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Some characterizations of the exceptional planar embedding of $\mathsf{W}(2)$

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Abstract

In this paper, we study the representation of $\mathsf{W}(2)$ in $\mathsf{PG}(2,4)$ related to a hyperoval. We provide a group-theoretic characterization and some geometric ones.

Keywords: Generalized quadrangle; Translation quadrangle; Projective embedding

1. Introduction

In the theory of embeddings of generalized quadrangles in projective spaces one usually assumes that the projective space has dimension at least three. This is obvious and even automatic if one considers natural additional conditions such as being full or polarized. In the lax case – so without additional requirements – the condition on the dimension of the projective space is necessary in order to be able to prove a partial classification; see [4]. Roughly, every embedded finite classical generalized quadrangle (different from a symplectic one in odd characteristic) in $\mathsf{PG}(d,q)$, $d \geq 3$, arises from its standard embedding by field extension and projection, or is a well-understood grumbling embedding of a quadrangle with small parameters. The proof of this heavily uses the assumption $d \geq 3$. In fact, this result is no longer true in dimension two ($d = 2$). Indeed, the hyperoval-embedding of $\mathsf{W}(2)$, the unique generalized quadrangle of order 2, in $\mathsf{PG}(2,4)$ does not arise from any embedding in $\mathsf{PG}(d,q)$ by projection, with $d \geq 3$ and $q = 4^e$ (e a positive integer). However, no other examples of this phenomenon are known. So, in order to start a theory of planar embeddings of generalized quadrangles (and later on, more generally, generalized polygons), it seems worthwhile to study this exceptional embedding of $\mathsf{W}(2)$ in $\mathsf{PG}(2,4)$. The characterizations we will prove will point at the exceptional character of this embedding, and feeds the conjecture that it might be “almost unique” (there are more exceptional planar embeddings of $\mathsf{W}(2)$ that do not occur for other classical quadrangles).

In order to state our results precisely, we give our definitions and notation.

A *generalized quadrangle* (GQ) of order (s,t) is a point–line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ consisting of a set $\mathcal{P}$ of points, a set $\mathcal{L}$ of lines, and a symmetric incidence relation $\mathtt{I}$ satisfying the following conditions.

- Every line is incident with precisely $s+1$ points and every point with precisely $t+1$ lines.

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- Two distinct points are never incident with two distinct lines.
- For every point x and every line L not incident with x, there exist a unique point y and a unique line M such that $x \mathtt{I} M \mathtt{I} y \mathtt{I} L$.

We will only be interested in finite generalized quadrangles, which is equivalent to restricting to finite s and t. We will use the following terminology. Two points (lines) incident with the same line (point) are *collinear (concurrent)*; two elements not incident with the same element are *opposite*. A *spread* of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a set of lines of $\mathcal{S}$ such that every point of $\mathcal{S}$ is incident with exactly one member of the spread. If we view the lines of $\mathcal{S}$ as sets of points incident with them, then a spread is a partition of $\mathcal{P}$ into lines. An *ovoid* is the dual notion, i.e., we interchange the role of points and lines in the definition of spread. It is well-known (see e.g. [2]) that every ovoid and every spread of a GQ of order (s, t) contains precisely $st + 1$ elements.

A *collineation* φ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a pair of permutations of $\mathcal{P}$ and $\mathcal{L}$ (both denoted by φ; this does not cause any confusion) such that both φ and its inverse preserve the incidence relation $\mathtt{I}$. The GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ will be called a *translation generalized quadrangle* (TGQ) with respect to the element $X \in \mathcal{P} \cup \mathcal{L}$ if there is a (necessarily unique) commutative group G of collineations of $\mathcal{S}$ fixing all elements incident with X and acting sharply transitively on the set of elements opposite X. The group G will be called the *translation group* with respect to X.

