Two new forms of half-discrete Hilbert inequality

L.E. Azar *

Department of Mathematics, Al al-Bayt University, P.O. Box 130095, Mafraq, Jordan

Received 18 February 2013; revised 28 March 2013; accepted 29 June 2013
Available online 12 August 2013

Abstract In this paper, we introduce two new forms of the half-discrete Hilbert inequality. The first form is a sharper form of the half-discrete Hilbert inequality and is related to Hardy inequality. In the second one, we give a differential form of this inequality.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 47A07; 26D10; 26D15

Keywords Hilbert inequality; Hölder inequality; Hardy inequality

1. Introduction

If \( f(x), g(y) > 0 \), \( 0 < \int_0^\infty f(x)dx < \infty \), and \( 0 < \int_0^\infty g(t)dy < \infty \), then the Hardy-Hilbert’s inequality may be written as

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin \left( \frac{\pi}{2} \right)} \left\{ \int_0^\infty f^p(x)dx \right\}^\frac{1}{p} \left\{ \int_0^\infty g^q(y)dy \right\}^\frac{1}{q},
\]

(1.1)

where the constant \( \frac{\pi}{\sin \left( \frac{\pi}{2} \right)} \) is the best possible [1].

Recently, many generalizations of (1.1) were given. Yang et al. [2] obtained the following extension of (1.1) as

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < B \left( \frac{1}{p}, \frac{1}{q} \right) \left\{ \int_0^\infty x^{\frac{1}{p} - 1}p(x)dx \right\}^\frac{1}{p} \times \left\{ \int_0^\infty y^{\frac{1}{q} - 1}g^q(y)dy \right\}^\frac{1}{q},
\]

(1.2)

where \( p > 0 \) and the constant \( B \left( \frac{1}{p}, \frac{1}{q} \right) \) (the Beta function) is the best possible. The following general inequality was given in [3]

\[
\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy < k(p,A_2) \left\{ \int_0^\infty x^{\frac{1}{p} - 1}p(x)dx \right\}^\frac{1}{p} \times \left\{ \int_0^\infty y^{\frac{1}{q} - 1}g^q(y)dy \right\}^\frac{1}{q},
\]

where \( k(p,A_2) = \int_0^\infty K(1, r)r^{-\frac{d}{p} \frac{A_2}{r}} \) is the best possible constant, \( K(x, y) \) is a homogeneous function of degree \( -\lambda(\lambda > 0) \), \( A_1 \in \left( \frac{1}{\lambda}, \frac{1}{\lambda - 1} \right) \), \( A_2 \in \left( \frac{1}{\lambda - 1}, \frac{1}{p-\lambda} \right) \) and \( pA_2 + qA_1 = 2 - \lambda. \) In [4] the following two new forms of (1.1) were proved:

For \( f, g > 0 \), \( f, g \in L(0, \infty) \), define \( F(x) = \int_0^x f(u)du \) and \( G(x) = \int_0^x g(u)du \), then for \( \lambda > 0 \)

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)}dxdy \leq \frac{\lambda^2}{p} B \left( \frac{1}{p}, \frac{1}{q} \right) \left( \int_0^\infty x^{\frac{1}{p} - 1}F^p(x)dx \right)^\frac{1}{p} \times \left( \int_0^\infty y^{\frac{1}{q} - 1}G^q(y)dy \right)^\frac{1}{q},
\]

(1.3)

For \( \lambda > n \max(p, q), n = 0, 1, \ldots \), and assuming that \( f, g \) satisfy the conditions of Lemma 2.1 (see Section 2.2), then:
where the constant factors in both (1.3) and (1.4) are the best possible.

Refinements of some Hilbert-type inequalities by virtue of various methods are obtained in [5–7]. A survey of some recent results concerning Hilbert and Hilbert-type inequalities can be found in [8].

In [9] Yang introduced the following half-discrete Hilbert’s inequality
\[
\int_0^\infty \frac{f(x)g(y)}{(x+y)^r} \, dx \leq \frac{\Gamma\left(\frac{1}{r} - 1\right)}{\Gamma\left(\frac{1}{r}\right)} \left( \int_0^\infty x^{p(\frac{1}{r} - 1)} \varphi(x) \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(\frac{1}{r} - 1)} \psi(y) \, dy \right)^{\frac{1}{q}},
\]
(1.4)

here, \( \lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = \lambda, \, 0 < \lambda_1 < 1, \) and the constant \( B(\lambda_1, \lambda_2) \) is the best possible. In particular if we set \( \lambda_1 = \frac{\lambda}{2}, \lambda_2 = \frac{\lambda}{2} \) we get from (1.5)
\[
\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^r} \, dx < B\left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left( \int_0^\infty x^{p(\frac{1}{r} - 1)} \varphi(x) \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(\frac{1}{r} - 1)} \psi(y) \, dy \right)^{\frac{1}{q}},
\]
(1.6)

For extensions and other half-discrete Hilbert’s inequalities see for example [10,11].

