \frac{\partial}{\partial t} \bigg|_{t=0} \{ \log \lambda_W(x, y, t) \} = \frac{\partial}{\partial t} \bigg|_{t=0} \frac{\{ \lambda_W(x, y, t) \}}{\lambda_W(x, y, 0)} = \frac{\partial}{\partial t} \bigg|_{t=0} \{ \lambda_W(x, y, t) \} = \frac{1}{|y - x|} \frac{\partial}{\partial t} \bigg|_{t=0} \left( \frac{|W(y, t) - W(x, t)|}{\sqrt{(\partial W/\partial s)_x} \sqrt{(\partial W/\partial s)_y}} \right). \]

Direct calculation (which we omit) of \( (\partial/\partial t)|_{t=0} \) in the last line together with multiple applications of the simplifying identity \( (\partial W/\partial s)|_{t=0} = 1 \) discussed above finally yields
\[
\frac{\partial}{\partial t}\bigg|_{t=0}\{\log \lambda_W(x, y, t)\} \\
= \left[ -\frac{1}{|y-x|} \frac{\partial W}{\partial y} \bigg|_y - \frac{\partial^2 W}{\partial s \partial t} \bigg|_x - \frac{1}{2} \frac{\partial^2 W}{\partial s \partial t} \bigg|_y \right]_{t=0}.
\]  

It is thus just a matter of substituting into the previous equation our known derivatives

\[
\frac{\partial W}{\partial t}(s, t) = \left[ -\frac{2i}{1 - w(s-i)(s+i)} \frac{\partial W}{\partial t}(s-i) \right].
\]

In fact, at \( t = 0 \) we find

\[
\frac{dW}{dt}(s, 0) = \left. \frac{2i}{1 - w(s-i)(s+i)} \frac{dz}{dt} \right|_{z = (s-i)(s+i)} = -\frac{i}{2} (s+i)^2 \frac{dz}{dt} \bigg|_{z = (s-i)(s+i)},
\]

where \( dz/dt \) is the coefficient of \( d/dz \) in the normalized trigonometric fields above. There remains a series of tedious calculations (for the various cases of normalized trigonometric fields which we omit) which lead to the derivation of the surprisingly simple (namely, rational) formulas presented below.

Turning briefly to the case that one of the points, say the point \( y \), lies at infinity, a parallel calculation to that above using the second part of Lemma A.1 leads to

\[
\frac{\partial}{\partial t}\bigg|_{t=0}\{\log \lambda_W(x, \infty, t)\} = \left. \frac{1}{2} \left( \frac{\partial^2 W}{\partial s \partial t} \bigg|_x + \frac{\partial^2 W}{\partial s \partial t} \bigg|_x \right) \right|_{t=0},
\]

and this is finally seen to be the limit of (\( \ast \)) above for fixed \( x \in \mathbb{R} \) as \( y \to \infty \).

**Theorem 14.** We have

\[
\frac{\partial}{\partial \theta} \left. \sum_{e \in \tau_*} \frac{(\xi^e \eta^e + 1)}{\xi^e \eta^e} \left( \frac{n(\xi^e + \eta^e)}{\eta - \xi} \right) \frac{\partial}{\partial \log \lambda_e} \right|_{\theta = \infty} \\
\frac{\partial}{\partial \theta} \left. \sum_{e \in \tau_*} \frac{(\xi^e \eta^e - 1)}{\xi^e \eta^e} \left( \frac{n(\xi^e + \eta^e)}{\eta - \xi} \right) \frac{\partial}{\partial \log \lambda_e} \right|_{\theta = \infty},
\]

where in each formula, \( e \in \tau_* \) has endpoints \( \xi, \eta \in S^1 \).
Let us just emphasize that the duality between sine and cosine which is
broken by normalization is nicely restored in these formulas, which are
furthermore happily found to be independent of the residue of $n$ modulo
four.

