ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 223 (2010) 405–415
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The g-periodic subvarieties for an automorphism g of positive entropy on a compact Kähler manifold

De-Qi Zhang [1]

Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore

Received 17 March 2009; accepted 17 August 2009

Available online 22 August 2009

Communicated by Gang Tian

Abstract

For a compact Kähler manifold X and a strongly primitive automorphism g of positive entropy, it is shown that X has at most $\rho(X)$ of g-periodic prime divisors. When X is a projective threefold, every prime divisor containing infinitely many g-periodic curves, is shown to be g-periodic (a result in the spirit of the Dynamic Manin–Mumford conjecture as in Zhang (2006) [17]).

MSC: 14J50; 37C25; 32H50

Keywords: Automorphism; Periodic subvariety; Topological entropy

1. Introduction

We work over the field $\mathbb{C}$ of complex numbers. Let X be a compact Kähler manifold and $g \in \mathrm{Aut}(X)$ an automorphism. The pair (X, g) is *strongly primitive* if it is not bimeromorphic to another pair (Y, g_Y) (even after replacing g by its power) having an equivariant fibration $Y \to Z$ with $\dim Y > \dim Z > 0$. g is of *positive entropy* if its *topological entropy*

$$h(g) := \max\left\{\log|\lambda|;\ \lambda \text{ is an eigenvalue of } g^* \,\Big|\, \bigoplus_{i \geqslant 0} H^i(X, \mathbb{C})\right\}$$

E-mail address: matzdq@nus.edu.sg.
[1] The author is supported by an ARF of NUS.

0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.08.010

is positive; see 2.1. We remark that every surface automorphism of positive entropy is automatically strongly primitive (cf. Lemma 2.4).

Theorems 1.1, 3.1 and 3.2 are our main results, where the latter determines the geometrical structure for those compact Kähler X with a strongly primitive automorphism. A subvariety $B \subset X$ is *g-periodic* if $g^s(B) = B$ for some $s > 0$. Let $\rho(X)$ be the *Picard number* of X.

Theorem 1.1. *Let X be a compact Kähler manifold, and $g \in \mathrm{Aut}(X)$ a strongly primitive automorphism of positive entropy. Then we have:*

(1) *X has at most $\rho(X)$ of g-periodic prime divisors.*
(2) *If X is a smooth projective threefold, then any prime divisor of X containing infinitely many g-periodic curves, is itself g-periodic (cf. [17, Conjecture 1.2.1]).*

Remark 1.2.

(1) Suppose that the X in Theorem 1.1(1) has $\rho(X)$ of g-periodic prime divisors, then the algebraic dimension $a(X) = 0$ by the proof, Theorem 3.2 and Remark 2.8. Suppose further that the irregularity $q(X) := h^1(X, \mathcal{O}_X) > 0$. Then the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X) =: Y$ is surjective and isomorphic outside a few points of Y, and $\rho(Y) = 0$. Conversely, we might realize such maximal situation by taking a complex n-torus T with $\rho(T) = 0$ and a matrix $H \in \mathrm{SL}_n(\mathbb{Z})$ with trace $> n$ so that H induces an automorphism $h \in \mathrm{Aut}(T)$ of positive entropy; if H could be so chosen that h has a few finite orbits O_i of a total ρ points $P_{ij} \in T$, then the blowup $a : X \to T$ along these ρ points lifts h to some $g \in \mathrm{Aut}(X)$ of positive entropy with $\rho = \rho(X)$ of g-periodic prime divisors $a^{-1}(P_{ij})$.
(2) When $\dim X = 2$, see [8, Proposition 3.1] or [14, Theorem 6.2] for results similar to Theorem 1.1(1). Meromorphic endomorphisms and fibrations are studied in [1].
(3) For a possible generalization of Theorem 1.1 to varieties over other fields, we remark that the Bertini type theorem is used in the proof, so the ground field might need to be of characteristic zero. Kähler classes are also used in the proof. The proof of Theorem 1.1(2) requires X to be projective in order to define nef reduction as in [2].

The following consequence of Lemma 2.11 or Theorem 3.2 and Lefschetz's fixed point formula, shows the practicality of the strong primitivity notion.

Theorem 1.3. *Let A be a complex torus of* $\dim A \geqslant 2$, *and $g \in \mathrm{Aut}_{\mathrm{variety}}(A)$ a strongly primitive automorphism of positive entropy (cf.* 2.1*). Then A has no g-periodic subvariety D with* $\mathrm{pt} \neq \mathrm{D} \subset \mathrm{A}$. *In particular, for every $s > 0$, the number* $\#\mathrm{Per}(g^s)$ *of g^s-fixed points (with multiplicity counted) satisfies*

$$\#\mathrm{Per}\big(g^s\big) = \sum_{i \geqslant 0} \mathrm{Tr}\big(g^s\big)^* | H^i(A, \mathbb{Z}).$$

2. Preliminary results

2.1. Most of the conventions are as in [10] and Hartshorne's book. Below are some more. In the following (till Lemma 2.4), X is a compact Kähler manifold of dimension $n \geqslant 2$.

(1) Denote by $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ the *Neron–Severi group*, and $\mathrm{NS}_B(X) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} B$ for $B = \mathbb{Q}, \mathbb{R}$, which is a B-vector space of finite dimension $\rho(X)$ (called the *Picard number*).

By abuse of notation, the *cup product* $L \cup M$ for $L \in H^{i,i}(X)$ and $M \in H^{j,j}(X)$ will be denoted as $L.M$ or simply LM. Two codimension-r cycles C_1, C_2 are *numerically equivalent* if $(C_1 - C_2)M_1 \cdots M_{n-r} = 0$ for all $M_i \in H^{1,1}(X)$. Denote by $[C_1]$ the equivalence class containing C_1, and $N^r(X)$ the $\mathbb{R}$-vector space of all equivalence classes $[C]$ of codimension-r cycles. By *abuse of notation*, we write $C_1 \in N^r(X)$ (instead of $[C_1] \in N^r(X)$). We remark that if C_1 and C_2 are cohomologous then C_1 and C_2 are numerically equivalent, but the converse may not be true if $r \leqslant n-2$. Our $N^{n-1}(X)$ coincides with the usual $N_1(X)$.

