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# A note on packing of three forests

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**Abstract**

We present the complete result concerning the packing (i.e. the edge-disjoint placement) of three forests into the complete graph $K_n$

## 1. Terminology

We shall use standard graph theory notation. A finite, undirected graph $G$ consists of a vertex set $V(G)$ and edge set $E(G)$. All graphs will be assumed to have neither loops nor multiple edges. If we sort the degrees of the vertices of $G$ in non-decreasing order, $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$, we denote the maximum degree $d_n$ by $\Delta(G)$, the minimum $d_1$ by $\delta(G)$ and $d_2$ by $\delta'(G)$. We denote by $G\backslash e$ any graph obtained by removing one edge from $G$.

For graphs $G$ and $H$ we denote by $G \cup H$ the vertex disjoint union of graphs $G$ and $H$ and $kG$ stands for the vertex disjoint union of $k$ copies of $G$. Suppose $G_1, \dots, G_k$ are graphs of order $n$. We say that there is a *packing* of $G_1, \dots, G_k$ (into the complete graph $K_n$) if there exist injections $\alpha_i : V(G_i) \to V(K_n)$, $i = 1, \dots, k$, such that $\alpha_i^*(E(G_i) \cap \alpha_j^*(E(G)) = \emptyset$ for $i \neq j$, where the map $\alpha_i^*\colon E(G_i) \to E(K_n)$ is the one induced by $\alpha_i$. From now on, we shall name *packing* the couple of functions $(\alpha : \bigcup V(G_i) \to V(K_n),\ \alpha^*\colon \bigcup E(G_i) \to E(K_n))$ induced by the $\alpha_i$, $\alpha_i^*$, and for the sake of brevity, we shall denote the couple by its first element $\alpha$. We shall also use the notation $\alpha(G_i)$ instead of $(\alpha(V(G_i)), \alpha^*(E(G_i)))$.

A packing of $k$ copies of a graph $G$ will be called a *k-placement* of $G$. A packing of two copies of $G$ (i.e. a 2-placement) is an *embedding* of $G$ (in its complement $\bar{G}$). So, an embedding of a graph $G$ is a permutation $\sigma$ on $V(G)$ such that if an edge $xy$ belongs to $E(G)$ then $\sigma(x)\sigma(y)$ does not belong to $E(G)$.

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The main references of this paper and of other packing problems are the last chapter of Bollobás's book [1], the 4th chapter of Yap's book [12] and the survey paper [13].

We shall need some additional definitions in order to formulate the results. Recall that $S_n$, $P_n$, respectively, denote the star, the path on $n$ vertices. Let $S'_n$ be the graph obtained by subdividing one of the edges of $S_{n-1}$. By analogy, denote by $S''_n$ the tree obtained by replacing one of the edges of $S_{n-2}$ by a path of length 3 and by $S'''_n$ the tree obtained by replacing one of the edges of $S_{n-3}$ by a path of length 4.

In a graph $G$ a vertex of degree one will be called an *end-vertex* or *a pendant vertex*. A pendant vertex in a tree is also called *a leaf*. An edge incident with an end-vertex is an *end-edge* or a *pendant edge*. If the tree is not $P_2$, the other extremity of a pendant edge is not an end-vertex, and we shall call it a *knot* of the tree.

The number of independent end-edges in a tree plays an important role in packing problems. Each non-star tree of order $\geqslant 4$ has at least two knots. The trees having exactly two knots are called *star-path-stars*. More precisely, the star-path-stars are the trees obtained from a path $a_0a_1\ldots a_r$, $r\geqslant 1$, by adding $q\geqslant 1$ edges $a_ry_i$, $1\leqslant i\leqslant q$, incident to one extremity of the path, and $p\geqslant \min\{q,2\}$ other edges $a_0z_j$, $1\leqslant j\leqslant p$, incident to the other extremity (with, obviously, $p+q+r=n-1$). We use for such a tree if $q\geqslant 2$ the notation $S^r_n(p,q)$, omitting the parameters $p$ and $q$ in the cases $n=6,7$ where there is only one possibility (so $S^1_6=S^1_6(2,2)$ and $S^1_7=S^1_7(3,2)$). In the case when $q=1$ this tree is called also a *comet* and we shall note it $S^{(r)}_n$ (and especially $S'$ for $r=1$, $S''$ for $r=2$ and $S'''$ for $r=3$). The vertices $a_0$ and $a_r$ will be called the *knots* of the star-path-star, $a_0$ being the *great* knot and $a_r$ the *small* one. In the case of a comet, we often shall say *the knot* for the vertex $a_0$ exclusively, but for the sake of generalization, it will remain possible to say that $a_r$ is the other knot (note that its end-neighbor is the effective extremity of the path of the comet). Observe that $S'''_6$ is simply $P_6$, the path of order 6.

