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Two new forms of half-discrete Hilbert inequality

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Abstract In this paper, we introduce two new forms of the half-discrete Hilbert inequality. The first form is a sharper form of the half-discrete Hilbert inequality and is related to Hardy inequality. In the second one, we give a differential form of this inequality.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 47A07; 26D10; 26D15

1. Introduction

If $f(x), g(y) > 0,\ 0 < \int_0^\infty f^p(x)dx < \infty$, and $0 < \int_0^\infty g^q(y)dy < \infty$, then the Hardy-Hilbert's inequality may be written as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty f^p(x)dx\right\}^{\frac{1}{p}} \left\{\int_0^\infty g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.1}$$

where the constant $\frac{\pi}{\sin(\frac{\pi}{p})}$ is the best possible [1].

Recently, many generalizations of (1.1) were given. Yang et al. [2] obtained the following extension of (1.1) as

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy < B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\int_0^\infty x^{p-1-\lambda} f^p(x)dt\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{q-1-\lambda} g^q(y)dt\right\}^{\frac{1}{q}}, \tag{1.2}$$

where $\lambda > 0$ and the constant $B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right)$ (the Beta function) is the best possible. The following general inequality was given in [3]

$$\int_0^\infty \int_0^\infty K(x,y)f(x)g(y)dxdy < k(pA_2)\left\{\int_0^\infty x^{pqA_1-1} f^p(x)dx\right\}^{\frac{1}{p}} \times \left\{\int_0^\infty y^{pqA_2-1} g^q(x)dx\right\}^{\frac{1}{q}},$$

where $k(pA_2) = \int_0^\infty K(1,t)t^{-pA_2}$ is the best possible constant, $K(x, y) \geqslant 0$ is a homogeneous function of degree $-\lambda(\lambda > 0)$, $A_1 \in \left(\frac{1-\lambda}{q}, \frac{1}{q}\right)$, $A_2 \in \left(\frac{1-\lambda}{p}, \frac{1}{p}\right)$ and $pA_2 + qA_1 = 2 - \lambda$. In [4] the following two new forms of (1.1) were proved:

For $f, g > 0$, $f, g \in L(0, \infty)$, define $F(x) = \int_0^x f(u)du$ and $G(x) = \int_0^x g(u)du$, then for $\lambda > 0$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leqslant \frac{\lambda^2}{pq} B\left(\frac{\lambda}{q}, \frac{\lambda}{p}\right) \left(\int_0^\infty x^{-\lambda-1} F^p(x)dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{-\lambda-1} G^q(y)dy\right)^{\frac{1}{q}}. \tag{1.3}$$

For $\lambda > n \max(p, q)$, $n = 0, 1, \ldots$, and assuming that f, g satisfy the conditions of Lemma 2.1 (see Section 2.2), then:

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dxdy \leqslant \frac{\Gamma\left(\frac{\lambda}{p}-n\right)\Gamma\left(\frac{\lambda}{q}-n\right)}{\Gamma(\lambda)} \times \left(\int_0^\infty x^{p(n+1)-\lambda-1}(f^{(n)}(x))^p dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{q(n+1)-\lambda-1}(g^{(n)}(y))^q dy\right)^{\frac{1}{q}}, \quad (1.4)$$

where the constant factors in both (1.3) and (1.4) are the best possible.

Refinements of some Hilbert-type inequalities by virtue of various methods are obtained in [5–7]. A survey of some recent results concerning Hilbert and Hilbert-type inequalities can be found in [8].

In [9] Yang introduced the following half-discrete Hilbert's inequality

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B(\lambda_1,\lambda_2)\left(\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right)^{\frac{1}{p}} \times \left(\sum_{n=1}^\infty n^{q(1-\lambda_2)-1}a_n^q\right)^{\frac{1}{q}}, \quad (1.5)$$

here, λ_1, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = \lambda$, $0 < \lambda_1 < 1$, and the constant $B(\lambda_1, \lambda_2)$ is the best possible. In particular if we set $\lambda_1 = \frac{\lambda}{p}, \lambda_2 = \frac{\lambda}{q}$, we get from (1.5)

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left(\int_0^\infty x^{p-\lambda-1}f^p(x)dx\right)^{\frac{1}{p}} \times \left(\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q\right)^{\frac{1}{q}}. \quad (1.6)$$

For extensions and other half-discrete Hilbert's inequalities see for example [10,11].