If \( p > 1, f(x) > 0, \) and \( F(x) = \int_0^x f(t) \, dt, \) then the famous Hardy inequality [1] is given as
\[
\int_0^\infty \left( \frac{F(x)}{x} \right) \, dx < \left( \frac{p-1}{p-1} \right) \int_0^\infty f(x) \, dx,
\]
(1.7)

the constant \( \left( \frac{p-1}{p-1} \right) \) is the best possible. A weighted form of (1.3) is given also by Hardy [1] as
\[
\int_0^\infty x^\alpha \left( \frac{F(x)}{x} \right)^\beta \, dx < \left( \frac{p}{p-1-a} \right)^\beta \int_0^\infty x^\alpha f(x) \, dx,
\]
(1.8)

where \( a < p-1 \) and the constant \( \left( \frac{p}{p-1-a} \right) \) is the best possible.

Inequality (1.7) was discovered by Hardy while he was trying to introduce a simple proof of Hilbert inequality. For more information about inequalities (1.7), (1.8) and their history and development, we refer the reader to the papers [12,13].

In this paper by estimating \( \int_0^\infty f(x) \sum_n a_n x^{(\frac{1}{r}-1)x} \, dx, \) we introduce two new inequalities with a best constant factor, similar to (1.3) and (1.4), the first one contained in Theorem 3.1 gives a relation between Hardy inequality and half-discrete Hilbert inequality, the second inequality contained in Theorem 3.2 gives a differential form of half-discrete Hilbert inequality.

2. Preliminaries and Lemmas

Recall that the Gamma function \( \Gamma(\theta) \) and the Beta function \( B(\mu, \nu) \) are defined, respectively, by
\[
\Gamma(\theta) = \int_0^\infty t^{\theta-1} e^{-t} \, dt, \quad 0 > 0,
\]
\[
B(\mu, \nu) = \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}} \, dt, \quad \mu, \nu > 0.
\]

By the definition of the Gamma function, the following equality holds
\[
\frac{1}{(x+y)^r} = \frac{1}{\Gamma(\lambda)} \int_0^\infty r^{1-r} e^{-(x+y)^r} \, dr.
\]
(2.1)

We will need the following three Lemmas (Lemmas 2.1 and 2.2 are given in [4]):

Lemma 2.1. Let \( r > 1, \frac{1}{p} + \frac{1}{q} = 1, \varphi > 0, \theta \in (0, \infty), \Phi(x) = \int_0^x \varphi(u) \, du, \) then for \( t, x > 0 \) we have
\[
\int_0^x \varphi(t) \, dt \leq \int_0^\infty \varphi(t) \, dt \left( \int_0^\infty x^{-(p+q)t} \Phi(x) \, dx \right)^{\frac{1}{p}},
\]

Lemma 2.2. Let \( r > 1, \frac{1}{p} + \frac{1}{q} = 1, \varphi > 0, \theta \in (0, \infty), \Phi(x) = \int_0^x \varphi(u) \, du, \) then for \( t, x > 0 \) we have
\[
\int_0^\infty \varphi(t) \, dt \leq \int_0^\infty x^{-(p+q)t} \Phi(x) \, dx \left( \int_0^\infty \varphi(t) \, dt \right)^{\frac{1}{q}},
\]

Lemma 2.3. Let \( r > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n > 0, \) then for \( t > 0 \)
\[
\sum_{n=1}^\infty e^{-ut} a_n < t^{1-r} \Gamma(1 - br) \left( \sum_{n=1}^\infty n^{\beta r} e^{-ut} a_n \right)^{\frac{1}{r}}.
\]

Proof. Using Hölder’s inequality, we get
\[
\sum_{n=1}^\infty e^{-ut} a_n = \sum_{n=1}^\infty \left( n^{-\beta} e^{-\frac{ut}{\beta}} \right) \left( n^\beta e^{-\frac{ut}{\beta}} a_n \right) \leq \left( \sum_{n=1}^\infty n^{-\beta} e^{-ut} a_n \right)^{\frac{1}{r}} \left( \sum_{n=1}^\infty n^\beta e^{-ut} a_n \right)^{\frac{1}{q}} \leq \left( \sum_{n=1}^\infty n^{-\beta} e^{-ut} a_n \right)^{\frac{1}{r}} \left( \sum_{n=1}^\infty n^\beta e^{-ut} a_n \right)^{\frac{1}{q}} = t^{1-r} \Gamma(1 - br) \left( \sum_{n=1}^\infty n^{\beta r} e^{-ut} a_n \right)^{\frac{1}{r}}.
\]