The trees $S^1_n(p,q)$ are called also *double stars*.

For $n\geqslant 6$ we denote by $X_n$ the tree on $n$ vertices obtained from the star $S_{n-2}$ by replacing two edges, each by a path of length 2.

By $Y_7$ we denote the tree on seven vertices obtained from the star $S_4$ by introducing three new vertices on three edges of $S_4$.

**Remark.** In the notation concerning members of the families of graphs such as trees or forests, the subscript *does not* denote, in general, the order of the graph.

## 2. Results

The following theorem was proved by Burns and Schuster in [2].

**Theorem 1.** *Let $G=(V,E)$ be a graph of order $n$. If $|E(G)|\leqslant n-2$, then $G$ can be embedded in its complement.*

This result has been improved in many ways. For instance, Sauer and Spencer proved in [8] the following improvement of Theorem 1.

**Theorem 2.** *Let $G$ and $H$ be two graphs of order $n$. If $|E(G)| \leqslant n-2$ and $|E(H)| \leqslant n-2$, then $G$ and $H$ are packable.*

The example of the star $S_n$ shows that neither Theorem 1 nor Theorem 2 can be improved by raising the size of $G$ even in the case when $G$ is a tree. However, in this case we have

**Theorem 3.** *Let $T$ be a tree of order $n$, $T \neq S_n$. Then $T$ is contained in its own complement.*

Theorem 3 was first proved by Straight (unpublished, cf. [4]). It is improved by the following theorem (cf. [4]).

**Theorem 4.** *Any two trees of order $n$, neither of which is a star, can be packed into $K_n$.*

The first theorem concerning packing of three trees was probably proved in connection with the following well-known conjecture stated by Gyárfás in [3], which remains open.

**Conjecture 5.** Let $T_i$ denote a tree of order $i$. The sequence of trees $T_2, T_3, \ldots, T_n$ can be packed into $K_n$.

The above conjecture is sometimes called the *Tree Packing Conjecture* (*TPC*). Note that if we add up the sizes of the trees, we obtain the size of the complete graph $K_n$.

Hobbs et al. [5] proved that

**Theorem 6.** *Any three trees of orders $n_1 < n_2 < n_3 \leqslant n$, respectively, can be packed into $K_n$.*

Inspired by the above theorem, a similar result has been obtained in [10].

**Theorem 7.** *Any three trees of order $n-1$ can be packed into $K_n$.*

The following conjecture of Bollobás and Eldridge [1] is related to Theorems 6 and 7.

**Conjecture 8.** Let $G_1, \ldots, G_k$ be $k$ graphs of order $n$. If $|E(G_i)| \leqslant n-k$, $i = 1, \ldots, k$, then $G_1, \ldots, G_k$ are packable into $k_n$.

The following theorem extends Theorem 1 [11].

**Theorem 9.** *Let $G = (V,E)$ be a graph of order $n$, $G \neq K_3 \cup 2K_1$, $G \neq K_4 \cup 4K_1$. If $|E(G)| \leqslant n-2$, then there exists a 3-placement of $G$.*

Motivated by this result, Wang and Sauer considered the 3-placement of a tree.

Observe now that if there is a 3-placement of a tree $T$ in $K_n$ then we have obviously $3(n-1) \leqslant \binom{n}{2}$ which implies that $n \geqslant 6$. Moreover, since the vertex $v \in V(T)$ such that $d(v) = \Delta(T)$ must be placed with two other vertices of degree at least one, we must assume that $\Delta(T) \leqslant n-3$.

However, these trivial necessary conditions are not sufficient as it is shown by the example of $S_6''$. This fact was observed by Huang and Rosa in [6].

Wang and Sauer [9] proved the following.