Now turning away from infinitesimal questions to global ones, let us
identify the Hilbert space $L^2(S^1)$ of complex square-integrable functions
on the circle with the Hilbert space $L^2([0, 2\pi])$ of complex square-integrable
functions on the interval in the usual way, so we regard $L^2(S^1)$ as the set
of periodic functions $R \to R$ of period $2\pi$.

Of course, $\sum_n c_n e^{in\theta} \in L^2([0, 2\pi])$ actually takes real values if and only if
$c_{-n} = \overline{c_n}$, for all $n$. Such a function $[0, 2\pi] \to R$ might enjoy various
topological and/or set-theoretic properties, for instance, it might be con-
tinuous or injective on $S^1$, and one would like to characterize such proper-
ties in terms of the Fourier modes $c_n$. We remark that there appear to be
no classical solutions to any analogous such basic problems (except, of
course, the well known results relating smoothness to rapid decay of
Fourier modes).

In particular, Dan Burghelea [Bu] posed to us the problem of charac-
terizing the Fourier modes $c_n$ of a homeomorphism of the circle; see also
Question c) at the beginning of Section I.4 in [Ka]. By suitably further
specializing, namely, by imposing further conditions on the homeo-
morphisms, we close with the partial solution to such a problem.

Let $R \subseteq \text{Homeo}_+$ denote the subspace of all homeomorphisms $f$ of
the circle satisfying conditions (i)-(iii) in Theorem 11, namely, $f$ is quasi-
symmetric and is smooth at the rational points of $S^1$ with derivatives there
uniformly near unity.

As before, we let $R_n \subseteq R$ denote the subspace of normalized homeo-
morphisms (i.e., fixing each of $-1, +1, -\sqrt{-1} \in S^1$) and define $\tilde{R}_n$ to be
the set of pairs $(f, \tilde{f})$, where $f \in R_n$ and $\tilde{f}$ is a homeomorphism of $\text{Tess}$
covering $f$. This time, $\tilde{R}_n$ is just a space, and

$$\tilde{R}_n \to R_n$$

$$(f, \tilde{f}) \mapsto f$$

is a continuous surjection.

Also as before and precisely because we demand differentiability at the
rationals, there is a section $\sigma: R_n \to \tilde{R}_n$ (this time, a continuous section)
defined in exactly the same manner as $\sigma: \text{Diff}_+ \to \text{Diff}_+$. Our final result is
then just a translation of Theorem 11 and the definition of $\sigma$ using both
parts of Lemma A.1 (i.e., there are again two parallel calculations, one of
which specializes to the other.
Theorem 15. Given $\sum c_ne^{an}$ where $c_{-n}=\tilde{c}_n$ define $f(z)=\sum c_nz^n$ for $z\in S^1$, and suppose that for each rational point $\zeta=(b+ia)/(b-ia)\in S^1$, where $b, a\in \mathbb{Z}$ are not both zero, the limits
\[
f(\zeta) = \sum c_n\zeta^n \quad \text{and} \quad f'(\zeta) = \sum nc_n\zeta^{n-1}
\]
exist and that $f$ fixes each of the points $-1, 1, -\sqrt{-1}\in S^1$. Then $f$ restricts to a quasi-symmetric homeomorphism of the circle which is smooth with bounded derivatives at the rational points (that is, $f\in R$) if there is some function $\varphi: X\rightarrow \mathbb{R}$ and some number $K>1$ so that
\[
K^{-1} < \chi \left( \frac{b}{a} \right) \left( \frac{d}{c} \right) \left( a^2 + b^2 \right) \left( c^2 + d^2 \right) \left| \frac{e^{f(b+ia)/(b-ia)} - e^{f(d+ie)/(d-ic)}}{f'(b+ia)/b-ia - f'(d+ie)/d-ic} \right| < K,
\]
for each $(a, b)\in \text{PSL}_2(\mathbb{Z})$, where we furthermore define $\chi(\infty) = 1$.

6. CLOSING REMARKS

We close with various remarks and speculations.