If $p > 1$, $f(x) > 0$, and $F(x) = \int_0^x f(t)dt$, then the famous Hardy inequality [1] is given as

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad (1.7)$$

the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible. A weighted form of (1.3) is *given* also by Hardy [1] as

$$\int_0^\infty x^a\left(\frac{F(x)}{x}\right)^p dx < \left(\frac{p}{p-1-a}\right)^p \int_0^\infty x^a f^p(x)dx, \quad (1.8)$$

where $a < p - 1$ and the constant $\left(\frac{p}{p-1-a}\right)^p$ is the best possible. Inequality (1.7) was discovered by Hardy while he was trying to introduce a simple proof of Hilbert inequality. For more information about inequalities (1.7), (1.8) and their history and development, we refer the reader to the papers [12,13].

In this paper by estimating $\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx$, we introduce two new inequalities with a best constant factor, similar to (1.3) and (1.4), the first one contained in Theorem 3.1 gives a relation between Hardy inequality and half-discrete Hilbert inequality, the second inequality contained in Theorem 3.2 gives a differential form of half-discrete Hilbert inequality.

2. Preliminaries and Lemmas

Recall that the Gamma function $\Gamma(\theta)$ and the Beta function $B(\mu, \nu)$ are defined, respectively, by

$$\Gamma(\theta) = \int_0^\infty t^{\theta-1}e^{-t}dt, \quad \theta > 0,$$

$$B(\mu,\nu) = \int_0^\infty \frac{t^{\mu-1}}{(t+1)^{\mu+\nu}}dt, \quad \mu,\nu > 0.$$

By the definition of the Gamma function, the following equality holds

$$\frac{1}{(x+y)^\lambda} = \frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}e^{-(x+y)t}dt. \quad (2.1)$$

We will need the following three Lemmas (Lemmas 2.1 and 2.2 are given in [4]):

Lemma 2.1. *Let* $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, $\varphi > 0$, $\varphi \in L(0,\infty)$, $\Phi(x) = \int_0^x \varphi(u)du$, *then for* t, $\alpha > 0$ *we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx \leqslant t^{\frac{1}{r}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\Phi^r(x)dx\right\}^{\frac{1}{r}}.$$

Lemma 2.2. *Let* $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, $\varphi > 0$, *the derivatives* φ', φ'', $\ldots$, $\varphi^{(k)}$ *exists and positive and* $\varphi^{(k)} \in L(0,\infty)$ *(*$k = 0, 1, \ldots$*)* *(*$\varphi^{(0)}:=\varphi$*), moreover, suppose that* $\varphi(0) = \varphi'(0) = \cdots = \varphi^{(k-1)}(0) = 0$, *then for* t, $\alpha > 0$ *we have*

$$\int_0^\infty e^{-tx}\varphi(x)dx \leqslant t^{-k-\frac{1}{s}-\alpha}\Gamma(\alpha s+1)^{\frac{1}{s}}\left\{\int_0^\infty x^{-\alpha r}e^{-tx}\left(\varphi^{(k)}(x)\right)^r dx\right\}^{\frac{1}{r}}.$$

Lemma 2.3. *Let* $r > 1$, $\frac{1}{r}+\frac{1}{s}=1$, $a_n > 0$, *then for* $t > 0$ *and* $0 \leqslant \beta < \frac{1}{r}$ *we have*

$$\sum_{n=1}^\infty e^{-nt}a_n < t^{\beta-\frac{1}{r}}\Gamma(1-\beta r)^{\frac{1}{r}}\left\{\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right\}^{\frac{1}{s}}.$$