**Theorem 10.** *Let $T$ be a tree of order $n$, $n \geqslant 6$, $T \neq S_n$, $T \neq S_n'$ and $T \neq S_6''$. Then there exists a 3-placement of $T$.*

The general theorem about the packing of three trees of maximal size was given by Mahéo and the authors in [7].

**Theorem 11.** *If $n$ is an integer with $n \geqslant 6$, one can pack any triple of trees $\mathcal{T} = (T_1, T_2, T_3)$ of order $n$ and maximum degree at most $n-3$ into $K_n$, except for the following (up to a permutation):*

- *For any $n$, the triples $(S_n'', S_n^1(a,b), T_n)$ where $T_n$ is one of the trees $S_n^1(p,q)$, $S_n^2(p,q)$, $S_n^3(p,q)$, $S_n''$, $S_n'''$.*
- *For any odd $n = 2p+3$, the triple $(S_n'', S_n^2(p,p), S_n^2(p,p))$.*
- *For $n = 6$, the triples $(S_6'', S_6'', S_6'')$, $(P_6, X_6, S_6^1)$, $(P_6, S_6^1, S_6^1)$, $(X_6, X_6, S_6^1)$, and $(X_6, S_6^1, S_6^1)$.*
- *For $n = 7$, the triple $(Y_7, S_7^1, S_7^1)$.*

Observe that if we study the packing into the complete graph $K_n$, we can assume that all the graphs we pack are of order $n$. For, if we pack the graphs of order less than $n$, we always may add to them some isolated vertices. So, Theorems 6, 7, 10, 11 can be considered as theorems about the packing of forests.

In this note we present the general case of the packing of three forests, which generalizes all the above results concerning tree-packing. The proof, based mainly on Theorem 11, is given in the next section.

It will be convenient to say that a triple $(F_1', F_2', F_3')$ is a *subtriple* of $(F_1, F_2, F_3)$, if each $F_i'$ is a (partial) subgraph of $F_i$. We may specify *proper subtriple* if they are not equal.

**Theorem 12.** *Let $\mathcal{F} = (F_1, F_2, F_3)$ be a triple of forests of order $n$ such that the following necessary conditions are satisfied:*

(1) $|E(F_1)| + |E(F_2)| + |E(F_3)| \leqslant n(n-1)/2$.

(2) $\forall i = 1,2,3,\ \Delta(F_i) + \delta(F_{i+1}) + \delta(F_{i+2}) \leqslant n-1$ (*where the subscripts greater than 3 are taken modulo 3*).

(2′) *If for any* $i = 1,2,3$ *there is equality (with the same convention as in* (2)) $\Delta(F_i)+\delta(F_{i+1})+\delta(F_{i+2}) = n-1$, *then* $\delta(F_i)+\Delta(F_{i+1})+\delta'(F_{i+2})$ *and* $\delta(F_i)+\delta'(F_{i+1})+\Delta(F_{i+2})$ *are both* $\leqslant n-1$.

(3) *If two forests are isomorphic to* $K_1 \cup S_{n-1}$, *the third one has a component of order* $\leqslant 2$.

*Then there is a packing of these three forests into* $K_n$, *except, if* $n \geqslant 6$, *for the following (up to permutation):*

- *for all integers p and q such that* $2 \leqslant p \leqslant n-4$, $2 \leqslant q \leqslant n-2$, *any triple having as subtriple* $(S_n^1(p, n-p-2), K_2 \cup S_{n-2}, S_q \cup S_{n-q})$,
- *for n odd,* $n = 2p+3$, *any triple having as subtriple* $(S_n^2(p,p), S_n^2(p,p), K_2 \cup S_{n-2})$,
- *for* $n = 6,7$ *any other excluded triple of Theorem* 11.

Note that the first excluded triples $(S_n'', S_n^1(a,b), T_n)$ of Theorem 11 belong to the first case of Theorem 12, and the second family $(S_n'', S_n^2(p,p), S_n^2(p,p)$ to the second case.

## 3. Proof

In the two first subsections we study the hypothesis of the theorem and the excluded triples. Next, we give the proof, dividing the general case into subcases, according to the value of the greater degrees of the forests in the triple. The main tool is Theorem 11.

### 3.1. Necessary conditions

Conditions (1), (2) and (2′) are obviously necessary. We just have to explain condition (3), and we prove the theorem for this case.