A *duality* φ of a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a pair of bijections from $\mathcal{P}$ to $\mathcal{L}$ and from $\mathcal{L}$ to $\mathcal{P}$ (both denoted by φ; this does not cause any confusion) such that both φ and its inverse preserve the incidence relation $\mathtt{I}$. A GQ is called *self-dual* if it admits a duality. A *polarity* is a duality of order 2. A *self-polar* GQ is one that admits a polarity. A group of collineations and dualities will be called a *correlation group*.

Regarding collineations and dualities, we use the same terminology for projective spaces (so collineations preserve the dimension of subspaces while dualities and polarities of $\mathsf{PG}(d, q)$ interchange subspaces of dimension k with subspaces of dimension $d - k - 1$).

The GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s, t) is *(laxly) embedded in* $\mathsf{PG}(d, q)$, with $d \geq 2$, if $\mathcal{P}$ is a generating subset of the point set of $\mathsf{PG}(d, q)$, if $\mathcal{L}$ is a subset of the line set of $\mathsf{PG}(d, q)$, and if a point x of $\mathcal{S}$ is incident with a line L of $\mathcal{S}$ in $\mathsf{PG}(d, q)$ as soon as $x \mathtt{I} L$ in $\mathcal{S}$. The embedding is *full* if $s = q$; it is called *polarized* if, for every point $x \in \mathcal{P}$, the set of points of $\mathcal{S}$ collinear in $\mathcal{S}$ with x does not generate $\mathsf{PG}(d, q)$; it is called *grumbling* if both s and t are powers of the same prime p and p does not divide q. If $d = 2$, we call the embedding *planar*. If G is a collineation (correlation) group of $\mathcal{S}$, then we call the embedding *locally G-homogeneous* if every element of G is the restriction to $\mathcal{P} \cup \mathcal{L}$ of a collineation (collineation or duality) of $\mathsf{PG}(d, q)$. It is called *(globally) G-homogeneous* if G is the restriction to $\mathcal{P} \cup \mathcal{L}$ of a collineation (correlation) group G' of $\mathsf{PG}(d, q)$ and $|G| = |G'|$.

Planar embeddings of generalized quadrangles exist in abundance. Indeed, consider any embedding of the GQ $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ in $\mathsf{PG}(d, q)$. Possibly after extending $\mathsf{PG}(d, q)$ to $\mathsf{PG}(d, q^e)$, with e large enough, one can find a subspace U of (projective) dimension $d - 3$ with the properties that (1) no subspace of dimension $d - 2$ containing U meets $\mathcal{P}$ in at least two points, and (2) no hyperplane containing U contains at least two members of $\mathcal{L}$. Projecting $\mathcal{P} \cup \mathcal{L}$ from U onto a plane skew to U yields a planar embedding. Obviously, such embeddings can never be full or polarized. Also, one sees that usually s will be much smaller than q^e. An embedding that does not arise from a "proper" projection is called *dominant*.

The symplectic GQ $\mathsf{W}(q)$ is defined as follows. Its point set is the set of points of $\mathsf{PG}(3, q)$; its line set is the set of fixed lines of a (fixed) symplectic polarity (and the incidence relation is inherited from $\mathsf{PG}(3, q)$). This definition yields a full and polarized embedding of $\mathsf{W}(q)$ in $\mathsf{PG}(3, q)$. A zero-dimensional subspace U as above can only be found for $e > 2$. Nevertheless, for $q = 2$, there exists a planar embedding of $\mathsf{W}(2)$ in $\mathsf{PG}(2, 4)$. This embedding can be described as follows. Fix a hyperoval $\mathcal{H}$ in $\mathsf{PG}(2, 4)$. Then it is well-known (see e.g. [1]) that the 15 points of $\mathsf{PG}(2, 4)$ not in $\mathcal{H}$ and 15 secant lines define a geometry isomorphic to $\mathsf{W}(2)$. We will call this embedding the *hyperoval-embedding* of $\mathsf{W}(2)$. It is globally G-homogeneous, with G the full correlation group of $\mathsf{W}(2)$, which is isomorphic to the automorphism group of the symmetric group on 6 letters.