Codimension-r_i cycles C_i $(i = 1, 2)$ are *perpendicular* to each other if $C_1.C_2 = 0$ in $N^{r_1+r_2}(X)$.

(2) A class L in the closure of the Kähler cone of X is called *nef*; this L is *big* if $L^n \neq 0$.

For $g \in \mathrm{Aut}(X)$, the *i-th dynamical degree* is defined as

$$d_i(g) := \max\{|\lambda|;\ \lambda \text{ is an eigenvalue of } g^*|H^{i,i}(X)\}.$$

It is known that the *topological entropy* $h(g)$ equals $\max_{1 \leqslant i \leqslant n} \log d_i(g)$. We say that g is of *positive entropy* if $h(g) > 0$. Note that $h(g) > 0$ if and only if $d_i(g) > 1$ for some i and in fact for all $i \in \{1, \ldots, n-1\}$, if and only if $h(g^{-1}) > 0$. We refer to [5] for more details.

By the generalized Perron–Frobenius theorem in [3], there are non-zero nef classes $L_g^{\pm}$ such that $g^* L_g^+ = d_1(g) L_g^+$ and $(g^{-1})^* L_g^- = d_1(g^{-1}) L_g^-$ in $H^{1,1}(X)$. When X is a projective manifold, we can choose $L_g^{\pm}$ to be in $\mathrm{NS}_{\mathbb{R}}(X)$.

An irreducible subvariety Z of X is *g-periodic* if $g^s(Z) = Z$ for some $s \geqslant 1$.

(3) When a cyclic group $\langle g \rangle$ acts on X, we use $g|X$ or g_X to denote the image of g in $\mathrm{Aut}(X)$. The pair $(X, g|X)$ is loosely denoted as (X, g).

(4) Suppose that a cyclic group $\langle g \rangle$ acts on compact Kähler manifolds X, X_i, Y_j. A morphism $\sigma : X_1 \to X_2$ is *g-equivariant* if $\sigma \circ g = g \circ \sigma$. Two pairs (Y_1, g) and (Y_2, g) are *bimeromorphically equivariant* if there is a decomposition $Y_1 = Z_1 \overset{\sigma_1}{\dashrightarrow} Z_2 \cdots \overset{\sigma_r}{\dashrightarrow} Z_{r+1} = Y_2$ into bimeromorphic maps such that for each i either σ_i or σ_i^{-1} is a g-equivariant bimeromorphic morphism.

(X, g) or simply $g|X$, is *non-strongly-primitive* (resp. *non-weakly-primitive*) if (X, g^s), for some $s > 0$, is bimeromorphically equivariant to some (X', g^s) and there is a g^s-equivariant surjective morphism $X' \to Z$ with Z a compact Kähler manifold of $\dim X > \dim Z > 0$ (resp. of $\dim X > \dim Z > 0$ and $g^s|Z = \mathrm{id}$). We call (X, g) *strongly primitive* (resp. *weakly primitive*) if (X, g) is not non-strongly-primitive (resp. not non-weakly-primitive).

(5) For a complex torus A, the (variety) automorphism group $\mathrm{Aut}_{\mathrm{variety}}(A)$ equals $T_A \rtimes \mathrm{Aut}_{\mathrm{group}}(A)$, with T_A the group of translations and $\mathrm{Aut}_{\mathrm{group}}(A)$ the group of group-automorphisms.

The two results below are crucial and due to Dinh and Sibony [5], but we slightly reformulated. The second result is from [5, Corollaire 3.2], with [12, Appendix A, Lemma A.4] used to weaken the assumption a bit.

Lemma 2.2. *(Cf. [5, Lemme 4.4].) Let X be a compact Kähler manifold of dimension $n \geqslant 2$, $g : X \to X$ a surjective endomorphism, and M_1, M_2, L_i $(1 \leqslant i \leqslant m;\ m \leqslant n-2)$ nef classes. Suppose that in $N^{m+1}(X)$ we have $L_1 \cdots L_m M_i \neq 0$ $(i = 1, 2)$ and $g^*(L_1 \cdots L_m M_i) = \lambda_i (L_1 \cdots L_m M_i)$ for some (positive real) constants $\lambda_1 \neq \lambda_2$. Then $L_1 \cdots L_m M_1 M_2 \neq 0$ in $N^{m+2}(X)$.*

Lemma 2.3. *(Cf. [5, Corollaire 3.2] [12, Appendix A, Lemma A.4].) Let X be a compact Kähler manifold with nef classes L, M. Then $L = 0$ in $N^i(X)$ if and only if $L = 0$ in $H^{i,i}(X, \mathbb{R})$. If $LM = 0$ in $N^2(X)$, then L and M are parallel in $H^{1,1}(X, \mathbb{R})$.*

We frequently use the (5) below. In particular, bimeromorphically equivariant automorphisms have the same dynamical degrees (and hence entropy).

Lemma 2.4. *Let X be a compact Kähler manifold of dimension n, and $g \in \mathrm{Aut}(X)$ an automorphism of positive entropy. Then the following are true.*

(1) *We have $n \geqslant 2$. If $n = 2$, then g is strongly primitive.*
(2) *All $d_i(g^{\pm})$ $(1 \leqslant i \leqslant n-1)$ are irrational algebraic integers.*
(3) *Let L_i $(1 \leqslant i \leqslant n-1)$ be in the closure $\overline{P^i(X)}$ of the Kähler cone $P^i(X)$ of degree i in the sense of* [12, *Appendix* A, *Lemma* A.9, *the definition before Lemma* A.3] *such that $g^* L_i = d_i(g) L_i$ in $H^{i,i}(X)$. Then no positive multiple of L_i is in $H^{2i}(X, \mathbb{Q})$.*
(4) *Every g-periodic curve is perpendicular to L_1.*
(5) *We have $d_i(g) = d_i(g|Y)$ $(1 \leqslant i \leqslant n)$ if there is a g-equivariant generically finite surjective morphism either from X to Y or from Y to X. Here g is not assumed to be of positive entropy.*

Proof. For (1), apply Lemma 2.2 or [14, Lemma 2.12] to L_g^+ and the fiber of an equivariant fibration (cf. also (5)). For the existence of the L_i in (3), we used the generalized Perron–Frobenius theorem in [3] for the closed cone $\overline{P^i(X)} \subset H^{i,i}(X, \mathbb{R})$. Now (3) follows from (2) by considering the cup product.