Let $F_1$ and $F_2$ be both equal to $K_1 \cup S_{n-1}$. There are, up to isomorphism, only two ways to pack them together into $K_n$.

— The first way is to pack the knot of each star with the isolated vertex of the other forest. This packing lets free in $K_n$ the edges of a $K_{n-2}$ and another independent edge. Therefore, this packing allows any forest with a $K_2$-component to be packed with the pair $F_1, F_2$.

— The second way is to pack the knot of the star-component of $F_1$ with a leaf of $F_2$. This packing lets free the edges of a $K_{n-2}$ and another edge incident to some vertex of this $K_{n-2}$. This allows now any forest having a $K_1$-component.

From now on, we will assume that the triples are not of the previous form.

### 3.2. Excluded triples

We inspect the excluded triples, other than those given in Theorem 11. Let us consider a packing $\alpha$ of $(F_1, F_2)$ with $F_1 = S_n^1(p, n-p-2)$, $F_2 = K_2 \cup S_{n-2}$ into $K_n$. Denote by $(x_1, y_1)$ the knots of $F_1$ and by $x_2$ the knot of the star-component in

$F_2$. We must have $\alpha(x_2) = \alpha(z_1)$ where $z_1$ is a leaf, say of $x_1$ in $F_1$, implying that $\alpha(K_2) = \alpha(x_1)\alpha(t_1)$ where $t_1$ is a leaf of $y_1$. Now, in the set $\alpha\{x_1, y_1, z_1, t_1\}$ only the edge $\alpha(z_1)\alpha(t_1)$ remains free. It is therefore impossible to pack also $F_3 = S_q \cup S_{n-q}$ since $\alpha(z_1)$ should be covered by a knot of $F_3$ packed at $\alpha(t_1)$, and $\alpha\{x_1, y_1\}$ should be covered by the other knot of $F_3$. But for every other vertex $u \in K_n$, either $u\alpha(x_1)$ or $u\alpha(y_1)$ is in $\alpha(F_1)$.

Let us now pack, for $n = 2p + 3$, $(F_1, F_2)$ with $F_1 = F_2 = S_n^2(p, p)$ into $K_n$. We name, for $i = 1, 2$, $(x_i, y_i)$ the knots of these trees, and $z_i$ their common neighbor. We denote moreover by $t_1$ a leaf of $x_1$, by $u_1, u_1'$ two leaves of $y_1$, in order to describe the only three possible (up to an isomorphism) packings $\alpha$ of these two trees:

— The first one is such that $\alpha(x_2) = \alpha(t_1)$, $\alpha(y_2) = \alpha(u_1)$, $\alpha(z_2) = \alpha(z_1)$.

— The second one is such that $\alpha(x_2) = \alpha(t_1)$, $\alpha(y_2) = \alpha(u_1)$, $\alpha(z_2) = \alpha(u_1')$.

— The third one is such that $\alpha(x_2) = \alpha(x_1)$, $\alpha(y_2) = \alpha(t_1)$, $\alpha(z_2) = \alpha(y_1)$.

If we want to pack now $K_2 \cup S_{n-2}$ its star-component's knot should be packed with a leaf of each $F_i$, but then we cannot pack the $K_2$-component since the eventual edges $\alpha(x_1)\alpha(x_2), \alpha(x_1)\alpha(y_2), \alpha(y_1)\alpha(y_2)$ of $K_n$ belong to $\alpha(F_1 \cup F_2)$.

### *3.3. The general case*

One can easily pack the triples satisfying the necessary conditions for $n \leqslant 5$, so we leave this to the reader, and we shall assume that $n \geqslant 6$.

We prove the possibility of packing all triples satisfying the necessary conditions, other than the triples already considered. We may sort these triples in such a way that we have $\Delta_1 \geqslant \Delta_2 \geqslant \Delta_3$.

We distinguish several cases, according to the values of maximum degrees.

*Case* 1: $\Delta_1 = n - 1$. In this case, $F_1 = S_n$ and we must have $F_2 = K_1 \cup F_2'$, $F_3 = K_1 \cup F_3'$.

If $\Delta_2 = n - 2$, then $F_2' = S_{n-1}$ implying by (2′) that $F_3 = 2K_2 \cup F_3''$, and the triple is packable by Theorem 7.

Otherwise, we have $\Delta_2 \leqslant n - 3$, and $F_2', F_3'$ are subforests of non-star trees, the triple is then packable according to Theorem 4.