The GQ $\mathsf{W}(2)$ has another natural embedding, from which all other embeddings in $\mathsf{PG}(d, q)$ with q even follow (by extension and projection as explained above). This embedding arises from a nonsingular quadric $\mathsf{Q}(4, 2)$ in $\mathsf{PG}(4, 2)$. Note that every ovoid of $\mathsf{W}(2)$ arises in this representation from the intersection with a(n elliptic) hyperplane (i.e., a hyperplane meeting $\mathsf{Q}(4, 2)$ in an elliptic quadric $\mathsf{Q}^-(3, 2)$).

The hyperoval-embedding of $\mathsf{W}(2)$ has interesting geometric properties. For instance, every ovoid is contained in a line of $\mathsf{PG}(2, 4)$ and the lines of any spread of $\mathsf{W}(2)$ meet in a fixed point of the hyperoval $\mathcal{H}$. This is in accordance with the fact that there are precisely 6 ovoids and 6 spreads of $\mathsf{W}(2)$, and precisely 6 external lines of $\mathcal{H}$ (which form a dual hyperoval).

In the present paper we will show the following results.

Theorem 1. *Let $\mathsf{W}(2)$ be embedded in $\mathsf{PG}(2, q)$.*

(i) *If at least two ovoids of $\mathsf{W}(2)$ are contained in a line of $\mathsf{PG}(2, q)$, then q is even and the embedding is dominant. If at least three ovoids of $\mathsf{W}(2)$ are contained in a line of $\mathsf{PG}(2, q)$, then $q = 4^e$ and there is a subplane $\mathsf{PG}(2, 4)$ of $\mathsf{PG}(2, q)$ containing all points and lines of $\mathsf{W}(2)$ such that the embedding in $\mathsf{PG}(2, 4)$ is the hyperoval-embedding.*

(ii) *Dually, if the lines of at least two spreads of $\mathsf{W}(2)$ contain a respective common point of $\mathsf{PG}(2, q)$, then q is even and the embedding is dominant. If the lines of at least three spreads of $\mathsf{W}(2)$ contain a respective common point of $\mathsf{PG}(2, q)$, then $q = 4^e$ and there is a subplane $\mathsf{PG}(2, 4)$ of $\mathsf{PG}(2, q)$ containing all points and lines of $\mathsf{W}(2)$ such that the embedding in $\mathsf{PG}(2, 4)$ is the hyperoval-embedding.*

(iii) *If at least two ovoids of $\mathsf{W}(2)$ are contained in a line of $\mathsf{PG}(2, q)$, and the lines of some spread of $\mathsf{W}(2)$ contain a common point of $\mathsf{PG}(2, q)$, then the lines of some second spread of $\mathsf{W}(2)$ contain a common point of $\mathsf{PG}(2, q)$.*

(iv) *Let x be a point of $\mathsf{W}(2)$ and let L be a line of $\mathsf{W}(2)$. Suppose that the two ovoids of $\mathsf{W}(2)$ containing x are contained in a respective line of $\mathsf{PG}(2, q)$, and that the lines of the two spreads of $\mathsf{W}(2)$ containing L contain a common respective point of $\mathsf{PG}(2, q)$. If p is not incident with L, then $q = 4^e$ and there is a subplane $\mathsf{PG}(2, 4)$ of $\mathsf{PG}(2, q)$ containing all points and lines of $\mathsf{W}(2)$ such that the embedding in $\mathsf{PG}(2, 4)$ is the hyperoval-embedding. If p is incident with L in $\mathsf{W}(2)$, then the embedding can be such that no other ovoid is contained in a line of $\mathsf{PG}(2, q)$.*

The connection of the above geometric results with groups is given in the following theorem. Note that $\mathsf{W}(2)$ is a TGQ with respect to every element.