(2) Since g^{-1} is also of positive entropy, we consider only g. Since g^* acts on $H^i(X, \mathbb{Z})$ and each $d_i(g)$ is known to be an eigenvalue of $H^i(X, \mathbb{C}) = H^i(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, all dynamical degrees $d_i(g) > 1$ are algebraic integers. Suppose that $d_i(g)$ is rational. Then $d_i(g) \in \mathbb{Z}_{\geqslant 2}$. Take an eigenvector M_i in $H^{2i}(X, \mathbb{Z})$ with $g^* M_i = d_i(g) M_i$. Since the cup product is non-degenerate, we can find $N_{n-i} \in H^{2n-2i}(X, \mathbb{Z})$ such that $M_i.N_{n-i} = m_i \in \mathbb{Z} \setminus \{0\}$. Now $m_i/d_i(g)^s = (g^{-s})^* M_i.N_{n-i} \in \mathbb{Z}$ for all $s > 0$. This is absurd.

(4) Suppose that $g^s(C) = C$ for some $s > 0$ and a curve C. Then $L_1.C = (g^s)^* L_1.(g^s)^* C = d_1(g)^s L_1.C$. So $L_1.C = 0$ for $d_1(g) > 1$.

For (5), see [15, Lemma 2.6] and [12, Appendix A, Lemma A.8]. □

Lemma 2.5. *Let X and Y be compact Kähler manifolds with $n := \dim X \geqslant 2$, and $\pi : (X, g) \to (Y, g_Y)$ an equivariant surjective morphism.*

(1) *Suppose that a nef and big class M on X satisfies $g^* M = M$ in $H^{1,1}(X)$. Then a positive power of g is in $\mathrm{Aut}_0(X)$ and hence g is of null entropy.*
(2) *Suppose that g is of positive entropy and $\dim Y = n - 1$. Then no nef and big class M on Y satisfies $g_Y^* M = M$. In particular, $g_Y^* | H^{1,1}(Y)$ is of infinite order and hence no positive power of g_Y is in the identity connected component $\mathrm{Aut}_0(Y)$ of $\mathrm{Aut}(Y)$.*

Proof. (1) is a result of Lieberman [11, Proposition 2.2]; see [15, Lemma 2.23] (by Demailly–Paun, a nef and big class can be written as the sum of a Kähler class and a closed real positive current).

(2) If $g_Y^* | H^{1,1}(Y)$ is of finite order r, then g_Y^* stabilizes $\sum_{i=0}^{r-1} (g_Y^i)^* H$ with H a Kähler class. So we only need to rule out the existence of such M in the first assertion. Set $M_X := \pi^* M$.

We apply Lemma 2.2 repeatedly to show the assertion that $M_X^{k-1}.L_g^+ \neq 0$ in $N^k(X)$ for all $1 \leqslant k \leqslant n$. Indeed, $M_X.L_g^+$ is non-zero in $N^2(X)$ since $g^*M_X = M_X$ while $g^*L_g^+ = d_1(g)L_g^+$ with $d_1(g) > 1$; if $M_X^{j-1}.L_g^+ \neq 0$ in $N^j(X)$ for $j < n$, then $M_X^j.L_g^+ \neq 0$ in $N^{j+1}(X)$ because $g^*(M_X^{j-1}.L_g^+) = d_1(g)(M_X^{j-1}.L_g^+)$ with $d_1(g) > 1$, and $g^*M_X^j = M_X^j$ ($\neq 0$ in $N^j(X)$), so the assertion is true. Now $\deg(g)(M_X^{n-1}.L_g^+) = g^*M_X^{n-1}.g^*L_g^+ = d_1(g)(M_X^{n-1}.L_g^+)$ implies a contradiction: $1 = \deg(g) = d_1(g) > 1$. Lemma 2.5 is proved. □

Lemma 2.6. *Let X be a compact Kähler manifold of dimension $n \geqslant 2$ and $q(X) = 0$, and $g \in \mathrm{Aut}(X)$ an automorphism of positive entropy. Then X has at most $\rho(X)$ of prime divisors D_j perpendicular to either one of L_g^+ and L_g^- in $N^2(X)$. Further, such D_j are all g-periodic.*

Proof. We only need to show the first assertion, since both $L_g^\pm$ are semi g^*-invariant and hence g permutes these D_j.

Suppose that X has $1 + \rho(X)$ of distinct prime divisors D_i with $L_g^+.D_i = 0$ in $N^2(X)$. The case L_g^- is similar by considering g^{-1}. Set $L := L_g^+$. Since these D_i are then linearly dependent, we may assume that $E_1 := \sum_{i=1}^{t_1} a_i D_i \equiv E_2 := \sum_{j=t_1+1}^{t_1+t_2} b_j D_j$ in $\mathrm{NS}_{\mathbb{Q}}(X)$ for some positive integers a_i, b_j, t_k. Since $q(X) = 0$, we may assume that $E_1 \sim E_2$ (linear equivalence) after replacing E_i by its multiple. Let $\sigma : X' \to X$ be a blowup such that $|\sigma^*E_1| = |M| + F$ with $|M|$ base point free and F the fixed component. Take a Kähler class H on X. Then $0 \leqslant \sigma^*L.M.\sigma^*(H^{n-2}) \leqslant \sigma^*L.(M + F).\sigma^*(H^{n-2}) = L.E_1.H^{n-2} = 0$. Hence $\sigma^*L.M.\sigma^*(H^{n-2}) = 0$. Thus, $\sigma^*L.M = 0$ in $H^{2,2}(X', \mathbb{R})$ by [12, Appendix A, Lemmas A.4 and A.5]. So, by Lemma 2.3, σ^*L equals M in $\mathrm{NS}_{\mathbb{Q}}(X')$, after replacing L by its multiple. Thus $L \in \mathrm{NS}_{\mathbb{Q}}(X)$, contradicting Lemma 2.4. This proves Lemma 2.6. □

Theorem 2.7 below effectively bounds the number of g-periodic prime divisors.