Proof. Using Hölder's inequality, we get

$$\sum_{n=1}^\infty e^{-nt}a_n = \sum_{n=1}^\infty \left\{n^{-\beta}e^{-\frac{nt}{r}}\right\}\left\{n^\beta e^{-\frac{nt}{s}}a_n\right\} < \left(\sum_{n=1}^\infty n^{-\beta r}e^{-nt}\right)^{\frac{1}{r}}\left(\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right)^{\frac{1}{s}} < \left(\int_0^\infty x^{-\beta r}e^{-tx}dx\right)^{\frac{1}{r}}\left(\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right)^{\frac{1}{s}} = t^{\beta-\frac{1}{r}}\Gamma(1-\beta r)^{\frac{1}{r}}\left\{\sum_{n=1}^\infty n^{\beta s}e^{-nt}a_n^s\right\}^{\frac{1}{s}}. \quad \square$$

3. Main results

In this section, we introduce the main two results in this paper. Theorem 3.1 gives a new form of the half-discrete Hilbert inequality (1.6) which is related to the famous Hardy inequality. In Theorem 3.2, we introduce another new form of the half-discrete Hilbert inequality, namely a differential form which is an extension of (1.6). Both of the obtained inequalities are with a best constant factor.

Theorem 3.1. *Let* $p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ \lambda > 0,\ f, a_n > 0,\ f \in L(0,\infty)$, *define* $F(x)=\int_0^x f(u)du$. *If* $\int_0^\infty x^{-\lambda-1}F^p(x)dx < \infty$ *and* $\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q < \infty$, *then*

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < C\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q\right)^{\frac{1}{q}}, \tag{3.1}$$

where the constant $C=\frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ *is the best possible. In particular for* $\lambda = 1,\ p = q = 2$

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{x+n}dx < \frac{\pi}{2}\left(\int_0^\infty \left(\frac{F(x)}{x}\right)^2 dx\right)^{\frac{1}{2}}\left(\sum_{n=1}^\infty a_n^2\right)^{\frac{1}{2}}.$$

Proof. By using (2.1) and applying Hölder inequality, we have

$$\begin{aligned}
I &= \int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx \\
&= \frac{1}{\Gamma(\lambda)}\int_0^\infty f(x)\sum_{n=1}^\infty a_n\left(\int_0^\infty t^{\lambda-1}e^{-(x+n)t}dt\right)dx \\
&= \frac{1}{\Gamma(\lambda)}\int_0^\infty \left(t^{\frac{\lambda-1}{p}}\int_0^\infty e^{-xt}f(x)dx\right)\left(t^{\frac{\lambda-1}{q}}\sum_{n=1}^\infty e^{-nt}a_n\right)dt \\
&\leqslant \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-xt}f(x)dx\right)^p dt\right)^{\frac{1}{p}} \\
&\quad \times \left(\int_0^\infty t^{\lambda-1}\left(\sum_{n=1}^\infty e^{-nt}a_n\right)^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.2}$$

By Lemma 2.2 for $r = p, s = q, \alpha = \frac{\lambda}{pq}$ and by Lemma 2.3 for $r = p, s = q, \beta = \frac{q-\lambda}{pq}$, we obtain, respectively,

$$\left(\int_0^\infty e^{-xt}f(x)dx\right)^p \leqslant t^{1-\frac{\lambda}{q}}\Gamma\left(\frac{\lambda}{p}+1\right)^{\frac{p}{q}}\int_0^\infty x^{-\frac{\lambda}{q}}e^{-tx}F^p(x)dx$$

$$\left(\sum_{n=1}^\infty e^{-nt}a_n\right)^q < t^{-\frac{\lambda}{p}}\Gamma\left(\frac{\lambda}{q}\right)^{\frac{q}{p}}\sum_{n=1}^\infty n^{\frac{q-\lambda}{p}}e^{-nt}a_n^q.$$

Substituting these two inequalities in (3.2) we have

$$\begin{aligned}
I &< \frac{\Gamma\left(\frac{\lambda}{p}+1\right)^{\frac{1}{q}}\Gamma\left(\frac{\lambda}{q}\right)^{\frac{1}{p}}}{\Gamma(\lambda)}\left(\int_0^\infty x^{-\frac{\lambda}{q}}F^p(x)\left(\int_0^\infty t^{\frac{\lambda}{p}}e^{-xt}dt\right)dx\right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{n=1}^\infty n^{\frac{q-\lambda}{p}}a_n^q\left(\int_0^\infty t^{\frac{\lambda}{q}-1}e^{-nt}dt\right)dy\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $\int_0^\infty t^{\frac{\lambda}{p}}e^{-xt}dt = x^{\frac{-\lambda}{p}-1}\Gamma\left(\frac{\lambda}{p}+1\right)$ and $\int_0^\infty t^{\frac{\lambda}{q}-1}e^{-nt}dt = n^{-\frac{\lambda}{q}}\Gamma\left(\frac{\lambda}{q}\right)$, we find

$$I < C\left(\int_0^\infty x^{-\lambda-1}F^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q\right)^{\frac{1}{q}},$$

where the constant $C=\frac{\Gamma\left(\frac{\lambda}{p}+1\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}=\frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$, here, we use the following formulas for the gamma function: $\Gamma(u+1) = u\Gamma(u)$ and $\frac{\Gamma\left(\frac{\lambda}{p}\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}=B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$. Inequality (3.1) is proved. We need to show that the constant factor $\frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ in (3.1) is the best possible. To do that we define $\widetilde{f}(x)=\frac{\lambda-\varepsilon}{p}x^{\frac{\lambda-\varepsilon}{p}-1}$ for $x \geqslant 1 (0<\varepsilon<\lambda)$, $\widetilde{f}(x)=0$ for $x\in(0,\ 1)$ and $\widetilde{a}_n = n^{\frac{\lambda-\varepsilon}{q}-1} (n \geqslant 1)$. Then, we get $\widetilde{F}(x)=x^{\frac{\lambda-\varepsilon}{p}}-1$ for $x \geqslant 1$, $\widetilde{F}\ (x) = 0$ for $x \in (0,\ 1)$. Suppose that $\frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$is not the best possible, then there exist $0 < K < \frac{\lambda}{p}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right)$ such that

$$\begin{aligned}
\widetilde{I} &:= \int_0^\infty \widetilde{f}(x)\sum_{n=1}^\infty \frac{\widetilde{a}_n}{(x+n)^\lambda}dx \\
&< K\left(\int_1^\infty x^{-\lambda-1}\widetilde{F}^p(x)dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q-\lambda-1}\widetilde{a}_n^q\right)^{\frac{1}{q}} \\
&= K\left(\int_1^\infty x^{-\lambda-1}\left(x^{\frac{\lambda-\varepsilon}{p}}-1\right)^p dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-\varepsilon-1}\right)^{\frac{1}{q}} \\
&< K\left(\int_1^\infty x^{-\lambda-1}\left(x^{\frac{\lambda-\varepsilon}{p}}\right)^p dx\right)^{\frac{1}{p}}\left(1+\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{q}} \\
&= \frac{K(\varepsilon+1)^{\frac{1}{q}}}{\varepsilon}.
\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}
\widetilde{I} &= \frac{\lambda-\varepsilon}{p}\int_1^\infty \sum_{n=1}^\infty \frac{x^{\frac{\lambda-\varepsilon}{p}-1}n^{\frac{\lambda-\varepsilon}{q}-1}}{(x+n)^\lambda}dx \\
&= \frac{\lambda-\varepsilon}{p}\sum_{n=1}^\infty n^{-\varepsilon-1}\int_{\frac{1}{n}}^\infty \frac{u^{\frac{\lambda-\varepsilon}{p}-1}}{(u+1)^\lambda}du \\
&= \frac{\lambda-\varepsilon}{p}\sum_{n=1}^\infty n^{-\varepsilon-1}\left\{\int_0^\infty \frac{u^{\frac{\lambda-\varepsilon}{p}-1}}{(u+1)^\lambda}du - \int_0^{\frac{1}{n}} \frac{u^{\frac{\lambda-\varepsilon}{p}-1}}{(u+1)^\lambda}du\right\} \\
&= \frac{\lambda-\varepsilon}{p}\left\{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)\sum_{n=1}^\infty n^{-\varepsilon-1} - \sum_{n=1}^\infty n^{-\varepsilon-1}\int_0^{\frac{1}{n}} \frac{u^{\frac{\lambda-\varepsilon}{p}-1}}{(u+1)^\lambda}du\right\} \\
&> \frac{\lambda-\varepsilon}{p}\left\{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)\int_1^\infty \theta^{-\varepsilon-1}d\theta - \sum_{n=1}^\infty n^{-\varepsilon-1}\int_0^{\frac{1}{n}} u^{\frac{\lambda-\varepsilon}{p}-1}du\right\} \\
&> \frac{\lambda-\varepsilon}{p}\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon} - \sum_{n=1}^\infty n^{-\frac{\varepsilon}{q}-\frac{\lambda}{p}-1} \\
&> \frac{\lambda-\varepsilon}{p}\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon} - \left(1+\int_1^\infty \theta^{-\frac{\varepsilon}{q}-\frac{\lambda}{p}-1}d\theta\right) \\
&= \frac{\lambda-\varepsilon}{p}\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon} - O(1).
\end{aligned} \tag{3.4}$$

Clearly, when $\varepsilon \to 0^+$ from (3.3) and (3.4) we obtain a contradiction. Thus the proof of the theorem is completed. □

Theorem 3.2. *If* $f, a_n > 0,\ p > 1,\ \frac{1}{p}+\frac{1}{q}=1,\ pk < \lambda \leqslant q,\ k = 0, 1, \ldots,$ *and* f *satisfies the conditions of Lemma 2.2 such that* $\int_0^\infty x^{p(k+1)-\lambda-1}f^{(k)}(x)^p dx < \infty$ *and* $\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q < \infty$, *then:*

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx < C\left(\int_0^\infty x^{p(k+1)-\lambda-1}\left(f^{(k)}(x)\right)^p dx\right)^{\frac{1}{p}} \times \left(\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q\right)^{\frac{1}{q}}, \tag{3.5}$$

where the constant factor $C = \frac{\Gamma\left(\frac{\lambda}{p}-k\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}$ is the best possible. In particular for $k = 1,\ \lambda = 2,\ p = \frac{3}{2},\ q = 3$

$$\int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^2}dx < \frac{2\pi}{\sqrt{3}}\left(\int_0^\infty (f'(x))^{\frac{3}{2}}dx\right)^{\frac{2}{3}}\left(\sum_{n=1}^\infty a_n^3\right)^{\frac{1}{3}}.$$

Proof. Using (2.1) and applying Hölder inequality as in Theorem 3.1, we get

$$\begin{aligned} I &= \int_0^\infty f(x)\sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}dx \\ &\leqslant \frac{1}{\Gamma(\lambda)}\left(\int_0^\infty t^{\lambda-1}\left(\int_0^\infty e^{-xt}f(x)dx\right)^p dt\right)^{\frac{1}{p}} \\ &\quad \times \left(\int_0^\infty t^{\lambda-1}\left(\sum_{n=1}^\infty e^{-nt}a_n\right)^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{3.6}$$

By Lemma 2.2 for $r = p, s = q, \alpha = \frac{\lambda - p(k+1)}{pq}$ and then by Lemma 2.3 for $r = p, s = q, \beta = \frac{q-\lambda}{pq}$ we obtain respectively,

$$\left(\int_0^\infty e^{-xt}f(x)dx\right)^p \leqslant t^{-k-\frac{\lambda}{q}}\Gamma\left(\frac{\lambda}{p}-k\right)^{\frac{p}{q}}\int_0^\infty x^{-\frac{\lambda-p(k+1)}{q}}e^{-tx}\left(f^{(k)}(x)\right)^p dx$$

$$\left(\sum_{n=1}^\infty e^{-nt}a_n\right)^q < t^{-\frac{\lambda}{p}}\Gamma\left(\frac{\lambda}{q}\right)^{\frac{q}{p}}\sum_{n=1}^\infty n^{\frac{q-\lambda}{p}}e^{-nt}a_n^q.$$