Theorem 2. *Let $\mathsf{W}(2)$ be embedded in $\mathsf{PG}(2, q)$. Let G be the translation group with respect to the element $X \in \mathcal{P} \cup \mathcal{L}$. If X is a point, then the two ovoids through X are contained in a respective line of $\mathsf{PG}(2, q)$ if the embedding is G-homogeneous. Conversely, if the two ovoids through X are contained in a respective line of $\mathsf{PG}(2, q)$, then the embedding is H-homogeneous, with H the subgroup of index 2 of G stabilizing the ovoids through X. Dually, if X is a line, then the lines of the two spreads containing X contain a common respective point if the embedding is G-homogeneous, and, conversely, if the lines of the two spreads of $\mathsf{W}(2)$ containing X are incident with a common respective point, then the embedding is H'-homogeneous, with H' the subgroup of index 2 of G stabilizing the spreads through X. Consequently, the embedding is contained in a subplane $\mathsf{PG}(2, 4)$ and is the hyperoval-embedding in it if and only if it is G-homogeneous for two translation groups G with respect to two elements of $\mathsf{W}(2)$ that are not incident.*

We also present a proof of the following result, which can also be found in [3].

Theorem 3. *Let the translation generalized quadrangle $\mathcal{S}$ with translation group G be G-homogeneously embedded in the projective plane $\mathsf{PG}(2, q)$. Then $\mathcal{S}$ is isomorphic to $\mathsf{W}(2)$ and we can apply the previous theorem.*

This theorem shows that $\mathsf{W}(2)$ really plays a special role in the theory of planar embeddings. It feeds the conjecture that no other (classical) generalized quadrangle admits a dominant planar embedding.

Remark. Some of the above results (in casu Theorem 3, and a part of Theorem 2) are also proved in [3]. The proofs presented here, however, are almost all purely geometric ones, unlike the ones in [3], and consequently shorter and more elegant. This also shows that ovoids play an important role in the theory of planar embeddings and in the theory of TGQs.

2. Proofs

We break up the results in small lemmas. Throughout we assume that we are given an embedding of $\mathsf{W}(2)$ in $\mathsf{PG}(2, q)$.

Lemma 4. *If two ovoids of* $\mathsf{W}(2)$ *are contained in respective lines of* $\mathsf{PG}(2, q)$*, then* q *is even.*

Proof. Suppose the two ovoids contained in a line of $\mathsf{PG}(2, q)$ are given by the sets of points $\{p, a, b, c, d\}$ and $\{p, a', b', c', d'\}$. Without loss of generality, we may choose the notation in such a way that a, b, c, d are opposite a', b', c', d', respectively. Then the other points of $\mathsf{W}(2)$ are given by $\{xy' \cap x'y \mid x, y \in \{a, b, c, d\}, x \neq y\}$. We denote $p_{xy} = xy' \cap x'y$, for $x, y \in \{a, b, c, d\}, x \neq y$. By Pappus' theorem the points p_{ab}, p_{ac} and p_{bc} are collinear. But these three points lie on different lines of $\mathsf{W}(2)$ incident with p. Hence the lines pp_{ab}, pp_{ac} and pp_{bc} of $\mathsf{PG}(2, q)$ are all distinct.

Let q_{xy} be the intersection of the lines xx' and yy' in $\mathsf{PG}(2, q)$, for $x, y \in \{a, b, c, d\}$, $x \neq y$. The theorem of the complete quadrilateral implies that the quadruple of lines $(pa, pa', pp_{ab}, pq_{ab})$ is harmonic. Since $pp_{ab} = pp_{cd}$ (since $\{p, p_{ab}, p_{cd}\}$ forms a line of $\mathsf{W}(2)$), we deduce that p, q_{ab} and q_{cd} are collinear. Similarly p, q_{ac}, q_{bd} and p, q_{ad}, q_{bc} are collinear. Since pp_{ab}, pp_{ac} and pp_{ad} are three distinct lines of $\mathsf{W}(2)$, the lines pq_{ab}, pq_{ac} and pq_{ad} are distinct. If $q_{ab} = q_{cd}$, then $q_{ab} = q_{cd} = q_{ac} = q_{ad}$ in contradiction with the fact that pq_{ab} and pq_{ac} are distinct. Hence we now see that the points $p, q_{ab}, q_{ac}, q_{ad}, q_{bc}, q_{bd}$ and q_{cd} are pairwise distinct and form a subplane of order 2 of $\mathsf{PG}(2, q)$, implying the assertion. □