Theorem 2.7. *Let X be a compact Kähler manifold of dimension $n \geqslant 2$ and $q(X) = 0$, and $g \in \mathrm{Aut}(X)$ a weakly primitive automorphism of positive entropy. Then we have:*

(1) *X has none or only finitely many g-periodic prime divisors D_i ($1 \leqslant i \leqslant r$; $r \geqslant 0$).*
(2) *If $r > \rho(X)$, then $n \geqslant 3$ and (after replacing g by its power and X by its g-equivariant blowup) there is an equivariant surjective morphism $\pi : (X, g) \to (Y, g_Y)$ with connected fibers, Y rational and almost homogeneous, $\dim Y \in \{1, \ldots, n-2\}$, and $g_Y \in \mathrm{Aut}_0(Y)$.*
(3) *If g is strongly primitive, then X has at most $\rho(X)$ of g-periodic prime divisors.*

Proof. Let D_i ($1 \leqslant i \leqslant r$; $r > \rho := \rho(X)$) be distinct g-periodic prime divisors of X. Then D_i's are linearly dependent. Replacing g by its power, we may assume that $g(D_i) = D_i$ for all $i \leqslant r$. By the reasoning in Lemma 2.6, the Iitaka D-dimension $\kappa := \kappa(X, \sum_{i=1}^r D_i) \geqslant 1$. If $\kappa = n$, then replacing X by its g-equivariant blowup, we may assume that some positive combination M of D_i is nef and big and $g^*M = M$, contradicting Lemma 2.5. Thus, $1 \leqslant \kappa < n$.

Take $E_1 := \sum_{i=1}^t a_i D_i$ with a_i non-negative integers such that $\Phi_{|E_1|} : X \dashrightarrow \mathbb{P}^N$ has the image Y with $\dim Y = \kappa$, and the induced map $\pi : X \dashrightarrow Y$ has connected general fibers. Since $g(E_1) = E_1$, replacing X by its g-equivariant blowup and removing redundant components in E_1, we may assume that $\mathrm{Bs}\,|E_1| = \emptyset$, π is holomorphic, Y is smooth projective, and g descends to an automorphism $g_Y \in \mathrm{Aut}(Y)$; further we can write $E_1 = \pi^*A$, where $g_Y(A)$

equals A and is a nef and big Cartier divisor with $\mathrm{Bs}\,|A| = \emptyset$ (notice that A may not be ample because we have replaced Y by its blowup). Hence $g_Y \in \mathrm{Aut}_0(Y)$ after g is replaced by its power, so $\dim Y \neq n-1$; see Lemma 2.5. Therefore, $1 \leqslant \kappa = \dim Y \in \{1, \dots, n-2\}$.

By the assumption on g, we have $\mathrm{ord}(g_Y) = \infty$. Since $q(Y) \leqslant q(X) = 0$, our $\mathrm{Aut}_0(Y)$ is a linear algebraic group; see [11, Theorem 3.12] or [7, Corollary 5.8]. Let H be the identity component of the closure of $\langle g_Y \rangle$ in $\mathrm{Aut}_0(X)$, and we may assume that $g_Y \in H$ after replacing g by its power. Let $\tau : Y \cdots \to Z = Y/H$ be the quotient map; see [7, Theorem 4.1]. Replacing Y, Z, X by their equivariant blowups, we may assume that Y and Z are smooth and τ is holomorphic. By the construction, $g \in \mathrm{Aut}(X)$ and $g_Y \in \mathrm{Aut}(Y)$ descend to $\mathrm{id}_Z \in \mathrm{Aut}(Z)$. The assumption on g implies that $\dim Z = 0$. So Y has a Zariski-open dense H-orbit H_y. In other words, Y is almost homogeneous. Since H is abelian (and a rational variety by a result of Chevalley), Y is bimeromorphically dominated by H (each stabilizer subgroup H_y being normal in H), so Y is rational (and smooth projective). (2) and (3) are proved.

To prove (1), suppose that X has infinitely many distinct g-periodic prime divisors D_i $(i \geqslant 1)$. We may assume that $\kappa := \kappa(X, \sum_{i=1}^{r} D_i) = \max\{\kappa(X, \sum_{i=1}^{s} D_i) \mid s \geqslant 1\} \geqslant 1$ for some $r > 0$, and use the notation above. In particular, $1 \leqslant \kappa \leqslant n-2$. We assert that $(*)$ all D_j $(j > r)$ are mapped to distinct g_Y-periodic prime divisors $D'_j \subset Y$ by the map $\pi : X \to Y$, after replacing $\{D_i\}$ by an infinite subsequence. Since π is smooth (and hence flat) outside a codimension one subset of X and the π-pullback of a prime divisor has only finitely many irreducible components, we have only to consider the case where $D_{j_1}, D_{j_2}, \dots$ (with $j_v > r$) is an infinite sequence of divisors each dominating Y, and show that this case is impossible. Replacing g by its power and X by its g-equivariant blowup, we may assume that $|E_3|$ is base point free for some $E_3 = b_{j_1} D_{j_1} + \cdots + b_{j_u} D_{j_u}$ with $b_{j_v} \in \mathbb{Z}_{\geqslant 1}$, and D_{j_1} dominates Y (notice that some components of E_3 are in the exceptional locus of the blowup). By the maximality of κ, we have $\kappa(X, E_1 + E_3) = \kappa(X, E_1)$ and hence $\Phi_{|E_1+E_3|}$ is holomorphic onto a variety W of dimension κ with $E_1 + E_3$ the pullback of an ample divisor $A_W \subset W$. Thus taking a Kähler class M on X, we obtain a contradiction:

$$0 = M^{n-1-\kappa}(E_1 + E_3)^{\kappa+1} \geqslant M^{n-1-\kappa}.E_1^{\kappa}.E_3 \geqslant M^{n-1-\kappa}.E_1^{\kappa}.D_{j_1} = M^{n-1-\kappa}.B > 0$$

where $E_1 = \pi^* A$ with A nef and big as above, and $B = (\pi^* A|D_{j_1})^{\kappa}$ is a sum of A^k of $(n-1-\kappa)$-dimensional general fibers of the surjective morphism $\pi|D_{j_1} : D_{j_1} \to Y$. The assertion $(*)$ is proved.