Substituting these two inequalities in (3.6) we have

$$\begin{aligned} I &\leqslant \frac{\Gamma\left(\frac{\lambda}{p}-k\right)^{\frac{1}{q}}\Gamma\left(\frac{\lambda}{q}\right)^{\frac{1}{p}}}{\Gamma(\lambda)}\left(\int_0^\infty x^{-\frac{\lambda-p(k+1)}{q}}\left(f^{(k)}(x)\right)^p\left(\int_0^\infty t^{\frac{\lambda}{p}-k-1}e^{-xt}dt\right)dx\right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{n=1}^\infty n^{\frac{q-\lambda}{p}}a_n^q\int_0^\infty t^{\frac{\lambda}{q}-1}e^{-nt}dt\right)^{\frac{1}{q}}, \\ &= \frac{\Gamma\left(\frac{\lambda}{p}-k\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}\left(\int_0^\infty x^{p(k+1)-\lambda-1}\left(f^{(k)}(x)\right)^p dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{q-\lambda-1}a_n^q\right)^{\frac{1}{q}}. \end{aligned}$$

Inequality (3.5) is proved. Define $\tilde{f}(x) = \frac{x^{\frac{\lambda-\varepsilon}{p}-1}}{\left(\frac{\lambda-\varepsilon}{p}-n\right)_n} = \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-n\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)}x^{\frac{\lambda-\varepsilon}{p}-1}$ for $x \geqslant 1 (0 < \varepsilon < \lambda)$ and $\tilde{f}(x) = 0$ for $x \in (0,\ 1)$, and $\tilde{a}_n = n^{\frac{\lambda-\varepsilon}{q}-1} (n \geqslant 1)$ where $(\gamma)_r = \gamma(\gamma+1)\ldots(\gamma+r-1) = \frac{\Gamma(\gamma+r)}{\Gamma(\gamma)}$ is the Pochhammer symbol. Therefore, we find $\tilde{f}^{(n)}(x) = x^{\frac{\lambda-\varepsilon}{p}-n-1}$ for $x > 1$. Suppose that $\frac{\Gamma\left(\frac{\lambda}{p}-k\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}$ is not the best possible, then there exist $0 < K < \frac{\Gamma\left(\frac{\lambda}{p}-k\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}$ such that

$$\begin{aligned} \tilde{I} &:= \int_0^\infty \tilde{f}(x)\sum_{n=1}^\infty \frac{\tilde{a}_n}{(x+n)^\lambda}dx \\ &< K\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty n^{-\varepsilon-1}\right)^{\frac{1}{q}} < K\frac{(\varepsilon+1)^{\frac{1}{q}}}{\varepsilon} \end{aligned} \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} \tilde{I} &= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-k\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)}\sum_{n=1}^\infty n^{\frac{\lambda-\varepsilon}{q}-1}\int_1^\infty \frac{x^{\frac{\lambda-\varepsilon}{p}-1}}{(x+n)^\lambda}dx \\ &= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-k\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)}\sum_{n=1}^\infty n^{-\varepsilon-1}\int_{\frac{1}{n}}^\infty \frac{u^{\frac{\lambda-\varepsilon}{p}-1}}{(u+1)^\lambda}du \\ &= \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-k\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)}\left\{\sum_{n=1}^\infty n^{-\varepsilon-1}\int_0^\infty \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du - \sum_{n=1}^\infty n^{-\varepsilon-1}\int_0^{\frac{1}{x}} \frac{u^{\frac{\lambda-\varepsilon}{q}-1}}{(u+1)^\lambda}du\right\} \\ &> \frac{\Gamma\left(\frac{\lambda-\varepsilon}{p}-k\right)}{\Gamma\left(\frac{\lambda-\varepsilon}{p}\right)}\left[\frac{B\left(\frac{\lambda-\varepsilon}{q},\frac{\lambda-\varepsilon}{p}\right)}{\varepsilon} - O(1)\right]. \end{aligned} \tag{3.8}$$

Let $\varepsilon \to 0^+$, then by (3.7) and (3.8) we have

$$K \geqslant \frac{\Gamma\left(\frac{\lambda}{p}-k\right)}{\Gamma\left(\frac{\lambda}{p}\right)}B\left(\frac{\lambda}{q},\frac{\lambda}{p}\right) = \frac{\Gamma\left(\frac{\lambda}{p}-k\right)\Gamma\left(\frac{\lambda}{q}\right)}{\Gamma(\lambda)}.$$

The Theorem is proved. □

Remark 3.1. If we apply the weighted Hardy inequality (1.8) to (3.1), we get inequality (1.6). Also if we put $k = 0$ in (3.5) we obtain (1.6).

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