First of all from the algebraic point of view, the importance of the Lie algebra $\text{homeo}_+$ should ultimately depend upon the richness of its representation theory. A major obstruction to the development of this theory at present is the absence of a reasonable triangular decomposition of $\text{homeo}_+$. This is in some ways reminiscent of another well-known challenge: the representation theory of a double loop algebra.

We exhibit a small spanning set for $\text{homeo}_+$, namely left hyperfans, but do not have the structure constants given explicitly. Instead, we have an algorithm to bracket two left hyperfans. In this respect, our algebra resembles a W-algebra, whose structure constants are also not known explicitly (but there is an algorithm to calculate the bracket of two fields using the formalism of the operator product expansion). Is there perhaps something deep behind this point of similarity?

Although morally and geometrically $\text{homeo}_+$ is closely related to the Virasoro algebra, it is algebraically and surprisingly more reminiscent of a loop algebra. For example, there are many commutative subalgebras. For instance, consider a collection of disjoint intervals and a collection of elements of $\text{homeo}_+$, one element for each interval, vanishing outside this interval.

Another point of view on $\text{homeo}_+$ is suggested by recent works on “Lie algebras graded by finite root systems” as discussed in [BM] and [BZ] for instance. Indeed, $\text{homeo}_+$ contains $sl_2$, and under the adjoint action of $sl_2$,
any element of homeo_+ generates a submodule isomorphic to a direct sum of several copies of sl_2. Thus, homeo_+ is sl_2 graded, and it seems that this particular example of an algebra graded by a finite root system has not been noticed before.

Turning to more geometric considerations, it seems likely that Homeo_+ and Homeo_+\textbackslash/M\ddot{o}b admit natural complexifications. Indeed in further work [P3], the second-named author has computed the Hilbert transform (i.e., the boundary values on the circle of the conjugate of the harmonic extension to the disk of a specified scalar function on the circle) in lambda length coordinates.

In the same spirit, we wonder about the quantization of our universal two-form described in Section 3 above and what might be its relation to the Gelfand–Fuchs cocycle [Fu] and/or the combinatorial Godbillon-Vey class [GS]; furthermore, our Lie bracket on homeo_+ together with our two-form on \mathcal{F}_{\text{ess}} should combine in the usual way to produce a new Poisson algebra. The relationship of our current constructions with the universal Ptolemy group (cf. [P1],[P2]) is clear, and we wonder what might be the relationship between homeo_+ and Grothendieck’s vision [Gr] of the absolute Galois group.

Finally, as is well known [Ah], the Beltrami equation has its own amazing regularity theory known from Ahlfors-Bers-Sullivan theory: While the input (the Beltrami differential) is given only measure-theoretically on D or S^1, the output (the solution to the Beltrami equation) is actually a homeomorphism of D or S^1. Insofar as we have here developed a deformation theory of homeomorphisms, we might dream of meaningful solutions to partial differential equations using our new coordinates.

APPENDIX: LAMBDA LENGTHS IN UPPER HALF SPACE

We collect here several elementary calculations which are required in this paper, namely, we derive several formulas involving lambda lengths working in the upper half space model \mathcal{U} = \mathbb{R} \times \mathbb{R}_+ of hyperbolic geometry. The proofs of these several results are entirely routine and tedious and are therefore only sketched. As a technical point (in order to conform to standard usage), we shall use the left action of M\ddot{o}b on \mathcal{U} in this appendix (rather then the right action as in the body of the paper).

Suppose that x \in \mathbb{R} is an ideal point of \mathcal{U}, so a horocycle centered at x is simply given by a Euclidean circle, say a circle of Euclidean diameter A, tangent to the real axis at x; we shall let h = h(x, A) \subseteq \mathcal{U} denote this horocycle. Extending this notation to the case x = \infty (i.e., the point at infinity in \mathcal{U}), let h(\infty, H) denote the horocycle about \infty at Euclidean height H.
Our first result simply expresses lambda lengths in terms of this data.