*Case* 2: $\Delta_1 = \Delta_2 = n - 2$. In the extremal case where $F_1 = F_2 = S_n'$ we must have by (2) and (2′), $F_3 = 2K_1 \cup F_3''$. Since $K_n \backslash E(F_1 \cup F_2) = 2K_1 \cup (K_{n-2} \backslash e)$, any forest $F_3''$ of order $n - 2$ ($\geqslant 4$) is packable with the two previous ones.

It remains, by the above assumptions, the case $F_1 = S_n'$, $F_2 = K_1 \cup S_{n-1}$, implying by (2) $F_3 = K_1 \cup F_3'$ in which we may assume that $F_3' \neq S_{n-1}$. Therefore it suffices to consider the case where $F_3'$ is a non-star tree of order $n - 1$. Let $x_1 y_1 z_1$ be the path of length 2 in $F_1$ where $x_1$ is the knot of degree $n - 2$. By packing $x_1$ with the isolated vertex of $F_2$, and the knot of the star $S_{n-1} \subset F_2$ with any leaf $t_1$ of $x_1$, there remain free in $K_n$ the edges of a $K_{n-2} \backslash e$ (namely $e = \alpha(y_1 z_1)$), plus another edge $\alpha(x_1)\alpha(z_1)$ incident with $e$. Therefore, any non-star tree of order $n - 1$ is packable with $F_1, F_2$.

*Case* 3: $\Delta_1 = n-2$, $\Delta_2 \leqslant n-3$. By (2), we may assume that $\delta_3 = 0$, and it suffices to prove the theorem when $F_1 = S_n'$, $F_2$ is any tree different from $S_n, S_n'$ and $F_3 = K_1 \cup F_3'$

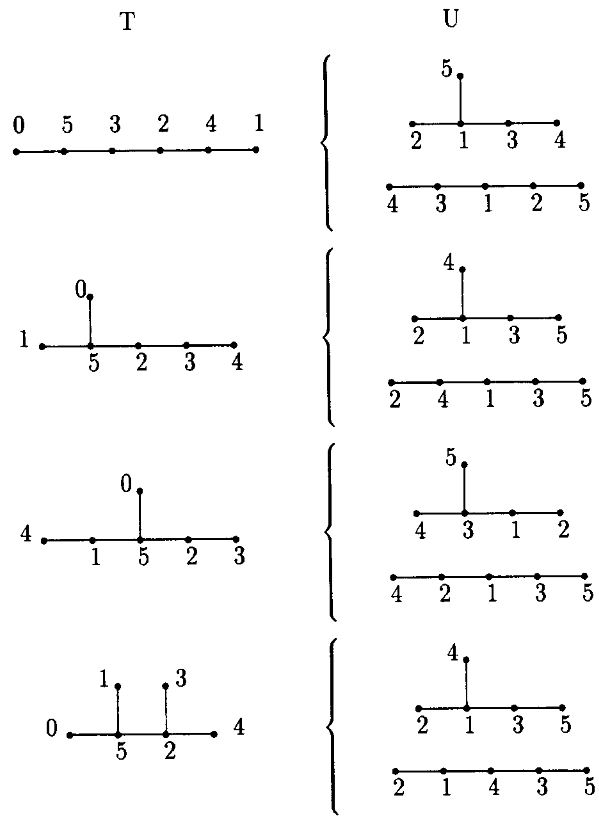


Fig. 1. The case $n = 6$.

where $F_3'$ is a tree of order $n-1$ distinct from $S_{n-1}$. The property comes from the following lemma:

**Lemma 13.** *For $n \geqslant 6$, let $T$ be a tree of order $n$ different from $S_n, S_n'$, and $U$ a non-star tree of order $n-1$. Then the triple $(S_n', T, U)$ is packable into $K_n$.*

**Proof.** Let us first consider the case $n = 6$. There are four non-isomorphic admissible trees $T$, and two possibilities for $U$. We label the vertices of $K_6$, using indices from 0 to 5, pack in 0 the vertex of maximum degree of $S_6'$, its leaves on 1,2,3, its remaining neighbor (the second knot of $S_6'$) on 4 and its leaf on 5. Now Fig. 1 gives (by the positions of the vertices) a packing for all the possible couples $(T, U)$ corresponding to $n = 6$.