3. Main Results

In this section, we introduce the main two results in this paper. Theorem 3.1 gives a new form of the half-discrete Hilbert inequality (1.6) which is related to the famous Hardy inequality. In Theorem 3.2, we introduce another new form of the half-discrete Hilbert inequality, namely a differential form which is an extension of (1.6). Both of the obtained inequalities are with a best constant factor.
Theorem 3.1. Let $p > 1$, $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, $\lambda > 0$, $f, a_n > 0$, $f \in L(0, \infty)$, define $F(x) = \int_0^x f(u) du$. If $\int_0^\infty x^{\lambda - 1} F(x) dx < \infty$ and $\sum_{n=0}^\infty n^{\lambda - 1} a_n < \infty$, then

$$\int_0^\infty f(x) \sum_{n=0}^\infty \frac{a_n}{(x+n)^2} dx < C \left( \int_0^\infty x^{\lambda - 1} F(x) dx \right)^{\frac{1}{2}} \left( \sum_{n=0}^\infty n^{\lambda - 1} a_n \right)^{\frac{1}{2}},$$

(3.1)

where the constant $C = \frac{\lambda}{p} B\left(\frac{\lambda}{p}, \frac{1}{2}\right)$ is the best possible. In particular for $\lambda = 1$, $p = q = 2$

$$\int_0^\infty f(x) \sum_{n=0}^\infty \frac{a_n}{x+n} dx < \pi \left( \int_0^\infty \left( \frac{F(x)}{x} \right)^2 dx \right)^{\frac{1}{2}} \left( \sum_{n=0}^\infty a_n^2 \right)^{\frac{1}{2}}.$$  

Proof. By using (2.1) and applying Hölder inequality, we have

$$I = \int_0^\infty f(x) \sum_{n=0}^\infty \frac{a_n}{(x+n)^2} dx$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty f(x) \sum_{n=0}^\infty a_n \left( \int_0^\infty x^{\lambda - 1} e^{-x(n+1)} dx \right) dt$$

$$\leq \frac{1}{\Gamma(\lambda)} \left( \int_0^\infty \left( \int_0^\infty e^{-x} f(x) dx \right)^p dt \right)^{\frac{1}{p}} \times \left( \int_0^\infty \left( \int_0^\infty e^{-x} a_n dx \right)^q dt \right)^{\frac{1}{q}}. $$

By Lemma 2.2 for $r = p, s = q, \alpha = \frac{1}{p}$ and by Lemma 2.3 for $r = p, s = q, \beta = \frac{1}{q}$ we obtain, respectively,

$$\left( \int_0^\infty e^{-x} f(x) dx \right)^p \leq \Gamma\left(\frac{p}{\beta} + 1\right) \Gamma\left(\frac{p}{\alpha} + 1\right) \int_0^\infty x^{\lambda - 1} F(x) dx$$

$$\left( \sum_{n=0}^\infty n^{\lambda - 1} a_n \right)^q < \Gamma\left(\frac{q}{\beta} + 1\right) \sum_{n=0}^\infty n^{\lambda - 1} a_n^q.$$  

Substituting these two inequalities in (3.2) we have

$$I < \frac{\Gamma\left(\frac{\lambda}{\beta} + 1\right) \Gamma\left(\frac{\lambda}{\alpha} + 1\right)}{\Gamma(\lambda)} \left( \int_0^\infty x^{\lambda - 1} F(x) dx \right)^{\frac{1}{2}} \times \left( \sum_{n=0}^\infty n^{\lambda - 1} a_n^q \right)^{\frac{1}{2}}.$$  

Since $\int_0^\infty \hat{f}(x) dx = x^{\lambda - 1} \Gamma(\lambda + 1)$ and $\int_0^\infty \hat{e}(x) dx = n^{\lambda - 1} \Gamma\left(\frac{\lambda}{\beta} + 1\right)$, we find

$$I < C \left( \int_0^\infty x^{\lambda - 1} F(x) dx \right)^{\frac{1}{2}} \left( \sum_{n=0}^\infty n^{\lambda - 1} a_n^q \right)^{\frac{1}{2}},$$