**Lemma A.1.** Provided \( x_0, x_1 \in \mathbb{R} \), the lambda length determined by the two horocycles \( h_0 = h(x_0, \Lambda_0) \) and \( h_1 = h(x_1, \Lambda_1) \) in upper half space is given by

\[
\lambda(h_0, h_1) = \frac{2}{\sqrt{\Lambda_0 \Lambda_1}} |x_0 - x_1|.
\]

On the other hand, for \( h_0 = h(\infty, H) \), \( x_1 \in \mathbb{R} \), and \( h_1 = h(x_1, \Lambda_1) \), we find

\[
\lambda(h_0, h_1) = \frac{2H}{\sqrt{\Lambda_1}}.
\]

**Proof.** Let \((u_i, v_i) \in \mathbb{U}\), for \( i = 0, 1 \), denote the respective points of intersection between \( h_i \) and the geodesic in \( \mathbb{U} \) with ideal endpoints \( x_0, x_1 \). The hyperbolic distance between \((u_0, v_0)\) and \((u_1, v_1)\) in \( \mathbb{U} \) is of course

\[
\delta = \begin{cases} 
\ln \left| \frac{v_0}{v_1} \right| & \text{if } u_0 = u_1; \\
\ln \left| \frac{(u_0 - c + r) v_1}{(u_1 - c + r) v_0} \right| & \text{if } u_0 \neq u_1, \text{where} \\
r = |x_1 - x_0|/2, \quad c = (x_0 + x_1)/2,
\end{cases}
\]

and in particular if \( x_0 = \infty \), then the claim follows easily from the definition of lambda lengths \( \lambda = \sqrt{2e^{2 \delta}} \).

Continuing with the argument now under the assumption that \( x_0, x_1 \in \mathbb{R} \), one computes directly that

\[
(u_i, v_i) = \left( x_i + A_i^2(x_j - x_i)/A_i^2 + (x_j - x_i)^2, \frac{A_i(x_j - x_i)^2}{A_i^2 + (x_j - x_i)^2} \right), \quad \text{for} \quad \{i, j\} = \{0, 1\},
\]

and a short calculation (plugging these values into the expressions above for \( \delta \) and \( \lambda \)) completes the proof. Q.E.D.

Our next result shows that diameters of horocycles scale by first derivatives under Möbius transformations.

**Lemma A.2.** Suppose that \( x \in \mathbb{R} \), \( h = h(x, \Lambda) \) is a horocycle in \( \mathbb{U} \), and

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Möb}
\]

is a Möbius transformation so that \( \gamma(x) \in \mathbb{R} \). Then \( \gamma(h) \) is the horocycle centered at \( \gamma(x) \) with Euclidean diameter \( \Lambda(cx + d)^{-2} \). Likewise, if \( \gamma(x) = \infty \), then \( \gamma(h) \) is the horocycle about \( \infty \) of Euclidean height \( H = c^{-2}A^{-1} \).
Proof. Suppose first that $\gamma(x) \neq \infty$. Setting $z = u + \sqrt{-1}v \in \mathbb{H}$, we have

$$
\gamma(z) = \frac{az + b}{cz + d} = \frac{(au + b)(cu + d) + av}{(cu + d)^2 + (cv)^2} + \sqrt{-1} \frac{v}{(cu + d)^2 + (cv)^2}.
$$

Furthermore, a parametrization of $h$ is given by

$$
t \mapsto \left( x + \frac{A}{2} \cos t, \frac{A}{2} (1 + \sin t) \right), \quad \text{for } t \in \mathbb{R}
$$

and the imaginary part of the image $\gamma(h)$ is thus given by

$$
I(t) = \frac{A(1 + \sin t)}{2(cx + d)(cx + d + A \cos t) + c^2 A^2 (1 + \sin t)}, \quad \text{for } t \in \mathbb{R}.
$$

A direct but somewhat involved exercise in calculus then shows that $I(t)$ has a unique extremum for $t \in \mathbb{R}$, namely, a maximum value $A(cx + d)^{-2}$, as desired.

An analogous but easier argument (which we omit) handles the case that $\gamma(x) = \infty$, completing the proof. Q.E.D.

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REFERENCES