Lemma 5. *If three ovoids of* $\mathsf{W}(2)$ *are contained in respective lines of* $\mathsf{PG}(2, q)$*, then all ovoids are.*

Proof. We use the same notation as in the proof of the previous lemma. In addition, we assume that the points d, d', p_{ab}, p_{ac} and p_{bc} are collinear (these points form a third ovoid in $\mathsf{W}(2)$, and all "third" ovoids play the same role). We first claim that a, a' and p_{bc} are collinear. Indeed, from the proof of the previous lemma, we know that the points p, q_{ad} and q_{bc} are collinear in $\mathsf{PG}(2, q)$, and that the quadruple $(pa, pa', pp_{bc}, pq_{bc})$ is harmonic. Since q is even – again by Lemma 4 – this implies that $pp_{bc} = pq_{bc} = pq_{ad}$ (noting that $pa = pa'$ would lead to the fact that all points of $\mathsf{W}(2)$ would be contained in the line pa of $\mathsf{PG}(2, q)$, a contradiction). Hence the point p_{bc} is incident with both lines pq_{ad} and dd', implying $p_{bc} = q_{ad}$, and the claim follows.

Now we claim that the other points p_{bd} and p_{cd} are also incident in $\mathsf{PG}(2, q)$ with the line aa'. Indeed, applying Pappus' theorem as in the previous proof, we see that p_{bc}, p_{bd} and p_{cd} are collinear. Similarly, applying Pappus' theorem on the hexagon formed by the points $p, p_{ab}, b, d', c, p_{ac}$, we deduce that the points a', p_{bd} and p_{cd} are collinear in $\mathsf{PG}(2, q)$. The claim follows. Similarly one shows that also the other ovoids are contained in lines of $\mathsf{PG}(2, q)$. □

Lemma 6. *If all ovoids of* $\mathsf{W}(2)$ *are contained in respective lines of* $\mathsf{PG}(2, q)$*, then the lines of every spread are incident with a common point of* $\mathsf{PG}(2, q)$*.*

Proof. Let S be a spread of $\mathsf{W}(2)$. We leave it to the reader to verify that, without loss of generality, we may choose the notation in the proof of Lemma 4 in such a way that S contains the lines ab', bc', cd', da' and pp_{ac}. Note that the ovoids of $\mathsf{W}(2)$ lie on the lines aa', bb', cc', dd', pa and pa'. Hence the point p_{ab} is the intersection of the lines cc' and dd' (since, as a point of $\mathsf{W}(2)$, it belongs to the corresponding ovoids). Consequently, the perspectivity with center p_{ab} mapping the point row pa to the point row pa' maps a to b', b to a', c to c' and d to d'. Applying similarly next the perspectivity from pa' to pa with center p_{ac}, and finally the one from pa to pa' with center p_{ad}, we see that this composition is a perspectivity (since it is a projectivity which fixes p) that maps a to b', b to c', c to d' and d to a', proving that the lines ab', bc', cd' and da' of S are incident with a common point x (the center of the composite perspectivity). In the complete quadrilateral $\{a, b, b', c'\}$, the diagonal points p, p_{ac} and x are collinear (since q is even), and so also the fifth line of the spread S is incident with x. □

Lemma 7. *Suppose that every ovoid of* $\mathsf{W}(2)$ *is contained in a line of* $\mathsf{PG}(2, q)$*, and that the lines of every spread of* $\mathsf{W}(2)$ *contain a common point of* $\mathsf{PG}(2, q)$*. Then* $q = 4^e$ *and there is a subplane* $\mathsf{PG}(2, 4)$ *of* $\mathsf{PG}(2, q)$ *containing all points and lines of* $\mathsf{W}(2)$ *such that the embedding in* $\mathsf{PG}(2, 4)$ *is the hyperoval-embedding.*

