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Explicit upper bounds for values at $s = 1$ of Dirichlet L-series associated with primitive even characters

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Abstract

Let S be a given finite set of pairwise distinct rational primes. We give an explicit constant κ_S such that for any even primitive Dirichlet character χ of conductor $q_\chi > 1$ we have

$$\left|\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi)\right| \leqslant \frac{1}{2}\left\{\prod_{p\in S}\left(1-\frac{1}{p}\right)\right\}(\log q_\chi + \kappa_S) + o(1),$$

where $o(1)$ is an explicit error term which tends rapidly to zero when q_χ goes to infinity.

MSC: primary 11M20

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1. Introduction

Let S be a given finite set of pairwise distinct rational primes. In [Lou1], by using integral representations of Dirichlet L-functions, we proved that there exists a computable constant κ_S such that for any primitive Dirichlet character χ of

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conductor $q_\chi > 1$ we have

$$\left|\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi)\right| \leqslant \frac{1}{2}\left\{\prod_{p\in S}\left(1-\frac{1}{p}\right)\right\}(\log q_\chi + \kappa_S) + o(1),$$

where $o(1)$ is an explicit error term which tends to zero when q_χ goes to infinity. However, our proof did not yield good values for these constants κ_S (e.g. see Remark 7 below). In [Lou6, Theorem 4], by using bounds on the characters sums $\sum_{a=1}^{n}\sum_{b=1}^{a}\chi(b)$, we gave in the special case of even characters a new proof of this result yielding better values for κ_S. According to this new proof, for primitive even Dirichlet characters we may choose

$$\kappa_S = \kappa_0 + \omega\log 4 - 2\sum_{i=1}^{\omega}\log\left(1-\frac{1}{p_i}\right) + 2\sum_{p\in S}\frac{\log p}{p-1},$$

where $\kappa_0 = 2\gamma - 1 = 0.154431$, where $p_1, \ldots, p_\omega$ are the primes $p_i \in S$ which do not divide q_χ (we may have $\omega = 0$) and where $\gamma = 0.577\cdots$ denotes Euler's constant. Here, by generalizing the method introduced in [Lou5], we improve upon this result (see Theorem 5 below for a more complicated but explicit result which implies the following one, and note also that [Lou6, Theorem 1] is a special case of the present Theorem 1 (for $\omega = 0$)):

Theorem 1. *Let S be a given finite set of pairwise distinct rational primes, and set $\kappa_{\text{even}} := 2 + \gamma - \log(4\pi) = 0.046191\cdots$. Then, for any even primitive Dirichlet character χ of conductor $q_\chi > 1$ we have*

$$\left|\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi)\right| \leqslant \frac{1}{2}\left\{\prod_{p\in S}\left(1-\frac{1}{p}\right)\right\}\left(\log q_\chi + \kappa_{\text{even}} + \omega\log 4 + 2\sum_{p\in S}\frac{\log p}{p-1}\right) + o(1),$$

where $\omega \geqslant 0$ is the number of primes $p \in S$ which do not divide q_χ, and where $o(1)$ is an explicit error term which tends rapidly to zero when q_χ goes to infinity. Moreover, if $S = \emptyset$ or if $S = \{2\}$, then this error term $o(1)$ is always less than or equal to zero, and if none of the prime in S divides q_χ then this error term $o(1)$ is less than or equal to zero for q_χ large enough.

Corollary 2. *Let χ be an even primitive Dirichlet character of conductor $q_\chi > 1$. Set*

$$\begin{cases} \kappa_1 := 2 + \gamma - \log(4\pi) = 0.04619\cdots \\ \kappa_2 := 2 + \gamma - \log(\pi) = 1.43248\cdots \\ \kappa_3 := 2 + \gamma - \log(\pi/4) = 2.81878\cdots. \end{cases}$$

Then,

$$|L(1,\chi)| \leqslant \begin{cases} (\log q_\chi + \kappa_1)/2 & \textit{in all cases}, \\ (\log q_\chi + \kappa_2)/4 & \textit{if } q_\chi \textit{ is even}, \\ (\log q_\chi + \kappa_3)/(2|2-\chi(2)|) & \textit{if } q_\chi \textit{ is odd}. \end{cases} \tag{1}$$

In particular, *if* χ *is quadratic then*

$$L(1,\chi) \leqslant \begin{cases} (\log q_\chi + \kappa_1)/2 & \textit{if } \chi(2) = +1, \\ (\log q_\chi + \kappa_2)/4 & \textit{if } \chi(2) = 0, \\ (\log q_\chi + \kappa_3)/6 & \textit{if } \chi(2) = -1. \end{cases}$$

We refer the reader to [Le,Lou6, Section 5; Lou7,Mos,MP; Ram,SSW, Corollary 2] for various applications of such explicit bounds for $|L(1,\chi)|$. We also refer the reader to [Lou4,Ram] for slight improvements on the first two bounds in (1).

Of course, by using Burgess' results it is possible to obtain better asymptotic bounds for $|L(1,\chi)|$. For example, it follows from [Toy] that for any non-trivial quadratic Dirichlet character χ modulo q a cube-free positive integer we have

$$\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi) \leqslant \frac{1+o(1)}{4}\left\{\prod_{p\in S}\left(1-\frac{1}{p}\right)\right\}\log q$$

(see also [GS,Pin] for even better bounds in the case that $S=\emptyset$). