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The evolution of invariant manifolds in Hamiltonian–Hopf bifurcations☆

Patrick D. McSwiggen and Kenneth R. Meyer*

Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221-0025, USA

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Abstract

We consider the evolution of the stable and unstable manifolds of an equilibrium point of a Hamiltonian system of two degrees of freedom which depends on a parameter, ν. The eigenvalues of the linearized system are complex for $\nu<0$ and pure imaginary for $\nu>0$. Thus, for $\nu<0$ the equilibrium has a two-dimensional stable manifold and a two-dimensional unstable manifold, but for $\nu>0$ these stable and unstable manifolds are gone. If the sign of a certain term in the normal form is positive then for small negative ν the stable and unstable manifolds of the system are either identical or must have transverse intersection. Thus, either the system is totally degenerate or the system admits a suspended Smale horseshoe as an invariant set.

Keywords: Stable manifold; Bifurcation; Strömgren's conjecture; Restricted three-body problem

MSC: 34C30; 34C37; 34C23; 70F07

1. Introduction

We consider a 1-parameter family of Hamiltonian systems of two degrees of freedom for which there is an equilibrium point that changes type from hyperbolic to elliptic. We can assume this equilibrium point is at the origin for all values of the parameter, ν, and the change in type occurs at $\nu = 0$. The linearization of such a system at the origin has a coefficient matrix $A(\nu)$ which is a 4×4 Hamiltonian

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*Corresponding author. Fax: +513-556-3417.

E-mail addresses: pat.mcswiggen@uc.edu (P.D. McSwiggen), ken.meyer@uc.edu (K.R. Meyer).

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matrix, so its eigenvalues are symmetric with respect to both the real and imaginary axis [24]. Therefore, the eigenvalues change from complex numbers of the form $\pm\alpha\pm\beta i$, $\alpha, \beta \neq 0$ when $v<0$ to two pairs of pure imaginary eigenvalues of the form $\pm\omega_1 i$, $\pm\omega_2 i$, ω_1, $\omega_2 \neq 0$ when $v>0$. Clearly $A(0)$ must have a single pair of pure imaginary eigenvalues of multiplicity two, i.e. its eigenvalues are of the form $\pm\omega i$, $\pm\omega i$. We assume that $A(0)$ is nondegenerate in the sense that its Jordan canonical form has nonzero off-diagonal entries.

Much is known about the local geometry of the flow in the two cases when $v<0$ and $v>0$. When $v<0$ the equilibrium point is a saddle point with two-dimensional stable and unstable manifolds [12], and when $v>0$ the Liapunov Center Theorem [21] assures that two families of periodic solutions emanate from the equilibrium point. How do these structures change as the parameter passes through zero?

In 1971, Meyer and Schmidt [25] stated and proved the theorem that has become known as the Hamiltonian–Hopf Theorem which tells what happens to the Liapunov families of periodic solutions provided a certain quantity η is nonzero. The quantity η, defined below, is the coefficient of a particular term in the normal form expansion of H. When $\eta<0$ the two Liapunov families are globally connected for $v>0$ and shrink to the equilibrium as $v\to 0^+$. When $\eta>0$ the two Liapunov families detach from the equilibrium as a single family as v decrease from zero. Meyer and Schmidt [25] show the latter case occurs in the restricted three-body problem.

In [23] a similar study was carried out on the evolution of the stable and unstable manifolds for the truncated normal form. Superficially, the formal story sounds the same with the sign of η reversed. When $\eta<0$ the stable and unstable manifolds detach from the equilibrium as a single invariant manifold as v increases from zero. When $\eta>0$ the stable and unstable manifolds are globally connected for $v<0$ and shrink to the equilibrium as $v\to 0^-$.

In this paper, we treat the evolution of the stable and unstable manifolds for the full system when $\eta>0$. We show that in this case, when v is small and negative, the stable and unstable manifolds must intersect. Moreover, if the invariant manifolds are not globally identical, then there exists a transverse intersection. Thus, either the system is totally degenerate, or near the equilibrium there must be the suspension of a Smale horse shoe with all its dynamic ramifications [27].

Section 6 considers two well known examples to which our theorem applies. The first is the restricted three-body problem. This was shown in 1968 by Deprit and Henrard [8] to have $\eta>0$ for the Lagrange triangular libration point at the Routh critical value of the mass ratio parameter. This is related to the Strömgren conjecture [31]. The second example is the fourth-order equation $u^{iv} + P\ddot{u} + u - u^2 = 0$, which has been studied extensively in the literature [1,4,17,18]. This equation can be written as a reversible Hamiltonian system with an equilibrium at the origin. As the parameter P passes through 2 the exponents change from complex to pure imaginary and a computation of the normal form shows that the conditions of our theorem are satisfied.

Since 1971 there have been many papers on the 1:1 resonance case, see van Gils et al. [11] for a discussion of the unfolding of the general nonconservative, non-symmetric case, Iooss and Peroueme [19] in the time reversing case and the references therein.

2. The system of equations

Consider a Hamiltonian system of two degrees of freedom which depends on a parameter v which has an equilibrium point at the origin for all v. That is, a system of the form

$$\dot{z} = J\nabla_z H(z, v) = A(v)z + F(z, v), \tag{1}$$

where $z \in \mathbb{R}^4$, $t, v \in \mathbb{R}$, $H : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ is smooth, J is the 4×4 skew symmetric matrix

$$J = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$A(v) = J\partial^2 H/\partial z^2(0, v)$, $F(z, v) = J\nabla_z H(z, v) - A(v)z$, and $\cdot = d/dt$. Since the equilibrium point is at the origin, $\nabla_z H(0, v) = F(0, v) = 0$ and since $A(v)$ is the linear part of the equation, $\partial F(0, v)/\partial z\ (0, v) = 0$.

The basic assumption is that when $v = 0$ the matrix A has eigenvalues $\pm\omega i$ of multiplicity two with nonelementary divisors and as v decreases from zero these eigenvalues move off the imaginary axis. By scaling time we may assume that $\omega = 1$.

For this problem Sokol'skii's normal form is appropriate. It depends on the quantities

$$\Gamma_1 = x_2 y_1 - x_1 y_2, \quad \Gamma_2 = \tfrac{1}{2}(x_1^2 + x_2^2),$$

$$\Gamma_3 = \tfrac{1}{2}(y_1^2 + y_2^2), \quad \Gamma_4 = x_1 y_1 + x_2 y_2, \tag{2}$$

where $z = (x_1, x_2, y_1, y_2)$. Hamiltonian (1) is in Sokol'skii's normal form if

$$H = \Gamma_1 + \delta\Gamma_2 + H^\dagger(\Gamma_1, \Gamma_3, v), \tag{3}$$

where $H^\dagger$ is at least quadratic in Γ_1, Γ_3 and v and where $\delta = \pm 1$.