Now the infinitely many distinct g_Y-periodic prime divisors $D'_j \subset Y$ are squeezed in the complement of some Zariski-open dense H-orbit H_y of Y (for some general $y \in Y$, whose existence was mentioned early on). This is impossible. Thus, we have proved (1). The proof of Theorem 2.7 is completed. $\square$

Remark 2.8. Assume that the algebraic dimension $a(X) = \dim X$ in Theorem 2.7. Then X is projective since X is Kähler. If X has $\rho(X)$ of linearly independent g-periodic divisors, then (a power of) g^* stabilizes an ample divisor on X; so g is of null entropy by Lemma 2.5, absurd! Thus, by the proof, '$r > \rho(X)$' in Theorem 2.7(2) (resp. '$\rho(X)$' in Theorem 2.7(3)) can be replaced by '$r \geqslant \rho(X)$' (resp. '$\rho(X) - 1$').

Lemma 2.9. *Let X be a projective manifold of dimension $n \geqslant 2$, and $g \in \mathrm{Aut}(X)$ an automorphism of positive entropy. Let $L = L_g^+$ or L_g^-. Then the nef dimension $n(L) \geqslant 2$, and the nef* *reduction map $\pi : X \cdots \to Y$ in* [2] *can be taken to be holomorphic with Y a projective manifold, after X is replaced by its g-equivariant blowup.*

Proof. Since $L \neq 0$, we have $n(L) = \dim Y \geqslant 1$. The second assertion is true by the construction of the nef reduction in [2, Theorem 2.6], using the chain-connectedness equivalence relation defined by numerically L-trivial curves (and preserved by g). Consider the case $n(L) = 1$. For a general fiber F of π, we have $L|F = 0$ by the definition of the nef reduction. By Lemma 2.3, a multiple of L is equal to F in $\mathrm{NS}_{\mathbb{Q}}(X)$, contradicting Lemma 2.4. □

We remark that the hypothesis in Lemma 2.10 below is optimal and the hypothetical situation may well occur when $X \to Y$ is g-equivariant, Y is a surface, and D_j and $L_g^{\pm}$ are pullbacks from Y, e.g. when $X = Y \times$ (a curve) and $g = g_Y \times \mathrm{id}$.

Lemma 2.10. *Let X be a 3-dimensional projective manifold with $q(X) = 0$, and $g \in \mathrm{Aut}(X)$ an automorphism of positive entropy. Let D_i $(i \geqslant 1)$ be infinitely many pairwise distinct prime divisors such that $L_g^+.L_g^-.D_i = 0$. Then for both $L = L_g^+$ and $L = L_g^-$, we have $L^2 = 0$ in $N^2(X)$ and the nef dimension $n(L) = 2$.*

Proof. Note that $L_g^+.L_g^- \neq 0$ in $N^2(X)$ by Lemma 2.2 or 2.3. Set $L_1 := L_g^+$, $L_2 := L_g^-$ and $\lambda_1 := d_1(g) > 1$, $\lambda_2 := 1/d_1(g^{-1}) < 1$. Then $g^*L_i = \lambda_i L_i$. If $L_i^2 \neq 0$ in $N^2(X)$ for both $i = 1, 2$, then $L_i.L_i.L_j \neq 0$, where $\{i, j\} = \{1, 2\}$; see Lemma 2.2; applying g^*, we get $\lambda_i^2 \lambda_j = 1$, whence $1 < \lambda_1 = \lambda_2 < 1$, absurd.

To finish the proof of the first assertion, we only need to consider the case where $L_1^2 \neq 0$ and $L_2^2 = 0$ in $N^2(X)$, because we can switch g with g^{-1}. By Lemma 2.2, $L_1^2.L_2 \neq 0$. Now $L_1 + L_2$ is nef and big because $(L_1 + L_2)^3 \geqslant 3L_1^2 L_2 > 0$. So we can write $L_1 + L_2 = A + \Delta$ with an ample $\mathbb{R}$-divisor A and an effective $\mathbb{R}$-divisor Δ; see [15, Lemma 2.23] for the reference on such decomposition. By Lemma 2.6 and taking an infinite subsequence, we may assume that $L_i.D_j \neq 0$ in $N^2(X)$ for $i = 1$ and 2 and all $j \geqslant 1$, and D_j is not contained in the support of Δ for all $j \geqslant 1$. Now $L_1^2.D_j = (L_1 + L_2)^2.D_j = (L_1 + L_2).(A + \Delta).D_j \geqslant (L_1 + L_2).A.D_j \geqslant A^2.D_j > 0$. Thus $L_1|D_j$ is a nef and big divisor and $L_2|D_j$ is a non-zero nef divisor such that $(L_1|D_j).(L_2|D_j) = L_1.L_2.D_j = 0$. This contradicts the Hodge index theorem applied to a resolution of D_j. The first assertion is proved.

Let L be one of L_g^+ and L_g^-. By Lemma 2.9, we only need to show $n(L) \neq 3$. As in the proof of Theorem 2.7, we may assume that the Iitaka D-dimension $\kappa := \kappa(X, E_1) = \max\{\kappa(X, \sum_{i=1}^s D_i) \mid s \geqslant 1\} \geqslant 1$ for some $E_1 := \sum_{i=1}^t a_i D_i$ with positive integers a_i. If $\kappa(X, E_1) = 3$, then E_1 is big and hence a sum of an ample divisor and an effective divisor, whence $L_g^+.L_g^-.E_1 > 0$, contradicting the choice of D_j. Therefore, $\kappa = 1, 2$.

Case (1). $\kappa = 2$. Let $\sigma : X' \to X$ be a blowup such that $|\sigma^* E_1| = |M| + F$ with $|M|$ base point free and F the fixed component. Since $\kappa(X', M) = \kappa(X, E_1) = 2$, we have $M^2 \neq 0$. If $\sigma^* L.M^2 = 0$, then the projection formula implies that $L.C = 0$ for every curve $C = \sigma_*(M_1.M_2)$ with $M_i \in |M|$ general members. So the nef dimension $n(L) < 3$.

Suppose that $\sigma^* L.M^2 > 0$. Then $\sigma^* L + M$ is nef and big because $(\sigma^* L + M)^3 \geqslant 3\sigma^* L.M^2 > 0$. Since $\sigma^*(L + E_1)$ is larger than $\sigma^* L + M$, it is also big. So $L + E_1$ is big, too. Hence $0 < L.L'.(L + E_1) = L_g^+.L_g^-.E_1$, where $\{L, L'\} = \{L_g^{\pm}\}$, contradicting the choice of D_j and E_1.

Case (2). $\kappa = 1$. We may assume that $|E_1|$ has no fixed component and is an *irreducible* pencil parametrized by $\mathbb{P}^1$ (noting: $q(X) = 0$), after removing redundant D_j from E_1. Since $L_g^{\pm}$

are semi g^*-invariant, every $g(D_j)$, like D_j, is also perpendicular to $L_g^+.L_g^-$. After relabeling and expanding the sequence, we may assume that $g(E_1)$ is also a positive combination of D_j's. By Case (1), we may assume that $\kappa(E_1 + g(E_1)) = 1$. For general (irreducible) members $M_1 \in |E_1|$ and $M_2 \in |g(E_1)|$, the two-component divisor $M_1 + M_2$ is a reduced member of $|E_1 + g(E_1)|$.

Note that $N := h^0(E_1 + g(E_1)) \geqslant h^0(E_1) + h^0(g(E_1)) - 1 \geqslant 3$. The linear system $|E_1 + g(E_1)|$ gives rise to a rational map from X onto a curve B of degree $\geqslant N - 1$ in $\mathbb{P}^{N-1}$. Thus, each member of $|E_1 + g(E_1)|$ lying over $B \setminus \operatorname{Sing} B$, is a sum of $N - 1$ linearly equivalent non-zero effective divisors, since B is a rational curve; indeed, the genus $g(B)$ of B satisfies $g(B) \leqslant q(X) = 0$. So $E_1 \sim g(E_1)$. Replacing X by its g-equivariant blowup, we may assume that $|E_1|$ is base point free and hence E_1 is a nef eigenvector of g^*. Now $L_g^+.L_g^-.E_1 = 0$ infers a contradiction to Lemma 2.2, since L_g^+, L_g^- and E_1 correspond to distinct eigenvalues $d_1(g), 1/d_1(g^{-1}), 1$ of $g^*|\operatorname{NS}_{\mathbb{Q}}(X)$. This proves Lemma 2.10. □

Lemma 2.11. *Let A be a complex torus of dimension $n \geqslant 2$ and $f \in \operatorname{Aut}_{\text{variety}}(A)$ of infinite order such that $f(D) = D$ for some subvariety* $\mathrm{pt} \neq \mathrm{D} \subset \mathrm{X}$. *Then there is a subtorus $B \subset A$ with* $\dim B \in \{1, \dots, n-1\}$ *such that f descends, via the quotient map $A \to A/B$, to an automorphism $h \in \operatorname{Aut}_{\text{variety}}(A/B)$ having a periodic point in A/B.*

Proof. Write $f = T_a \circ g$ with $T_a \in T_A$ a translation and g a group automorphism.

Case (1). $\kappa(D) = \dim D$, i.e., D is of general type. Then $\operatorname{Aut}(D)$ is finite, so $f^s|D = \mathrm{id}_D$ for some $s > 0$. Since f^s fixes D pointwise, the identity component B of the pointwise fixed point set A^{g^s} (a subtorus) is a positive-dimensional subtorus; see [4, Lemma 13.1.1]. Write $f^s = T_c \circ g^s$ with $T_c \in T_A$. If $\dim B \geqslant n$, then $B = A$, $g^s = \mathrm{id}_A$ and $f^s = T_c$, so $f^s = \mathrm{id}$ for $f^s|D = \mathrm{id}_D$. This contradicts the assumption on f. Thus $1 \leqslant \dim B \leqslant n - 1$. Our g acts on A^{g^s}, so $g(B) \subset A^{g^s}$ is a coset in $A^{g^2}/B \leqslant A/B$. Thus $g(B) = \delta + B$ for some δ. So $g(B) = B$, because $(*)$: g is a group-automorphism and $0 \in B \leqslant A$. Now $f(x + B) = a + g(x) + g(B) = f(x) + B$. So f permutes cosets in A/B and f^s fixes those cosets $d + B$ with $d \in D$. Lemma 2.11 is true.

Case (2). The Kodaira dimension $\kappa(D) \leqslant 0$. Then $\kappa(D) = 0$ and $D = \delta + B$ with a subtorus B of A; see [13, Lemma 10.1, Theorem 10.3]. Now $\delta + B = D = f(D) = a + g(\delta) + g(B)$, thus $g(B)$ equals a coset in A/B and hence $g(B) = B$ by the reasoning $(*)$ in Case (1). Therefore, f permutes cosets in A/B as in Case (1), and fixes the coset $\delta + B$. So Lemma 2.11 is true.

Case (3). $\kappa(D) \in \{1, \dots, \dim D - 1\}$. By [13, Theorem 10.9], the identity connected component B of $B' := \{x \in A \mid x + D \subseteq D\}$ is a subtorus with $\dim B = \dim D - \kappa(D)$. We claim that f permutes cosets in A/B. Indeed, for every $b \in B$, we have $D = f(D) = f(b + D) = a + g(b) + g(D) = g(b) + f(D) = g(b) + D$, so $g(b) \in B'$. Thus $g(B) \leqslant B'$. Hence $g(B) = B$ and the claim is true, by the reasoning in Case (1). Further, the map $D \to D/B$ is bimeromorphic to the Iitaka fibration, and $\kappa(D/B) = \dim(D/B)$ (cf. [13, Theorem 10.9]). f descends to an automorphism $f' \in \operatorname{Aut}_{\text{variety}}(A/B)$ stabilizing $D/B \subset A/B$. Using Case (1), we are done for some quotient torus $(A/B)/(B'/B) \cong A/B'$. Lemma 2.11 is proved. □

3. Proof of Theorem 1.1 and Remark 1.2(1)

In this section, we prove Theorem 1.1 in the introduction and the two results below. Theorem 3.1 treats X with $q(X) = 0$, while Theorem 3.2 determines the geometrical structure of those Kähler X with a strongly primitive automorphism.

Theorem 3.1. *Let X be a compact Kähler manifold of dimension $n \geqslant 2$ and irregularity $q(X) = 0$, and $g \in \mathrm{Aut}(X)$ a weakly primitive automorphism of positive entropy. Then:*

(1) *X has finitely many prime divisors B_i $(1 \leqslant i \leqslant r;\ r \geqslant 0)$ such that: each B_i is g-periodic, and $\bigcup B_i$ contains every g-periodic prime divisor and every prime divisor perpendicular to L_g^+ or L_g^-.*
(2) *Suppose that g is strongly primitive. Then the r in* (1) *satisfies $r \leqslant \rho(X)$, and $r = \rho(X)$ holds only when the algebraic dimension $a(X) < n$.*
(3) *Suppose that X is a smooth projective threefold, and g is strongly primitive. Then $(L_g^+ + L_g^-)|D$ is nef and big for every prime divisor $D \neq B_i$ $(1 \leqslant i \leqslant r)$. In particular, if a prime divisor $D \subset X$ contains infinitely many curves each of which is either g-periodic or perpendicular to $L_g^+ + L_g^-$, then D itself is g-periodic.*

A compact Kähler manifold X is called *weak Calabi–Yau* if $\kappa(X) = 0 = q(X)$.

Theorem 3.2. *Let X be a compact Kähler manifold of dimension $n \geqslant 2$, and $g \in \mathrm{Aut}(X)$ a strongly primitive automorphism of positive entropy. Then the algebraic dimension $a(X) \in \{0, n\}$. Suppose further that $(*)$ either $\kappa(X) \geqslant 0$, or $q(X) > 0$, or $\kappa(X) = -\infty$, $q(X) = 0$ and X is projective and uniruled. Then* (1), (2) *or* (3) *below occurs.*

(1) *X is a weak Calabi–Yau manifold.*
(2) *X is rationally connected in the sense of Campana, Kollár–Miyaoka–Mori (so $q(X) = 0$).*
(3) *The Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is surjective and isomorphic outside a few points of $\mathrm{Alb}(X)$. There is no h-periodic subvariety of dimension in $\{1, \ldots, n-1\}$ for the (variety) automorphism h of $\mathrm{Alb}(X)$ induced from g.*

3.3. Proof of Theorem 3.1

The assertions (1) and (2) follow from Lemma 2.6, Theorem 2.7 and Remark 2.8. For (3), by Lemmas 2.10 and 2.9, our X has finitely many divisors D_j $(1 \leqslant j \leqslant s)$ such that $L_g^+.L_g^-.D_j = 0$ and $L_g^+.L_g^-.D > 0$ for every prime divisor $D \neq D_j$ $(1 \leqslant j \leqslant s)$. Since both $L_g^{\pm}$ are semi g^*-invariant, these D_j's are permuted by g and hence are all g-periodic. Thus $\{D_j\} \subset \{B_i\}$.

Suppose that $D \neq B_i$ $(1 \leqslant i \leqslant r)$ is a prime divisor of X. Then $M := L_g^+ + L_g^-$ is nef and $(M|D)^2 \geqslant 2L_g^+.L_g^-.D > 0$, so $M|D$ is nef and big. Thus D has none or only finitely many curves perpendicular to M, by the Hodge index theorem applied to a resolution of D. So D contains only finitely many g-periodic curves (cf. Lemma 2.4(4)). This proves (3) and also Theorem 3.1.

3.4. Proof of Theorem 3.2

As in the proof of [16, Lemma 2.16], a suitable algebraic reduction $X \to Y$, with $\dim Y = a(X)$, is holomorphic and g-equivariant. So $a(X) \in \{0, n\}$, since g is strongly primitive.

Consider the case $\kappa(X) \geqslant 1$. Let $\Phi = \Phi_{|mK_X|} : X \cdots\to \mathbb{P}^N$ be the Iitaka fibration. Replacing X by its g-equivariant blowup, we may assume that Φ is holomorphic and g-equivariant onto some smooth Z with $\dim Z = \kappa(X)$. Our g descends to an automorphism $g_Z \in \mathrm{Aut}(Z)$. Now $\mathrm{ord}(g_Z) < \infty$ (so $\dim Z < \dim X$ by Lemma 2.4(5)), by the generalization of [13, Theorem 14.10] to dominant meromorphic selfmaps on Kähler manifolds as in [12, Theorem A or Corollary 2.4]. This contradicts the strong primitivity of g. Therefore, $\kappa(X) \leqslant 0$.

Case (1). $q(X) > 0$. We will show that Theorem 3.2(3) holds. Consider the Albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ and let $Y = \mathrm{alb}_X(X)$ be its image. g descends to automorphisms $g|\,\mathrm{Alb}(X)$ and $h \in \mathrm{Aut}(Y)$. Since g is strongly primitive, $\dim Y = n$. Thus alb_X is generically finite onto Y and hence $0 \geqslant \kappa(X) \geqslant \kappa(Y) \geqslant 0$; see [13, Lemma 10.1]. So $\kappa(X) = \kappa(Y) = 0$. Hence alb_X is surjective and bimeromorphic, with E denoting the exceptional divisor; see [9, Theorem 24]. If alb_X is not an isomorphism, i.e., $E \neq \emptyset$, then $g(E) = E$ and $h(\mathrm{alb}_X(E)) = \mathrm{alb}_X(E)$ because g and h are compatible. By Lemma 2.11 and since g is strongly primitive, $\dim \mathrm{alb}_X(E) = 0$. So Theorem 3.2(3) holds by Lemma 2.11.

If $q(X) = 0 = \kappa(X)$, then X is weak Calabi–Yau by the definition. So we have only to consider the case where $q(X) = 0$ and $\kappa(X) = -\infty$, or the following case by the assumption.

Case (2). X is projective and uniruled. We will show that X is rationally connected. After g-equivariant blowups, we may assume that the maximal rationally connected fibration $\pi : X \to Y$ is holomorphic and g-equivariant, with Y smooth and $\dim Y < n$ (cf. [12, Theorem C]). Since g is strongly primitive, we have $\dim Y = 0$, so X is rationally connected. Theorem 3.2 is proved.

3.5. Proof of Theorem 1.1 and Remark 1.2(1)

For Theorem 1.1(1), by Theorem 3.1, we may assume that $q(X) > 0$, so Theorem 3.2(3) occurs. Suppose that X has $r \geqslant \rho := \rho(X)$ of g-periodic prime divisors D_i. Then each $\mathrm{alb}_X(D_i) \subset \mathrm{Alb}(X) =: Y$ is h-periodic, so it is a point, since we are in Theorem 3.2(3). Thus these D_i are irreducible components of the exceptional divisor E of $\mathrm{alb}_X : X \to Y$. We assert that $(**)$: $\mathrm{NS}_{\mathbb{Q}}(X)$ has a basis consisting of the irreducible components of E and the pullback of a basis of $\mathrm{NS}_{\mathbb{Q}}(Y)$. This is clear if alb_X is the blowup along a smooth center. The general case can be reduced to this special case by the weak factorization theorem of bimeromorphic maps due to Abramovich–Karu–Matsuki–Wlodarczyk (or by blowing up the indeterminacy of $Y \cdots\to X$ as suggested by Oguiso). Now the assertion $(**)$ implies that $r = \rho$, $E = \sum_{i=1}^{\rho} D_i$ and $\rho(Y) = 0$ (so $a(X) = 0$ by Theorem 3.2). This proves Theorem 1.1(1) and Remark 1.2(1).

For Theorem 1.1(2), let $D \subset X$ be a prime divisor containing infinitely many g-periodic curves C_i ($i \geqslant 1$). We may assume that $q(X) > 0$ by Theorem 3.1. The assumption $(*)$ of Theorem 3.2 follows from the successful good minimal model program for projective threefolds. So Theorem 3.2(3) occurs, and hence $\mathrm{alb}_X(C_i)$ is a point since it is h-periodic, noting that C_i is g-periodic and g and h are compatible. Thus, these C_i are contained in the exceptional divisor E of alb_X, and we may assume that the Zariski closure $\overline{\bigcup_j C_{m_j}}$ equals E_1 for some irreducible component E_1 of E and some infinite subsequence $\{C_{m_j}\} \subset \{C_i\}$. Thus $E_1 = D$, for $C_{m_j} \subset D$. Since g and h are compatible, we have $g(E) = E$ and hence $g^s(E_1) = E_1$ for some $s > 0$. So $D = E_1$ is g-periodic. This completes the proof of Theorem 1.1.

Acknowledgments

I thank Tien-Cuong Dinh and Nessim Sibony for the informative reference [6] and comments about g-periodic points, and the referee for the better exposition of the paper.

References

[1] E. Amerik, F. Campana, Fibrations méromorphes sur certaines variétés á fibré canonique triviale, Pure Appl. Math. Q. 4 (2) (2008) 509–545, part 1.

[2] T. Bauer, F. Campana, T. Eckl, S. Kebekus, T. Peternell, S. Rams, T. Szemberg, L. Wotzlaw, A reduction map for nef line bundles, in: Complex Geometry, Göttingen, 2000, Springer, 2002, pp. 27–36.

[3] G. Birkhoff, Linear transformations with invariant cones, Amer. Math. Monthly 74 (1967) 274–276.
[4] C. Birkenhake, H. Lange, Complex Abelian Varieties, 2nd ed., Grundlehren Math. Wiss., vol. 302, Springer, 2004.
[5] T.-C. Dinh, N. Sibony, Groupes commutatifs d'automorphismes d'une variété kählerienne compacte, Duke Math. J. 123 (2004) 311–328.
[6] T.-C. Dinh, N. Sibony, Super-potentials for currents on compact Kaehler manifolds and dynamics of automorphisms, arXiv:0804.0860v1.
[7] A. Fujiki, On automorphism groups of compact Kähler manifolds, Invent. Math. 44 (1978) 225–258.
[8] S. Kawaguchi, Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint, Amer. J. Math. 130 (1) (2008) 159–186.
[9] Y. Kawamata, Characterization of abelian varieties, Compos. Math. 43 (1981) 253–276.
[10] J. Kollár, S. Mori, Birational Geometry of Algebraic Varieties, Cambridge Tracts in Math., vol. 134, 1998.
[11] D.I. Lieberman, Compactness of the Chow scheme: Applications to automorphisms and deformations of Kähler manifolds, in: Lecture Notes in Math., vol. 670, Springer, 1978, pp. 140–186.
[12] N. Nakayama, D.-Q. Zhang, Building blocks of étale endomorphisms of complex projective manifolds, Proc. London Math. Soc., in press, also: arXiv:0903.3729.
[13] K. Ueno, Classification Theory of Algebraic Varieties and Compact Complex Spaces, Lecture Notes in Math., vol. 439, Springer, 1975.
[14] D.-Q. Zhang, Automorphism groups and anti-pluricanonical curves, Math. Res. Lett. 15 (2008) 163–183.
[15] D.-Q. Zhang, Dynamics of automorphisms on projective complex manifolds, J. Differential Geom., in press, also: arXiv:0810.4675.
[16] D.-Q. Zhang, A theorem of Tits type for compact Kähler manifolds, Invent. Math. 176 (2009) 449–459.
[17] S.-W. Zhang, Distributions in algebraic dynamics, in: Surv. Differ. Geom., vol. 10, 2006, pp. 381–430.