where the constant $C = \frac{\Gamma\left(\frac{\lambda}{\beta} + 1\right) \Gamma\left(\frac{\lambda}{\alpha} + 1\right)}{\Gamma(\lambda)} \frac{1}{\Gamma\left(\frac{\lambda}{\beta} + 1\right)} B\left(\frac{\lambda}{\beta}, \frac{\lambda}{\alpha}\right)$, here, we use the following formulas for the gamma function: $\Gamma(u + 1) = u \Gamma(u)$ and $\Gamma\left(\frac{\lambda}{\beta} + 1\right) = B\left(\frac{\lambda}{\beta}, \frac{\lambda}{\alpha}\right)$. Inequality (3.1) is proved. We need to show that the constant factor $\frac{1}{\pi} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ in (3.1) is the best possible. To do that we define $\tilde{f}(x) = \frac{1}{p} x^{\frac{\lambda}{\beta} - 1}$ for $x \geq 1(0 < e < \lambda)$, $\tilde{f}(x) = 0$ for $x \in (0, 1)$ and $\tilde{a}_n = n^{\lambda - 1}(n \geq 1)$. Then, we get $F(x) = x^{\frac{\lambda}{\beta} - 1}$ for $x \geq 1$, $\tilde{F}(x) = 0$ for $x \in (0, 1)$. Suppose that $\frac{\lambda}{p} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ is not the best possible, then there exist $0 < K < \frac{1}{\pi} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ such that

$$\tilde{I} := \int_0^\infty \tilde{f}(x) \sum_{n=0}^\infty \tilde{a}_n \frac{1}{(x+n)^2} dx$$

$$< \frac{K}{\pi} \left( \int_0^\infty x^{\lambda - 1} \tilde{F}(x) dx \right)^{\frac{1}{2}} \left( \sum_{n=0}^\infty \tilde{a}_n^2 \right)^{\frac{1}{2}}$$

$$= K \left( \int_0^\infty x^{\lambda - 1} (x^{\frac{\lambda}{\beta} - 1})^p dx \right)^{\frac{1}{2}} \left( \sum_{n=0}^\infty a_n^p \right)^{\frac{1}{2}}$$

$$< K \left( \int_0^\infty x^{\lambda - 1} (x^{\frac{\lambda}{\beta} - 1})^p dx \right)^{\frac{1}{2}} \left( 1 + \int_0^\infty x^{\lambda - 1} dx \right)^{\frac{1}{2}}$$

$$= K(\lambda + 1)^{\frac{1}{2}}.$$  

(3.3)

On the other hand, we have

$$\tilde{I} = \frac{\lambda - \epsilon}{p} \int_1^\infty \left( \sum_{n=1}^\infty \tilde{a}_n^{p+1} \tilde{a}_n^{-\frac{p}{\beta}} \right) dx$$

$$= \frac{\lambda - \epsilon}{p} \int_1^\infty \left( \sum_{n=1}^\infty \tilde{a}_n^{p+1} (u+1)^{-\frac{p}{\beta}} du \right)$$

$$= \frac{\lambda - \epsilon}{p} \int_1^\infty \left( \sum_{n=1}^\infty \tilde{a}_n^{p+1} (u+1)^{-\frac{p}{\beta}} du \right) - \int_1^\infty \tilde{a}_n^{p+1} (u+1)^{-\frac{p}{\beta}} du$$

$$> \frac{\lambda - \epsilon}{p} \left( \sum_{n=1}^\infty \tilde{a}_n^{p+1} (u+1)^{-\frac{p}{\beta}} du \right) \int_1^\infty 0^{\frac{p-1}{\beta}} dt$$

$$= \frac{\lambda - \epsilon}{p} \left( \sum_{n=1}^\infty \tilde{a}_n^{p+1} (u+1)^{-\frac{p}{\beta}} du \right)$$

(3.4)

Clearly, when $\epsilon \to 0^+$ from (3.3) and (3.4) we obtain a contradiction. Thus the proof of the theorem is completed. □

Theorem 3.2. If $f, a_n > 0$, $p > 1$, $\frac{1}{\beta} + \frac{1}{\alpha} = 1$, $pk = \lambda \leq q$, $k = 0, 1, \ldots$, and $f$ satisfies the conditions of Lemma 2.2 such that $\int_0^\infty x^{\lambda(k+1) - 1} f(x)^p dx < \infty$ and $\sum_{n=0}^\infty n^{\lambda - 1} a_n < \infty$, then:

$$\int_0^\infty f(x) \sum_{n=0}^\infty \frac{a_n}{(x+n)^2} dx < C \left( \int_0^\infty x^{\lambda(k+1) - 1} f(x)^p dx \right)^{\frac{1}{2}} \times \left( \sum_{n=0}^\infty n^{\lambda - 1} a_n^q \right)^{\frac{1}{2}},$$