To see which terms are the most important near the origin and when v is small we will use the scaling in [25] which was used to identify the important terms for the Hamiltonian–Hopf bifurcation. Scale the variables by

$$x_1 \to \varepsilon^2 x_1, \quad x_2 \to \varepsilon^2 x_2,$$

$$y_1 \to \varepsilon y_1, \quad y_2 \to \varepsilon y_2,$$

$$v \to \varepsilon^2 v, \tag{4}$$

which is symplectic with multiplier ε^{-3}. The Hamiltonian becomes

$$H = \Gamma_1 + \varepsilon\{\delta\Gamma_2 + vd\Gamma_3 + \eta\delta\Gamma_3^2\} + O(\varepsilon^2). \tag{5}$$

If $d \neq 0$ then by changing the definition of v as necessary we may assume that $d = \delta = \pm 1$.

3. Formal analysis

The scaling above indicates that the most important terms are those explicitly displayed in (5). A complete analysis of the truncated system

$$\hat{H} = \Gamma_1 + \delta\Gamma_2 + v\delta\Gamma_3 + \eta\delta\Gamma_3^2 \tag{6}$$

is given in [23,28], so we will summarize the salient points in the case when $\eta > 0$. The unfolding parameter is v, the coefficient of the nonlinearity is η, and $\delta = \pm 1$.

The linearized equations have a coefficient matrix

$$A(v) = \begin{bmatrix} 0 & 1 & v\delta & 0 \\ -1 & 0 & 0 & v\delta \\ -\delta & 0 & 0 & 1 \\ 0 & -\delta & -1 & 0 \end{bmatrix} \tag{7}$$

with eigenvalues $\lambda = \pm i(1 \pm \sqrt{v})$. Thus for small v, the eigenvalues are complex when $v < 0$ and pure imaginary when $v > 0$.

We shall follow Sokol'skii by using polar coordinates to study this system. Specifically, make the symplectic change of coordinates

$$x_1 = R\cos\theta - \frac{\Theta}{r}\sin\theta, \quad y_1 = r\cos\theta,$$

$$x_2 = R\sin\theta + \frac{\Theta}{r}\cos\theta, \quad y_2 = r\sin\theta. \tag{8}$$

Hamiltonian (3) becomes

$$\hat{H} = \Theta + \frac{\delta}{2}\left\{R^2 + \frac{\Theta^2}{r^2}\right\} + \frac{v\delta}{2}r^2 + \frac{\eta\delta}{4}r^4. \tag{9}$$

Thus, θ is an ignorable coordinate and its conjugate momentum Θ is an integral. Set $\Theta = c$ where c is an arbitrary constant and ignore θ at least temporarily. The usual conventions of polar coordinates hold; in particular, θ is arbitrary, so for fixed r, R, you have a circle if $r \neq 0$ or a point if $r = 0$. The problem is reduced to studying the one-degree-of-freedom problem defined by

$$\hat{H} = c + \frac{\delta}{2}\left\{R^2 + \frac{c^2}{r^2}\right\} + \frac{v\delta}{2}r^2 + \frac{\eta\delta}{4}r^4. \tag{10}$$

Thus, the analysis of the truncated system is reduced to the elementary plotting of the level curves of (10).

Since the stable and unstable manifolds lie in the $H = 0$ level set we shall only consider the flow on this level set. In (10) set $\hat{H} = 0$ and solve for R^2 to obtain

$$R^2 = -2c\delta - \frac{c^2}{r^2} - vr^2 - \frac{1}{2}\eta r^4. \tag{11}$$

Fixing v, δ and η fixes the parameters in the equation. r, R, θ, and c then sweep out the level set where $H = 0$. Since we are only interested in the stable and unstable manifolds, we need only consider the level set where $c = 0$:

$$R^2 = -vr^2 - \frac{1}{2}\eta r^4. \tag{12}$$

There are two cases depending on the sign of η and we are interested in the case when η is positive, which is illustrated in Fig. 1 for $v \to 0^-$.

Recall that these are illustrations of projections of the $H = 0$ level set onto the r, R-plane, and that θ is arbitrary. Over each point (r, R) with $r \neq 0$ there is a circle in $H = 0$, but these circles tend to zero as $r \to 0^+$, and above each point where $r = 0$ there is just a single point.

In summary: *In the case when* $\eta > 0$ *the stable and unstable manifolds of the formal system* (6) *are globally connected for* $v < 0$ *and shrink to the equilibrium as* $v \to 0^-$.

These statements hold only for the truncated system with Hamiltonian (6), but are a good first approximation of the local evolution of the stable and unstable manifolds for the full Hamiltonian.

We want to look at the invariant manifolds of the full system as perturbations of those of the truncated system. However, this is made difficult by the fact that for the truncated system they shrink to the equilibrium as $v \to 0^-$. To keep the invariant manifolds a fixed size, we renormalize a neighborhood of the equilibrium as $v \to 0^-$, using (4), with scaling factor ε tied to v. For $v < 0$, define ε by $v = -\frac{1}{2}\eta\varepsilon^2$. Using this scaling factor we obtain the new Hamiltonians

$$\hat{H} = \Gamma_1 + \varepsilon\delta(\Gamma_2 - \tfrac{1}{2}\eta\Gamma_3 + \eta\Gamma_3^2),$$

$$H = \hat{H} + O(\varepsilon^2). \tag{13}$$

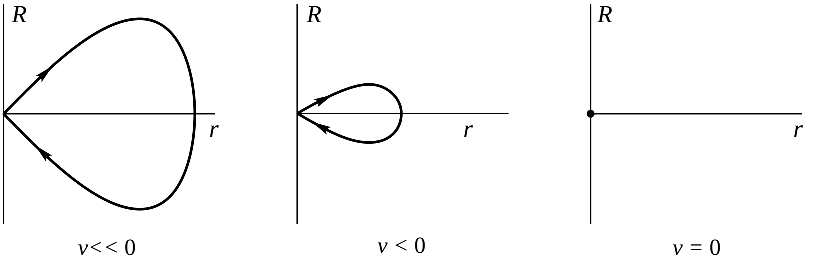

Fig. 1. Manifolds when $\eta > 0$ (orientation for $\delta = +1$).

We now use ε as the parameter, where $v \to 0^-$ as $\varepsilon \to 0$, and will write these as $\hat{H}(\varepsilon)$ and $H(\varepsilon)$ when we wish to explicitly emphasize the dependence on ε.