For $n = 7$ it suffices to extend these packings in all possible ways, adding to $K_6$ a new vertex labeled 6, and in each case adding to $\alpha(S_6')$ the edge 06, then modifying the two other trees, generally by adding an edge incident to 6, in order to obtain all the admissible couples $T, U$ corresponding to $n = 7$.

Namely, in the first case of $T$ in Fig. 1, we may modify it into a $P_7$ by replacing either the edge 42 by the path 462 or the edge 35 by the path 365. We may modify it into $S_7'''$ by adding either 46 or 56, into another tree by adding 26. Every case lets free all the extensions of $U$, which are achieved by adding one of the edges 16, 56, 36 or 46.

In the second case, in order to obtain other triples, we have to modify $T$ by adding one of the three edges 56,26,36. This allows all extensions of $U$, except for the addition of 36 which prevents that of the same edge to the first form of $U$. In this case we add it to this tree, and modify $T$ by replacing the edge 34 by 24 and 52 by the path 562.

In the third case, we have only to modify $T$ by adding the edge 56, or putting the vertex of maximum degree in 6, its leaf in 5, and adding the edge 06. This allows all modifications of $U$.

In the last case, it suffices to modify $T$ by adding either 56 or 26 and this allows all extensions for $U$.

We now proceed by induction on $n \geqslant 8$, distinguishing several cases. As in [7], we say that a tree $T$ of order $n$ is a *bi-extension* of another tree $T'$ of order $n-2$ if the former is obtained by adding to the latter two vertices, and two independent edges incident with these new vertices. The two other extremities of these new edges are called the *nodes* of the extension.

The general case of our induction is when $T$ and $U$ are both bi-extensions of trees statisfying (at the order $n-2$) the conditions of the lemma; we shall say that they are *admissible* bi-extensions.

(a) *$T$ and $U$ are admissible bi-extensions of $T', U'$ respectively.* By the induction hypothesis, there is a packing $\alpha$ of $S'_{n-2}, T', U'$ into $K_{n-2}$. Let $a$ be the knot of $S'_{n-2}$, $abc$ its path of length 2, and call $i, j$ the nodes of $T'$. Note that $a$ is packed only with an end-vertex of $T'$.

We take now two new vertices $x, y$ and add to $S'_{n-2}$ the edges $ax, ay$. There is no problem for extending $T'$ and $U'$ if $\alpha(a) \notin \alpha\{i,j\}$. So assume that $\alpha(a) = \alpha(i)$. Therefore $i$ is a leaf of a vertex $k \in V(T')$ such that $\alpha(k) = \alpha(c)$. If $k = j$, we have only to add to $T'$ the edges $\alpha(j)x, xy$ in order to obtain $T$, and the extension of $U'$ is obviously possible. Otherwise, we delete the edge $\alpha(ik) = \alpha(a)\alpha(c)$ from $\alpha(T')$, give to it the edge $\alpha(a)x$ and add the new edge $x\alpha(k)$. We give now to $\alpha(S'_{n-2})$ the two edges $\alpha(a)\alpha(c)$, $xy$ instead of $\alpha(a)x$, $\alpha(bc)$, and the extension of $U'$ is also possible.

(b) *$U$ is not an admissible bi-extension, therefore $U = S'_{n-1}$.* Let $x, y, z, t, u$ be five vertices of $K_n$ and call $a$ the knot of $S'_n$, $abc$ its path of length 2, $i$ the knot of $U = S'_{n-1}$, $ijk$ its path of length 2. We may define a first packing of the pair $(S'_n, U)$ by letting $\alpha(a) = x$, $\alpha(b) = \alpha(k) = y$, $\alpha(c) = z$, $\alpha(i) = t$, $\alpha(j) = u$. The free edges in $K_n \backslash \{x,t\}$ form a $K_{n-2} \backslash \{yz, yu\}$ and the edges $xz, yt$ remain also free. Therefore we can easily pack any tree $T_n$ distinct from $S''_n$ or $S^1_n(p,q)$ (which are the only admissible trees with only two adjacent knots). In order to allow these trees, we may define another packing of $(S'_n, U)$ by $\alpha(a) = x$, $\alpha(b) = y$, $\alpha(c) = \alpha(k) = z$, $\alpha(i) = t$, $\alpha(j) = u$, which lets free in $K_n \backslash \{x,t\}$ the edges of $K_{n-2} \backslash \{yz, zu\}$, and also the two edges $zx, zt$.