(3.5)
where the constant factor $C = \frac{r(\lambda - 1)}{\Gamma(\lambda)}$ is the best possible. In particular for $k = 1$, $\lambda = 2$, $p = \frac{1}{2}$, $q = 3$

\[
\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^2} dx < \frac{2\pi}{\sqrt{3}} \left( \int_0^\infty (f(x))^2 dx \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty a_n^2 \right)^{\frac{1}{2}}.
\]

**Proof.** Using (2.1) and applying Hölder inequality as in Theorem 3.1, we get

\[
I = \int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^2} dx
\leq \frac{1}{\Gamma(\lambda)} \left( \int_0^\infty t^{\lambda-1} \left( \int_0^\infty e^{-st} f(x) dx \right)^p dt \right)^{\frac{1}{p}}
\times \left( \int_0^\infty t^{\lambda-1} \left( \sum_{n=1}^\infty e^{-st} a_n \right) dt \right)^{\frac{1}{q}}.
\]

(3.6)

By Lemma 2.2 for $r = p, s = q, \alpha = \frac{r(\lambda - 1)}{\Gamma(\lambda)}$ and then by Lemma 2.3 for $r = p, s = q, \beta = \frac{r}{\Gamma(\lambda)}$ we obtain respectively,

\[
\left( \int_0^\infty e^{-st} f(x) dx \right)^p \leq \frac{1}{\Gamma(\lambda - 1)} \int_0^\infty x^{\lambda-1} e^{-st} (f(x))^p dx
\leq \frac{1}{\Gamma(\lambda - 1)} \sum_{n=1}^\infty n^{\lambda-1} a_n.
\]

Substituting these two inequalities in (3.6) we have

\[
I \leq \frac{r(\lambda - 1)}{\Gamma(\lambda)} \left( \int_0^\infty x^{\lambda-1} e^{-st} dx \right)^{\frac{1}{2}}
\times \left( \sum_{n=1}^\infty n^{\lambda-1} a_n \right)^{\frac{1}{2}}
\leq \frac{r(\lambda - 1)}{\Gamma(\lambda)} \left( \int_0^\infty x^{\lambda-1} e^{-st} dx \right)^{\frac{1}{2}}
\leq \frac{r(\lambda - 1)}{\Gamma(\lambda)} \left( \int_0^\infty x^{\lambda-1} e^{-st} dx \right)^{\frac{1}{2}}
\]

Inequality (3.5) is proved. Define $\tilde{f}(x) = e^{-st} f(x)$ for $x > 0 \leq \lambda$ and $\hat{f}(x) = 0$ for $x \in (0, 1)$, and $\hat{a}_n = n^{\lambda-1}$ where $t = (\gamma + 1) \ldots (\gamma + r - 1) = \Gamma(\lambda) / \Gamma(\lambda - 1)$ is the Pochhammer symbol. Therefore, we find $\tilde{f}(x) = x^{\lambda-1}$ for $x > 1$. Suppose that $\frac{r(\lambda - 1)}{\Gamma(\lambda)}$ is not the best possible, then there exist $0 < K < \frac{r(\lambda - 1)}{\Gamma(\lambda)}$ such that

\[
\tilde{I} = \int_0^\infty \tilde{f}(x) \sum_{n=1}^\infty \frac{\hat{a}_n}{(x+n)^2} dx
\leq K \left( \int_1^\infty x^{\lambda-1} dx \right)^{\frac{1}{2}} \left( \sum_{n=1}^\infty n^{\lambda-1} \right)^{\frac{1}{2}}
\leq K \frac{1}{\Gamma(\lambda)} \left( \int_0^\infty x^{\lambda-1} dx \right)^{\frac{1}{2}}
\]

On the other hand, we have

\[
\tilde{I} = \int_0^\infty \tilde{f}(x) \sum_{n=1}^\infty \frac{\hat{a}_n}{(x+n)^2} dx
\leq \frac{r(\lambda - 1)}{\Gamma(\lambda)} \sum_{n=1}^\infty n^{\lambda-1} \int_1^\infty \frac{\tilde{x}^{\lambda-1}}{(x+n)^2} dx
\]

Let $\varepsilon \rightarrow 0^+$, then by (3.7) and (3.8) we have

\[
K \geq \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2})} B \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \geq \frac{\Gamma(\frac{1}{2})}{\Gamma(1)} = 2.
\]

Theorem is proved. □

**Remark 3.1.** If we apply the weighted Hardy inequality (1.8) to (3.1), we get inequality (1.6). Also if we put $k = 0$ in (3.5) we obtain (1.6).

**Acknowledgement**

The author would like to thank the referees for their valuable comments which have improved the final version of the paper.

**References**