Scaling (4) in the context of the change to polar coordinates (8) translates to

$$r \to \varepsilon r, \quad \theta \to \theta,$$

$$R \to \varepsilon^2 R, \quad \Theta \to \varepsilon^3 \Theta.$$

Therefore, in these scaled coordinates, Eq. (12) becomes

$$R^2 = \tfrac{1}{2}\eta(r^2 - r^4). \tag{14}$$

This is the equation that dictates the shape of the stable and unstable manifolds for the truncated system in the scaled coordinates—and does not depend upon ε. Therefore, while the flow on the invariant manifolds changes with ε, the stable and unstable manifolds themselves are now fixed.

4. Convergence

We have shown that for the truncated system, $\hat{H}$, the stable and unstable manifolds of the origin always agree. We want to show that for the full system, H, while they may not agree, for ε sufficiently small they will still intersect. The first step will be to show that as $\varepsilon \to 0$ the stable and unstable manifolds of the origin for H converge to those for $\hat{H}$.

Let $\hat{W}^s = \hat{W}^u$ denote the (ε independent) stable and unstable manifolds of the origin for $\hat{H}$ ($\varepsilon \neq 0$) and let $\hat{\Delta}_s$ (respectively, $\hat{\Delta}_u$) denote a compact stable (respectively, unstable) disk.

Theorem 4.1. *Let $\hat{\Delta}_s$ be any local stable manifold of the origin for the Hamiltonian systems defined by the $\hat{H}(\varepsilon)$. For each $\varepsilon \neq 0$ one can choose a local stable manifold for $H(\varepsilon)$, $\Delta_s(\varepsilon)$, such that $\Delta_s(\varepsilon) \to \hat{\Delta}_s$ as $\varepsilon \to 0$ in the following sense. Let D be the unit disk in $\mathbb{R}^2$ and let $i : D \to \mathbb{R}^4$ be any embedding with $i(D) = \hat{\Delta}_s$. We can choose a one parameter family of embeddings $i_\varepsilon : D \to \mathbb{R}^4$ such that:*

1. *for each $\varepsilon \neq 0$, $i_\varepsilon(D)$ is a local stable manifold for $H(\varepsilon)$,*
2. *$i_0 = i$, and*
3. *for each k, $1 \leqslant k < \infty$, i_ε is a continuous function of ε in the C^k topology.*

The same holds for the local unstable manifolds.

The proof of this theorem consists largely of the application of standard tools in a slightly nonstandard context—nonstandard because we would like to use the continuity of the stable and unstable manifolds, but the origin is not hyperbolic when $\varepsilon = 0$. In particular, we will use the following version of the Stable Manifold Theorem excerpted from one in [15].

Let $E_1, E_2 \subset \mathbb{R}^n$ be complimentary subspaces and $E_i(r)$ denote the closed ball of radius r about $0 \in E_i$. For $0 < \lambda < 1$, define $\mathscr{H}_\lambda$ to be the set of all (hyperbolic) automorphisms T of $\mathbb{R}^n$ preserving this splitting for which $||T_1|| < \lambda$ and $||T_2^{-1}|| < \lambda$, where $T_i = T|_{E_i}$.

Theorem 4.2 (Local Stable Manifold Theorem [15, Theorem 2.3]). *Let $\mathscr{H}_\lambda$ be as above. There exists an $\epsilon > 0$, depending only on λ, and, for each $r > 0$, a $\delta > 0$ such that for any $T \in \mathscr{H}_\lambda$, if $\psi : E_1(r) \times E_2(r) \to \mathbb{R}^n$ satisfies $\mathrm{Lip}(\psi) < \epsilon$ and $||\psi(0)|| < \delta$, then the local stable manifold for $f \coloneqq T + \psi$, $\Delta_s(f)$, is the graph of a unique Lipschitz function $g_\psi : E_1(r) \to E_2(r)$ with $\mathrm{Lip}(g_\psi) \leqslant 1$. If ψ is C^k, then so is g_ψ and the assignment $\psi \to g_\psi$ is continuous in the C^k-topology. An analogous statement holds for the unstable manifold.*

Let $\mathscr{U} = \{\psi \in C^k(E_1(r) \times E_2(r), \mathbb{R}^n)\colon\ \mathrm{Lip}(\psi) < \epsilon(\lambda), ||\psi(0)|| < \delta(\lambda, r, \varepsilon)\}$. Note that $\mathscr{U}$ depends only on λ and r, not on a specific $T \in \mathscr{H}_\lambda$. The above theorem states that for any $T \in \mathscr{H}_\lambda$ and $\psi \in \mathscr{U}$, $\Delta_s(T + \psi)$ is the graph of some function $g \in C^k(E_1(r), E_2(r))$. Thus, for each T we have a function $G_T : \mathscr{U} \to C^k(E_1(r), E_2(r))$ defined by $G_T(\psi) = g_\psi$ and G_T is continuous as a map between C^k function spaces.

We claim that not only is each G_T continuous, but as a family $\{G_T\colon\ T \in \mathscr{H}_\lambda\}$ is an equicontinuous family. For this it is important to note that $\mathscr{H}_\lambda$ is defined in terms of a fixed coordinate system (in which the T are already block diagonal) and a fixed, adapted norm—two things that are readily modified at the beginning of most proofs of the stable manifold theorem. Here we assume any such necessary modifications have already been completed. If we were to consider instead all matrices which in *some* coordinate system satisfied the conditions defining $\mathscr{H}_\lambda$ (all matrices of so called skewness λ), then we would *not* end up with an equicontinuous family.

Theorem 4.3. *For any $T \in \mathscr{H}_\lambda$, the continuity of G_T at $\psi \in \mathscr{U}$ depends only on ψ and λ. Consequently, $\{G_T\colon\ T \in \mathscr{H}_\lambda\}$ is an equicontinuous family.*

Although perhaps not stated, it is obvious from any of the many proofs of the stable manifold theorem that the continuous dependence of the invariant manifold on the perturbation depends only very weakly on T itself, namely only on its strength of hyperbolicity as measured by λ. This is simply because once the coordinate system and norm have been adapted to T, the only things left of T that are actually used in the proof are the two norms $||T_1||$ and $||T_2^{-1}||$, both controlled by λ. From this the equicontinuity of the family $\{G_T\}$ would follow immediately. Below we give a direct a proof based on the same method used to derive the more general stable manifold theorems for hyperbolic sets and normally hyperbolic submanifolds from the fixed point case. See [15, pp. 149–151]. A more detailed exposition can be found in [29, Chapter 6].

Proof. To show equicontinuity of the family $\{G_T\}$ at $\psi_0 \in U$, it would certainly suffice to show $\sup_{T \in \mathscr{H}_\lambda} ||G_T(\psi) - G_T(\psi_0)||_{C^k} \to 0$ as $||\psi - \psi_0||_{C^k} \to 0$.

Consider $\pi : \mathscr{H}_\lambda \times \mathbb{R}^n \to \mathscr{H}_\lambda$ as a vector bundle. Let $\mathbb{E}$ be the Banach space of bounded section with the sup norm. If $\sigma \in \mathbb{E}$, then $\sigma(T) = (T, \sigma_T)$, for some $\sigma_T \in \mathbb{R}^n$, and $||\sigma|| \coloneqq \sup_{T \in \mathscr{H}_\lambda} ||\sigma_T||$. When convenient, we will freely ignore the distinction between $\sigma \in \mathbb{E}$ and the bounded function $\sigma_T : \mathscr{H}_\lambda \to \mathbb{R}^n$.

Let $\mathbb{E}_i$ denote the subspaces of $\mathbb{E}$ consisting of those section with images in the subbundles $\pi : \mathscr{H}_\lambda \times E_i \to \mathscr{H}_\lambda$, $i = 1, 2$. That is, $\sigma \in \mathbb{E}_i$ if and only if $\sigma_T \in E_i$ for all T. Clearly $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$. Define $\mathbf{T} \in L(\mathbb{E}, \mathbb{E})$ and, for any $\psi \in U$, $\Psi : \mathbb{E}_1(r) \oplus \mathbb{E}_2(r) \to \mathbb{E}$ by

$$[\mathbf{T}\sigma]_T = T\sigma_T,$$

$$[\Psi(\sigma)]_T = \psi(\sigma_T).$$

We make the following observations. $\mathbf{T}$ respects the splitting $\mathbb{E}_1 \oplus \mathbb{E}_2$ and, due to the uniform hyperbolicity in $\mathscr{H}_\lambda$, we have $||\mathbf{T}_1||, ||\mathbf{T}_2^{-1}|| < \lambda$. Because $E_1(r) \times E_2(r)$ is compact, the derivatives of ψ are uniformly continuous (what Irwin calls *uniformly* C^k [20]). As a consequence, Ψ is also uniformly C^k with derivatives given by

$$[(D^i\Psi)_\sigma(v_1, \dots, v_i)](T) = (T, (D^i\psi)_{\sigma_T}([v_1]_T, \dots, [v_i]_T)).$$

Moreover, $\mathrm{Lip}(\Psi) = \mathrm{Lip}(\psi) < \varepsilon$ and $||\Psi(0)|| = ||\psi(0)|| < \delta$.

The statement of Theorem 4.2 above is for finite dimensions because that is all we need for this paper. However, the original statements in [15,16,29] are for Banach spaces and the constraints on ε and δ do not depend on the space. Therefore, $\mathbf{F} \coloneqq \mathbf{T} + \Psi$ satisfies the conditions of the theorem for any $\psi \in U$. Consequently, there exists a C^k function $\mathbf{g}_\Psi : \mathbb{E}_1(r) \to \mathbb{E}_2$, depending continuously on Ψ, such that the graph of $\mathbf{g}_\Psi$ is the local stable manifold of $\mathbf{F}$. We now observe that since $[\mathbf{F}(\sigma, \mathbf{g}_\Psi(\sigma))]_T$ depends only on $[(\sigma, \mathbf{g}_\Psi(\sigma))]_T$, $(\sigma, \mathbf{g}_\Psi(\sigma))$ is in the stable manifold for $\mathbf{F}$ if and only if individually $[(\sigma, \mathbf{g}_\Psi(\sigma))]_T = (\sigma_T, [\mathbf{g}_\Psi(\sigma)]_T)$ is in the stable manifold for $T + \psi$ for each $T \in \mathscr{H}_\lambda$. However, the stable manifold of $T + \psi$ is unique and is the graph of $G_T(\psi)$. Therefore, $[\mathbf{g}_\Psi(\sigma)]_T = G_T(\psi)(\sigma_T)$. From this we get $||\mathbf{g}_\Psi - \mathbf{g}_{\Psi_0}||_{C^k} = \sup_{T \in \mathscr{H}_\lambda} ||G_T(\psi) - G_T(\psi_0)||_{C^k}$. Since $||\Psi - \Psi_0||_{C^k} = ||\psi - \psi_0||_{C^k} \to 0$, continuity of $\mathbf{g}_\Psi$ in Ψ implies $||\mathbf{g}_\Psi - \mathbf{g}_{\Psi_0}||_{C^k} \to 0$, proving equicontinuity of the family $\{G_T\}$. □

To prove Theorem 4.1 we need to eliminate the problem of the systems becoming less hyperbolic as $\varepsilon \to 0$. We do this by rescaling time. For the systems determined by H_ε and $\hat{H}_\varepsilon$, reparameterize time by a constant factor of ε^{-1}. This is equivalent to dividing the Hamiltonians by an additional factor of ε.

$$\hat{H} = \tfrac{1}{\varepsilon}\Gamma_1 + \delta(\Gamma_2 - \tfrac{1}{2}\eta\Gamma_3 + \eta\Gamma_3^2),$$

$$H = \hat{H} + O(\varepsilon). \tag{15}$$

Since reparameterizing time will not change the invariant manifolds, it will suffice to prove Theorem 4.1 for these new systems.

Dividing the Hamiltonian by ε also divides the eigenvalues by ε. Therefore, using that $v = -\frac{1}{2}\eta\varepsilon^2$, the eigenvalues are now $\lambda = \pm\sqrt{\eta/2} \pm i/\varepsilon$. Consequently, the eigenvalues no longer converge to the imaginary axis as $\varepsilon \to 0$, but rather go to infinity parallel to the imaginary axis. Thus, we have eliminated the problem of the systems becoming less hyperbolic, by replacing it with the problem that the rotational component goes to infinity.

Define a new Hamiltonian, $\hat{H}_0$, to be $\hat{H}$ with no rotational component (i.e., $\omega = 0$),

$$\hat{H}_0 = \delta(\Gamma_2 - \tfrac{1}{2}\eta\Gamma_3 + \eta\Gamma_3^2). \tag{16}$$

We want to express the flow for $\hat{H}$ in terms of the flow for $\hat{H}_0$. Let R_θ denote the rotation by θ in both the x and y planes,

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{17}$$

The Γ_i are invariant with respect to rotation by θ, $z \to R_\theta z$. Therefore, since $\hat{H}$ and $\hat{H}_0$ are expressed only in terms of the Γ_i, both the vector fields and the invariant manifolds associated to $\hat{H}$ and $\hat{H}_0$ are symmetric with respect to rotation by θ. Let φ_t, $\hat{\varphi}_t$, and $\hat{\varphi}_t^0$ be the flows for H, $\hat{H}$, and $\hat{H}_0$, respectively. Let $\hat{\Delta}_s(r)$ denote the stable disk of radius r for $\hat{H}$ (which, by the following lemma, will also be the stable disk of radius r for $\hat{H}_0$).

Lemma 4.1. *$\hat{\varphi}_t = R_{-t/\varepsilon}\hat{\varphi}_t^0$. Therefore, $W^s(\hat{H}_0) = W^s(\hat{H})$, and given any $r_1, r_2 > 0$, there is a fixed time τ, independent of ε, such that $\hat{\varphi}_\tau(\hat{\Delta}_s(r_1)) \supset \hat{\Delta}_s(r_2)$.*

The same holds for the unstable manifolds.

Proof. One can check directly that $R_{-t/\varepsilon}\hat{\varphi}_t^0$ satisfies the same differential equation as $\hat{\varphi}_t$ and hence these are equal. Since the two flows only differ by rotation and $W^s(\hat{H}_0)$ and $W^s(\hat{H})$ are rotationally symmetric, both are invariant under both flows. They are tangent since both flows have the same stable subspace. Hence, $W^s(\hat{H}_0) = W^s(\hat{H})$. Since $\hat{\varphi}_t^0$ does not depend on ε, we can choose τ such that $\hat{\varphi}_\tau^0(\hat{\Delta}_s(r_1)) \supset \hat{\Delta}_s(r_2)$ and by the rotational symmetry this will carry over to $\hat{\varphi}_\tau$. $\square$

Proposition 4.1. *Let $K \subset \mathbb{R}^4$ be a compact set. For any fixed $\tau \in \mathbb{R}$ and $k \in \mathbb{N}$,*

$$||(\varphi_\tau - \hat{\varphi}_\tau)|_K||_{C^k} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Let $\hat{X}$ and X denote the vector fields associated to $\hat{H}$ and H, respectively. Write X as $X = \hat{X} + \varepsilon X^{\dagger}(z, \varepsilon)$, where $X^{\dagger}$ is analytic in ε and z. Place these systems in an ε dependent rotating coordinate system, $w = R_{t/\varepsilon}z$. Relative to these coordinates we obtain two new vector fields, $\hat{Y}$ and Y. By Lemma 4.1, $\hat{Y}(w, \varepsilon) = \hat{X}_0(w)$. Therefore,

$$\begin{aligned} Y(t, w, \varepsilon) &= \hat{X}_0(w) + \varepsilon R_{t/\varepsilon} X^{\dagger}(R_{-t/\varepsilon}w, \varepsilon) \\ &= \hat{X}_0(w) + \varepsilon Y^{\dagger}(t, w, \varepsilon) \quad (\varepsilon \neq 0), \end{aligned} \tag{18}$$

where $Y^{\dagger}(t, w, \varepsilon) := R_{t/\varepsilon} X^{\dagger}(R_{-t/\varepsilon}w, \varepsilon)$ is analytic in (t, w, ε) for $\varepsilon \neq 0$.

Now extend Y to $\varepsilon = 0$ by $Y(t, w, 0) := \hat{X}_0(w)$. We claim Y and its partial derivatives with respect to w, $(D_2^i Y)$, $(i \leqslant k)$ are continuous in (t, w, ε), including at $\varepsilon = 0$. For $\varepsilon \neq 0$, Y is analytic. For $\varepsilon = 0$,

$$Y(t, w, \varepsilon) - Y(t', w', 0) = [\hat{X}_0(w) - \hat{X}_0(w')] + \varepsilon Y^{\dagger}(t, w, \varepsilon). \tag{19}$$

The first term is analytic, so we only need to show that $\varepsilon Y^{\dagger}$ and its partials with respect to w go to zero as $\varepsilon \to 0$.

Differentiating $Y^{\dagger}$ with respect to w and using that $||R_\theta|| = 1$, we have

$$(D_2^i Y^{\dagger})_{(t,w,\varepsilon)} = R_{t/\varepsilon}(D_1^i X^{\dagger})_{(R_{-t/\varepsilon}w,\varepsilon)}[(R_{-t/\varepsilon})^{(i)}]$$

and

$$||(D_2^i Y^{\dagger})_{(t,w,\varepsilon)}|| = ||(D_1^i X^{\dagger})_{(R_{-t/\varepsilon}w,\varepsilon)}||.$$

Consequently, for any $r, \varepsilon_0 > 0$,

$$\begin{aligned} &\sup\{||(D_2^i Y^{\dagger})_{(t,w,\varepsilon)}|| : t \in \mathbb{R}, ||w|| \leqslant r, \ 0 < |\varepsilon| \leqslant \varepsilon_0\} \\ &\quad = \sup\{||(D_1^i X^{\dagger})_{(w,\varepsilon)}|| : \ ||w|| \leqslant r, 0 < |\varepsilon| \leqslant \varepsilon_0\}. \end{aligned}$$

Since $X^{\dagger}$ is analytic, for any $r, \varepsilon_0 > 0$ and $k \in \mathbb{N}$, we can choose an $M > 0$ such that $||(D_1^i X^{\dagger})_{(w,\varepsilon)}|| \leqslant M$ for all $||w|| \leqslant r_0$, $|\varepsilon| \leqslant \varepsilon_0$, and $i \leqslant k$. Therefore, $||D_2^i(\varepsilon Y^{\dagger}(t, w, \varepsilon))|| \leqslant \varepsilon M \to 0$ as $(t, w, \varepsilon) \to (t', w', 0)$ for $0 \leqslant i \leqslant k$, which shows the claim.

By Theorem 4.1 of [13, p. 100], $\dot{w} = Y(t, w, \varepsilon)$ has unique solutions $\psi(t, w, \varepsilon)$ for each fixed ε, and $(D_2^i \psi)_{(t,w,\varepsilon)}$ is continuous in (t, w, ε) for $0 \leqslant i \leqslant k$, including at $\varepsilon = 0$. Of course, ψ is just the flow for X in rotated coordinates

$$\psi(t, w, \varepsilon) = R_{t/\varepsilon}\varphi_t(R_{-t/\varepsilon}w)$$

or

$$\varphi_t(z) = R_{-t/\varepsilon}\psi(t, R_{t/\varepsilon}z, \varepsilon).$$

Now fix $t = \tau$ and choose r such that K is contained in the ball of radius r. $\hat{\varphi}_\tau$ is the time τ map of the flow for $\hat{X}$, which in the rotating coordinates is $\hat{X}_0 = Y(t, w, 0)$. Therefore,

$$(\varphi_\tau - \hat{\varphi}_\tau)(z) = R_{-\tau/\varepsilon}[\psi(\tau, R_{\tau/\varepsilon} z, \varepsilon) - \psi(\tau, R_{\tau/\varepsilon} z, 0)].$$

Consequently,

$$D^i(\varphi_\tau - \hat{\varphi}_\tau)_z = R_{-\tau/\varepsilon}((D_2^i\psi)_{(\tau, R_{\tau/\varepsilon}z, \varepsilon)} - (D_2^i\psi)_{(\tau, R_{\tau/\varepsilon}z, 0)})[(R_{\tau/\varepsilon})^{(i)}]$$

and

$$\sup_{||z|| \leqslant r} ||D^i(\varphi_\tau - \hat{\varphi}_\tau)_z|| = \sup_{||w|| \leqslant r} ||(D_2^i\psi)_{(\tau, w, \varepsilon)} - (D_2^i\psi)_{(\tau, w, 0)}||$$

Since $(D_2^i\psi)$ is continuous and K compact, this completes the proof. $\square$

Proof of Theorem 4.1. Let X and $\hat{X}$ be, as before, the vector fields associated with H and $\hat{H}$, so $X = \hat{X} + O(\varepsilon)$. Let $A(\varepsilon)$ denote their common linear part $(DX)_0 = (D\hat{X})_0$;

$$A(\varepsilon) = \begin{bmatrix} 0 & \varepsilon^{-1} & -\frac{1}{2}\delta\eta & 0 \\ -\varepsilon^{-1} & 0 & 0 & -\frac{1}{2}\delta\eta \\ -\delta & 0 & 0 & \varepsilon^{-1} \\ 0 & -\delta & -\varepsilon^{-1} & 0 \end{bmatrix}. \tag{20}$$

The (block) diagonalizing matrix for $A(\varepsilon)$ is

$$S = \begin{bmatrix} 0 & \delta\kappa & 0 & -\delta\kappa \\ \delta\kappa & 0 & -\delta\kappa & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \tag{21}$$

where $\kappa = \sqrt{\eta/2}$. Note that S does not depend on ε. Consequently, we can make a linear change of variable (determined by S) such that $A(\varepsilon)$ is block diagonal for all ε:

$$A(\varepsilon) = \begin{bmatrix} -\kappa & -\varepsilon^{-1} & 0 & 0 \\ \varepsilon^{-1} & -\kappa & 0 & 0 \\ 0 & 0 & \kappa & -\varepsilon^{-1} \\ 0 & 0 & \varepsilon^{-1} & \kappa \end{bmatrix}. \tag{22}$$

Let $f,\ \hat{f},\ \hat{f}_0$ denote the time-one maps $\varphi_1,\ \hat{\varphi}_1,\ \hat{\varphi}_1^0$, respectively. Let

$$T_0 = \begin{bmatrix} e^{-\kappa} I_2 & 0 \\ 0 & e^{\kappa} I_2 \end{bmatrix} \tag{23}$$

and $T(\varepsilon) = T$ be the matrix $R_{1/\varepsilon} T_0 = e^{A(\varepsilon)}$. We then have $(D\hat{f}_0)_0 = T_0,\ (Df)_0 = (D\hat{f})_0 = T$ and, for this coordinate system, $T_0, T \in \mathscr{H}_\lambda$ for all ε, if we choose λ with $e^{-\kappa} < \lambda < 1$.

For any function f, let $\rho(f)$ be the remainder $\rho(f) := f - (Df)_0$ and let $\psi := \rho(\hat{f}_0)$. Since $\psi(0) = 0$, $(D\psi)_0 = 0$, and the ε in Theorem 4.2 does not depend upon r, we can choose r sufficiently small that ψ is in the set $\mathscr{U}$ defined below that theorem. Therefore, for this r, $\hat{f}_0$ satisfies the hypotheses of Theorem 4.2.

Since $\hat{f} = \hat{\varphi}_1 = R_{1/\varepsilon}\hat{\varphi}_1^0 = R_{1/\varepsilon}\hat{f}_0$, we have $\mathrm{Lip}(\rho(\hat{f})) = \mathrm{Lip}(\rho(\hat{f}_0))$, which implies $\rho(\hat{f}) \in \mathscr{U}$ and $\hat{f}$ also satisfies the hypotheses for all ε.

Finally,

$$\begin{aligned} \mathrm{Lip}(\rho(f)) &\leqslant \mathrm{Lip}(f - \hat{f}) + \mathrm{Lip}(\rho(\hat{f})) \\ &\leqslant \|(f - \hat{f})|_{E_1(r)\times E_2(r)}\|_{C^1} + \mathrm{Lip}(\rho(\hat{f})). \end{aligned}$$

By Proposition 4.1 (with $\tau = 1$), $\|(f - \hat{f})|_{E_1(r)\times E_2(r)}\|_{C^1} \to 0$ as $\varepsilon \to 0$. Therefore, $\mathrm{Lip}(\rho(f)) \to \mathrm{Lip}(\rho(\hat{f}))$ and we can choose ε_0 sufficiently small that $\rho(f) \in \mathscr{U}$ for all $\varepsilon \leqslant \varepsilon_0$, and so f satisfies the hypotheses of Theorem 4.2 for all $\varepsilon \leqslant \varepsilon_0$.

Consequently, we can conclude the local stable manifolds of f and $\hat{f}$, $W_r^s(f)$ and $\hat{W}_r^s$, are graphs of functions $g_\varepsilon, \hat{g} : E_1(r) \to E_2(r)$. Moreover, even though the linear part of f and $\hat{f}$, $T(\varepsilon)$, has a rotational component of angle $1/\varepsilon$ as $\varepsilon \to 0$, since the difference $\|\rho(f) - \rho(\hat{f})\|_{C^k} = \|f - \hat{f}\|_{C^k} \to 0$ as $\varepsilon \to 0$, the equicontinuity of the family $\{G_{T(\varepsilon)}\}$ guarantees that $\|g_\varepsilon - \hat{g}\|_{C^k} \to 0$ as $\varepsilon \to 0$ (each k).

Now let $\hat{\Delta}_s$ be any stable disk for the truncated system and $i : D \to \mathbb{R}^4$ any embedding with $i(D) = \hat{\Delta}_s$. By Lemma 4.1, there is a time τ, independent of ε, such that $\hat{\varphi}_\tau(\hat{W}_r^s) \supset \hat{\Delta}_s$ for all ε. However, $\hat{\varphi}_\tau(\hat{W}_r^s)$ is the image of the embedding $\hat{\varphi}_\tau \circ \mathrm{gr}(\hat{g})$, where $\mathrm{gr}(\hat{g})$ is the function $x \mapsto (x, \hat{g}(x))$. Likewise, $\Delta_s := \mathrm{image}(\varphi_\tau \circ \mathrm{gr}(g_\varepsilon))$ is a stable disk for H. Therefore, it will be sufficient to show $\|\varphi_\tau \circ \mathrm{gr}(g_\varepsilon) - \hat{\varphi}_\tau \circ \mathrm{gr}(\hat{g})\|_{C^k} \to 0$ as $\varepsilon \to 0$.