(c) *$U$ is an admissible bi-extension, but $T$ is not, therefore is $S_n''$, $S_n^1(p,2)$ (with $p = n-4$)* or $X_n$. Assume that $U$ is an admissible bi-extension of $U'$ (of order $n-2$), and put $T' = S_{n-2}''$, $S_{n-2}^1(p-2,2)$, $X_{n-2}$ respectively. So we can recover $T$ from $T'$ by adding two edges incident to the same node. By induction hypothesis, we may pack into $K_{n-2}$ the triple $(S_{n-2}', T', U')$. Note that together with the knot of $S_{n-2}'$ is packed only a leaf of $T'$. Thus there is no difficulty to make the extensions, if no node of $U'$ is packed with the (double) node of $T'$. We now assume the contrary, and distinguish three cases, according to the nature of $T'$. We call $a_1$ the knot of $S'$, $a_1b_1c_1$ its path of length 2, $a_2$ the double node of $T'$, $a_3, b_3$ the nodes of $U'$, and assume $\alpha(a_3) = \alpha(a_2)$. We name $x, y$ the two new vertices. In every case, $a_3$ must be an end-vertex of $U'$, say the leaf of a vertex $c_3$ (eventually $c_3 = b_3$), and we replace in $\alpha(U')$ the edge $\alpha(a_3c_3)$ by the path $\alpha(a_3)x\alpha(c_3)$, and add the edge $\alpha(b_3)y$.

(c1) $T' = S_{n-2}''$. Let $a_2b_2c_2d_2$ be the path of $T'$. We have $\alpha(c_3) \in \alpha\{c_2, d_2\}$.

If $\alpha(c_3) = \alpha(d_2)$ we delete in $\alpha(T')$ the edge $\alpha(b_2c_2)$ and add the edges $\alpha(a_2)\alpha(c_2)$, $\alpha(a_2)y$, $\alpha(c-2)x$ (observe that we have $\alpha(a_1) \notin \alpha\{b_2, c_2, d_2\}$).

If $\alpha(c_3) = \alpha(c_2)$ but $\alpha(a_1) \notin \alpha(d_2)$, we delete in $\alpha(T')$ the edge $\alpha(b_2c-2)$ and add $\alpha(a_2)\alpha(c_2), \alpha(a_2)y, \alpha(d_2)x$.

If $\alpha(c_3) = \alpha(c_2)$ and $\alpha(a_1) = \alpha(d_2)$, then we have $\alpha(c_1) = \alpha(c_2)$ and $b_1$ must be packed with a leaf of $a_2$. We delete in $\alpha(T')$ the path $\alpha(b_2c_2d_2)$, add the edges $\alpha(a_2)\alpha(c_2), \alpha(a_2)y$, and the path $\alpha(b_2)x\alpha(d_2)$. In $\alpha(S_{n-2})$ we delete the edge $\alpha(b_1c_1)$, and add the edges $\alpha(b_1)x, \alpha(a_1)\alpha(c_2), \alpha(a_1)y$.

(c2) $T' = S_{n-2}^1(n-4,2)$. Let $b_2$ be the other knot of $T'$. Then $c_3$ must be packed with a leaf $c_2$ of $b_2$. We delete in $\alpha(T')$ the edge $\alpha(b_2c_2)$ and add the edges $\alpha(a_2)\alpha(c_2)$, $\alpha(a_2)y, \alpha(b_2)x$.

(c3) $T' = X_{n-2}$. Let $a_2b_2c_2$ be the path of length 2 such that $\alpha(c_3) = \alpha(c_2)$. We have $\alpha(a_1) \notin a\{b_2, c_2\}$. We delete in $\alpha(T')$ the edge $\alpha(b_2c_2)$ and add the edges $\alpha(a_2)\alpha(c_2)$, $\alpha(a_2)y$, $\alpha(b_2)x$.