Proof. It is easily checked that the 15 points and 15 lines of $\mathsf{W}(2)$ together with the 6 lines of $\mathsf{PG}(2, q)$ containing the ovoids of $\mathsf{W}(2)$ and the 6 points of $\mathsf{PG}(2, q)$ incident with the lines of the spreads of $\mathsf{W}(2)$ constitute, with the natural incidence, a subplane of order 4 of $\mathsf{PG}(2, q)$ in which the latter 6 points form a hyperoval. □

Lemma 8. *Suppose that the two ovoids through the point p of $\mathsf{W}(2)$ are contained in a line of $\mathsf{PG}(2, q)$, and that the lines of some spread S are incident with a common point of $\mathsf{PG}(2, q)$. Let L be the member of S incident with p. Then also the lines of the second spread containing L are incident with a common point of $\mathsf{PG}(2, q)$.*

Proof. As before, we may choose the notation in such a way that the ovoids are $\{p, a, b, c, d\}$ and $\{p, a', b', c', d'\}$, with x opposite x', for all $x \in \{a, b, c, d\}$, and S consists of the lines ab', bc', cd', da' and pp_{ac} (see above for the notation). The second spread through $L = pp_{ac}$ contains the lines ba', cb', dc' and ad'. Since q is even, the theorem of the complete quadrilateral applied to $\{a, c, b', d'\}$ and $\{b, d, a', c'\}$ implies that the intersection point r of ad' and cb' and the intersection point s of ba' and dc' are incident with L. Considering the quadrilateral $\{a, b, b'c'\}$, we see that ac' and bb' meet on L. Since also ac' and $a'c$ meet on L, the lines bb' and $a'c$ meet on L. Considering the quadrilateral $\{b, c, a', b'\}$, we deduce that ba' and cb' meet on L, i.e., $r = s$. $\square$

Lemma 9. *Let p be a point of $\mathsf{W}(2)$ and let G be the translation group with respect to the translation point p. Suppose that the embedding is globally G-homogeneous. Then each of the two ovoids through p is contained in a line of $\mathsf{PG}(2, q)$.*

Proof. Since G is an elementary 2-group, the intersection $G \cap \mathsf{PGL}_3(q)$ has size 4 or 8. Hence there are at least four "linear" collineations in G. So we may consider a subgroup H of G of order 4 entirely contained in $\mathsf{PGL}_3(q)$. Let $\{a, b, c, d\}$ be an orbit of H of order 4 in $\mathsf{W}(2)$ of points opposite p. Notice that each element of H fixes all lines of $\mathsf{PG}(2, q)$ through p. If ab were a line of $\mathsf{W}(2)$, then the third point on ab must be p (since ab contains p in $\mathsf{PG}(2, q)$), a contradiction. So $\{p, a, b, c, d\}$ is an ovoid and all its points are contained in a line of $\mathsf{PG}(2, q)$. Similarly for the other orbit. $\square$

Lemma 10. *If at least two ovoids of $\mathsf{W}(2)$ are contained in respective lines of $\mathsf{PG}(2, q)$, then the embedding is dominant.*

Proof. Suppose the embedding is not dominant. Then it arises from a projection of another embedding, and by the results of [4], it arises from a projection of the standard embedding in $\mathsf{PG}(4, 2)$ (obtained from the embedding in $\mathsf{PG}(4, 2)$ after field extension) from a line L. Every ovoid of $\mathsf{W}(2)$ embedded in $\mathsf{PG}(4, q)$ is contained in a 3-space of $\mathsf{PG}(4, q)$. Hence L must be contained in the intersection π of two such 3-spaces. But π is also contained in a 3-space that meets $\mathsf{W}(2)$ in the lines incident with the common point of the two ovoids. Consequently, these three lines are projected onto the same line and we obtain a contradiction. $\square$

Now Theorem 1 follows from the previous lemmas and their duals.