We just showed $\|\mathrm{gr}(g_\varepsilon) - \mathrm{gr}(\hat{g})\|_{C^k} = \|g_\varepsilon - \hat{g}\|_{C^k} \to 0$ as $\varepsilon \to 0$, by Proposition 4.1 $\|(\varphi_\tau - \hat{\varphi}_\tau)|_{E_1(r)\times E_2(r)}\|_{C^k} \to 0$, and composition is continuous in the C^k-topology [20, Theorem 11]. However, this is not enough to conclude $\varphi_\tau \circ \mathrm{gr}(g_\varepsilon) \to \hat{\varphi}_\tau \circ \mathrm{gr}(\hat{g})$ since both φ_τ and $\hat{\varphi}_\tau$ depend on ε (whereas $\hat{g}$ is fixed).

Once again rotate by $R_{\tau/\varepsilon}$:

$$\begin{aligned} \|\varphi_\tau \circ \mathrm{gr}(g_\varepsilon) - \hat{\varphi}_\tau \circ \mathrm{gr}(\hat{g})\|_{C^k} &= \|R_{\tau/\varepsilon}(\varphi_\tau \circ \mathrm{gr}(g_\varepsilon) - \hat{\varphi}_\tau \circ \mathrm{gr}(\hat{g}))\|_{C^k} \\ &= \|(R_{\tau/\varepsilon}\varphi_\tau) \circ \mathrm{gr}(g_\varepsilon) - \hat{\varphi}_\tau^0 \circ \mathrm{gr}(\hat{g})\|_{C^k}. \end{aligned}$$

Now neither $\hat{\varphi}^0_\tau$ nor $\hat{g}$ depend on ε and

$$\begin{aligned}||((R_{\tau/\varepsilon}\varphi_\tau) - \hat{\varphi}^0_\tau)|_K||_{C^k} &= ||R_{\tau/\varepsilon}(\varphi_\tau - \hat{\varphi}_\tau)|_K||_{C^k} \\ &= ||(\varphi_\tau - \hat{\varphi}_\tau)|_K||_{C^k} \to 0.\end{aligned}$$

Consequently, by continuity of the composition map for a compact domain, $\varphi_\tau \circ \mathrm{gr}(g_\varepsilon) \to \hat{\varphi}_\tau \circ \mathrm{gr}(\hat{g})$ in the C^k-topology, which is what we want and concludes the proof. $\square$

Now undo the reparametrization of time by ε^{-1} and return to the scaled Hamiltonians (13). We have shown the following. For the truncated system $\hat{H}(\varepsilon)$, the stable and unstable manifolds of the origin agree and, as a set, do not depend upon ε. The flow for the truncated system decomposes as independent angular and radial components. The angular rotation does not depend upon ε, and the radial component goes to zero and $\varepsilon \to 0$. Consequently, as $\varepsilon \to 0$ the system is progressively less hyperbolic and at $\varepsilon = 0$ the invariant set becomes the union of closed orbits. The full system, $H(\varepsilon)$, is converging to the same rotation at $\varepsilon = 0$. For $\varepsilon \neq 0$ we do not know the global properties of the stable and unstable manifolds for the full Hamiltonian, but we know that for any compact local stable or unstable disk, $\hat{\Delta}$, of the truncated system, we can choose stable or unstable disks $\Delta(\varepsilon)$ for the full system such that $\Delta(\varepsilon) \to \hat{\Delta}$ as $\varepsilon \to 0$.

5. Intersection

The stable and unstable manifolds being Lagrangian almost forces their intersection by the Arnold theory [2] based on the work of Weinstein [32] on the intersection theory of Lagrangian manifolds. These general theories do not apply directly to our problem. So, in order to show that the stable and unstable manifolds must intersect, we first reduce the problem to an area-preserving diffeomorphism of a planar region. We then note that the original argument of Poincaré [26], as exposed in [22], shows that the manifolds indeed intersect and, if not identical, must possess a topologically transverse intersection.

Use the polar coordinates introduced in Section 3 for the scaled Hamiltonian (13). The equation for θ is $\dot{\theta} = -\partial \hat{H}/\partial \Theta = -1 + O(\varepsilon)$ and so $\theta = 0$ is a cross section for the flow for small ε. The stable/unstable manifold of the Poincaré map for the truncated system is a one-dimensional loop. Choose a point P on this invariant manifold and overlapping, compact local stable and unstable manifolds, $\hat{W}^{\mathrm{s}}$ and $\hat{W}^{\mathrm{u}}$, containing P in their interior. Choose a compact 2-disk, Σ, in $\theta = 0$ transverse to $\hat{W}^{\mathrm{s}}$ and $\hat{W}^{\mathrm{u}}$ at P. See Fig. 2(a). We can then choose neighborhoods $\mathscr{U}$ of Σ and $\mathscr{V}_{\mathrm{s}}$, $\mathscr{V}_{\mathrm{u}}$ of $\hat{W}^{\mathrm{s}}$, $\hat{W}^{\mathrm{u}}$ (as embeddings) such that anything in $\mathscr{U}$ would intersect transversally everything in $\mathscr{V}_{\mathrm{s}}$ and $\mathscr{V}_{\mathrm{u}}$.

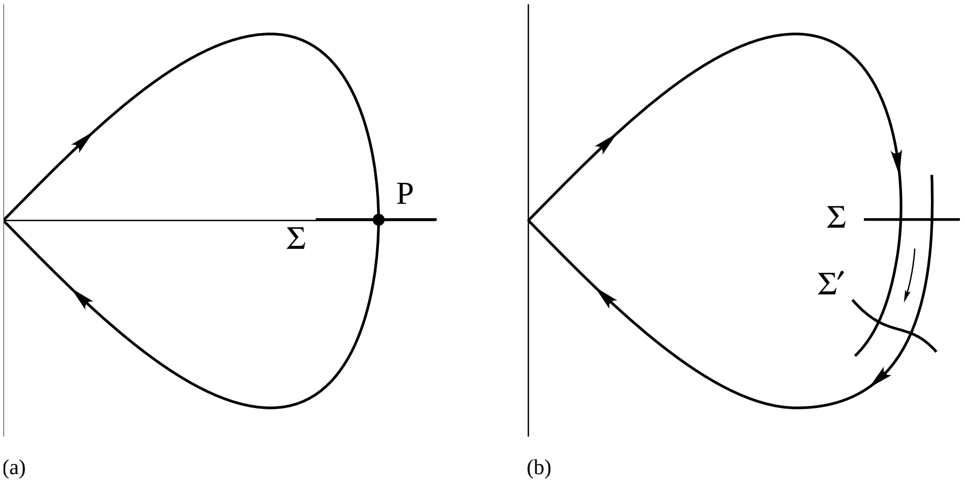

Fig. 2. (a) Truncated system and (b) non-area-preserving perturbation.