*Case* 4: $\Delta_1 \leqslant n-3$. Then we have to consider subtriples of the triples of trees studied in Theorem 11. It is therefore sufficient to prove our theorem, in the case where $(F_1, F_2, F_3)$ is a proper subtriple of an excluded triple of Theorem 11. Those which contain as subtriple either $(S_n^1(p, n-p-2), K_2 \cup S_{n-2}, S_q \cup S_{n-q})$ or $(S_n^2(p,p), S_n^2(p,p), K_2 \cup S_{n-2})$ are already excluded by our study. Note that $K_2 \cup S_{n-2}$ is obtained from $S_n''$ as well as $S_q \cup S_{n-q}$ (for $q \geqslant 3$) from $S_n^1(q-1, n-q-1)$, by removing the edge joining the two knots. In order to obtain a forest $S_q \cup S_{n-q}$ from $S_n''$ or $S_n^2$, we must remove a non-pendant edge, from $S_n'''$ we have to remove a non-pendant edge non-incident to the knot of maximum degree (for $n \geqslant 7$), and from $S_n^3$ (for $n \geqslant 8$) we have to remove the edge non-incident to any knot.

Let $(F_1, F_2, F_3)$ be an admissible subtriple of some triple $(S_n^1, S_n'', T_n)$ of Theorem 11. In the case where $T_n$ is $S_n'''$, we call $e_1$ the non-pendant edge incident to the knot of degree 2, and if $T_n = S_n^3$, $e_1$ will denote any non-pendant edge incident to a knot. First assume $F_1 = S_n^1$. Therefore $(F_2, F_3)$ is a subcouple either of $(S_n''\backslash e, T_n)$, with $e$ denoting an end-edge, or of $(S_n'', T_n\backslash e)$ with also $e$ an end-edge, or, if $T_n$ is $S_n'''$ or $S_n^3$, $e = e_1$. Then it is easy to see that the triple $(F_1, F_2, F_3)$ is also a subtriple of a packable triple of trees. The same property obviously holds if we assume $F_3 = T_n$ in the case where $T_n$ is a $S_n^1$.

Now assume $F_1$ (and also $F_3$) is different from $S_n^1$. Thus $(F_1, F_2, F_3)$ is a subtriple of $(S_n^1 \backslash e, S_n'', T_n)$ with $T_n$ not an $S_n^1$, or of $(S_n^1 \backslash e, S_n'', S_n^1 \backslash e')$. If $e$ is a pendant edge of $S_n^1$, then $S_n^1 \backslash e$ is a subforest of a tree with three knots (of $X_6$ if $n = 6$), and if $e$ is the edge joining the two knots, $S_n^1 \backslash e$ is a subforest of $S_n^2(p, n-p-3)$ or $S_n^3(p-1, n-p-3)$ (if $n = 6$ of $P_6$ or $S_6''$). Therefore it is always possible to consider $(F_1, F_2, F_3)$ as a subtriple of a packable triple of trees.

The proof is quite analogous for an admissible subtriple $(F_1, F_2, F_3)$ of $(S_n'', S_n^2(p, p), S_n^2(p, p))$, in which either at least one of $F_2, F_3$ must be a proper subforest of $S_n^2(p, p)$, or $F_1$ must be a subforest of $S_n'' \backslash e$, with $e$ an end-edge.

In order to complete the proof, we have to inspect the subtriples of the little special triples excluded by Theorem 13, namely $(S_6'', S_6'', S_6'')$, $(P_6, X_6, S_6^1)$, $(P_6, S_6^1, S_6^1)$, $(X_6, X_6, S_6')$, $(X_6, S_6^1, S_6^1)$ and $(Y_7, S_7^1, S_7^1)$. It is sufficient to consider the subforests obtained by deleting only one edge, and the triples with only one proper subforest.

For $n = 6$ observe that $S_4 \cup K_2$ is a subforest only of $S_6''$, $S_5' \cup K_1$ is a subforest of $S_6''$, $X_6$ and $S_6^1$, $2P_3$ is a subforest of $P_6$, $S_6''$, $S_6^1$, $P_4 \cup K_2$ of $P_6$, $X_6$ and finally $P_5 \cup K_1$ of $P_6$, $S_6''$, $X_6$. The packings at subtriples which are not also subtriples of packable triples are left to the reader.

For $n = 7$, as already noticed in general, there is no forest exclusively a subforest of $S_7^1$. Since $Y_7 \backslash e$ is also a subforest of $X_7$ or $P_7$, any proper subtriple of $(Y_7, S_7^1, S_7^1)$ is also a subtriple of a packable triple. □

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