In order to finish the proof of Theorem 2, we cannot continue in a purely geometric way, as we did up to now. Here, we must coordinatize. We assume that two ovoids of $\mathsf{W}(2)$ are contained in lines of $\mathsf{PG}(2, q)$ and we use the notation of the proof of Lemma 4. We may assign coordinates as follows.

$$
\begin{aligned}
p &= (1, 0, 0) \qquad & a' &= (0, 0, 1)\\
a &= (0, 1, 0) & b' &= (1, 0, 1)\\
b &= (1, 1, 0) & c' &= (\alpha', 0, 1)\\
c &= (\alpha, 1, 0) & d' &= (\beta', 0, 1)\\
d &= (\beta, 1, 0) & &
\end{aligned}
$$

We can now compute the coordinates of all points p_{xy}, $x, y \in \{a, b, c, d\}$, $x \neq y$, and write down the conditions under which p, p_{ab}, p_{cd} are collinear, p, p_{ac}, p_{bd} and p, p_{ad}, p_{bc} are collinear, and these three lines are different. Without going into (the elementary) details, we content ourself with mentioning that these conditions reduce to $\beta = 1 + \alpha$, $\beta' = 1 + \alpha'$ and $\alpha \neq \alpha'$, $|\{\alpha, \alpha'\} \cap \{0, 1\}| = 0$. But then it is easily checked that the collineation of $\mathsf{PG}(2, q)$ induced by the mapping $(x, y, z) \mapsto (\alpha + y + z, y, z)$ preserves $\mathsf{W}(2)$ and defines an element of the translation group G with respect to p interchanging a with b, a' with b', c with d and c' with d'. Similarly (or by recoordinatization and then using the same arguments) we can find linear collineations of G interchanging a with c, and interchanging a with d.

Also, one can check that the lines of the spread $\{pp_{ac}, ab', bc', cd', da'\}$ go through a common point if and only if $\alpha + \alpha' = \alpha\alpha'$. The embedding is the hyperoval-embedding in a subplane if and only if $\alpha + \alpha' = \alpha\alpha' = 1$, as one can easily calculate.

This completes the proof of Theorem 2.

We now prove Theorem 3, using ideas developed above.

Let $\mathcal{S}$ be a TGQ of order (s, t) with translation group G and translation point x, and let it be G-homogeneously embedded in $\mathsf{PG}(2, q)$. Every element of G fixes all lines of $\mathcal{S}$ through x. Also, G is an elementary abelian p-group, for some prime p. Clearly, G cannot fix all lines of $\mathsf{PG}(2, q)$ through x, so not all elements of G are linear. But since the automorphism group of the Galois field $\mathsf{GF}(q)$ is cyclic, G contains a linear subgroup H of prime index, that prime being necessarily equal to p. As we showed above for $\mathsf{W}(2)$, one sees that the orbits of H on the set of points of $\mathcal{S}$ opposite x are sets of s^2t/p pairwise opposite points. Since there are at most st pairwise opposite points opposite x (together with x forming an ovoid in that extreme case), we must have $p = s$. Consequently $t \in \{p, p^2\}$. Since quadrangles of order (p, p^2) do not admit ovoids, we deduce $t = p$ and $\mathcal{S}$ is isomorphic to the dual of $\mathsf{W}(p)$ (see [2] for these claims). Note that the orbits of H on the set of points opposite x are ovoids, if completed with x. Consider such an ovoid O.

Now suppose p is odd. We recall from [2] that x is an anti-regular point, i.e., if two opposite points opposite x are collinear with some point collinear to x, then they are collinear with exactly two such points. Considering two points y, z of $O \setminus \{x\}$ collinear with the same point u collinear with x (this certainly exists by the definition of ovoid), we see that there is a second point v collinear with all of x, y, z, and the element of H mapping y to z has order p and must hence fix all points of $\mathcal{S}$ – and hence of $\mathsf{PG}(2, q)$ – on the lines xu and xv. This is a contradiction.

Theorem 3 is proved. $\square$

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