Since the flow for the full system is converging to a rotation as $\varepsilon \to 0$, the Poincaré map is converging to the identity. Consequently, the image of Σ under the Poincaré map, Σ', is inside $\mathscr{U}$ for ε sufficiently small. The convergence of the invariant manifolds to those of the truncated system carries over to the Poincaré maps. Therefore, we can choose local stable and unstable manifolds, $W^{\mathrm{s}}(\varepsilon)$ and $W^{\mathrm{u}}(\varepsilon)$, of the Poincaré map for the full system such that $W^{\mathrm{s}}(\varepsilon) \in \mathscr{V}_{\mathrm{s}}$ and $W^{\mathrm{u}}(\varepsilon) \in \mathscr{V}_{\mathrm{u}}$ for all ε sufficiently small. Consequently, for ε small both $W^{\mathrm{s}}(\varepsilon)$ and $W^{\mathrm{u}}(\varepsilon)$ intersect transversally both Σ and Σ'.

Finally, the intersection of $H = 0$ and $\theta = 0$ (excluding the origin) is a two-dimensional region containing the stable and unstable manifolds of the Poincaré map. This two-dimensional region is invariant under the Poincaré map. Σ and Σ' are transverse to the stable and unstable manifolds and so also to $H = 0$. Therefore, Σ and Σ' intersect this region in one-dimensional curves transverse to the stable and unstable manifolds. The flow preserves the symplectic structure which in polar coordinates is $\Omega = dr \wedge dR + d\theta \wedge d\Theta$. Using r, R as coordinates in the section $\theta = 0$, the Poincaré map preserves $dr \wedge dR$ and so it is area preserving. This is enough to be able to apply Poincaré's argument to conclude that $W^{\mathrm{s}}(\varepsilon)$ and $W^{\mathrm{u}}(\varepsilon)$ must intersect between Σ and Σ'.

Poincaré's argument is based on the observation that if the stable and unstable manifolds are initially identical, then a small perturbation which disconnects them, as in Fig. 2(b), must be area increasing or decreasing. The argument applies equally well when the perturbed manifolds are not identical and do intersect, but no intersections are topologically transverse. Poincaré's argument was criticized by Cherry and Wintner (see [6] for details). However, Poincaré's argument was absolutely sound and a detailed discussion can be found in [22]. Applying these arguments to our situation in light of the theorems in Section 4 shows that either the stable and unstable manifolds remain identical or they have a topologically transverse intersection ε small.

If stable and unstable manifolds have a transverse intersection, the cross-section map admits a Smale horseshoe with all the chaos this implies about the flow. The area-preserving argument of Poincaré only assures a topologically transverse intersection. By a result of Burns and Weiss [5], this is sufficient for there to exist a set which is invariant under some iterate of the cross-section map and on which this iterate is semiconjugate to a Smale horseshoe. Moreover, Conley [7] proved that, for the case of two dimensions (here a codimension 1 cross-section in a three-dimensional energy surface), in every neighborhood of a point of nontransverse intersection there exists a point of transverse intersection. Consequently, there is an even higher iterate of the cross-section map fixing a set on which it is actually conjugate to the Smale horseshoe. Hence, the cross-section map is at least as chaotic as a Smale horseshoe. In particular, for the flow, the homoclinic orbits for the equilibrium point are in the closure of the periodic orbits.

6. Applications

Strömgren [31] conjectured, based on numerical evidence, that there are orbits doubly asymptotic to $\mathscr{L}_4$, the Lagrange equilateral equilibrium point, in the restricted three body problem and that these doubly asymptotic orbits are the limit of periodic orbits with long periods (the blue sky catastrophe). Henrard [14] and Devaney [9,10] establish general theorems which would verify Strömgren conjecture provided the stable and unstable manifolds at $\mathscr{L}_4$ intersect transversally in the $H =$ constant level set. Buffoni [3] dropped the transversality condition and obtained a similar result.

The Hamiltonian of the restricted problem at $\mathscr{L}_4$ can be considered as a perturbation of Hamiltonian (3) with $v = \mu_1 - \mu$ where μ_1 is Routh's critical mass ratio parameter. The normal form for the restricted problem at $\mathscr{L}_4$ at μ_1 was first discussed and computed by Sokol'skii [30]—also see [8,24,25]. These computations show that $\eta > 0$ in the restricted problem and our theorem applies.

As another example consider the fourth-order equation

$$\ddddot{u} + P\ddot{u} + u + f(u) = 0, \tag{24}$$

where P is a parameter and f is a nonlinearity. In [1,4,17,18] $f(u) = -u^2$ but in some others $f(u) = -u^3$. Eq. (24) can be written as a Hamiltonian system with Hamiltonian

$$H = p_2^2 + p_1 p_2 - \frac{1}{2}\left(q_2 - \frac{1}{2}(P+2)q_1\right)^2 + \frac{1}{2}q_1^2 + F(q_1), \tag{25}$$

where $F'(u) = f(u)$ and $u = q_1$. When $P = 2$ the exponents are $\pm i$ with multiplicity 2. Let $P = 2 + \mu$. The coefficient matrix of the linearized equations at $\mu = 0$ is

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 3 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{bmatrix}.$$

The symplectic matrix

$$R = \frac{1}{4}\begin{bmatrix} -1 & -1 & 2i & -2i \\ -5 & -5 & 2i & -2i \\ 5i & -5i & 2 & 2 \\ -i & i & -2 & -2 \end{bmatrix}$$

reduces A to complex normal form

$$R^{-1}A_2R = \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 1 & i & 0 \\ 1 & 0 & 0 & -i \end{bmatrix}.$$

In the case when $f(u) = -u^2$, $F(u) = -(1/3)u^3$ the complex normalized system is

$$H = i(z_1z_3 - z_2z_4) - z_1z_2 - \frac{\mu}{4}z_3z_4 - \frac{5}{192}(z_3z_4)^2 + \cdots ,$$

changing to real symplectic (multiplier 1/2) coordinates by

$$z_1 = x_1 - ix_2, \quad z_2 = x_1 + ix_2,$$

$$z_3 = y_1 + iy_2, \quad z_4 = y_1 - iy_2$$

the Hamiltonian becomes

$$\begin{aligned} H &= (x_2y_1 - x_1y_2) - \frac{1}{2}(x_1^2 + x_2^2) - \frac{\mu}{8}(y_1^2 + y_2^2) - \frac{5}{384}(y_1^2 + y_2^2)^2 + \cdots \\ &= \Gamma_1 + \delta\Gamma_2 + v\delta\Gamma_3 + \eta\delta\Gamma_3^2 + \cdots \end{aligned}$$

where $\delta = -1$, $v = \mu/4$, $\eta = 5/96$. Thus our theory applies to this example as well.

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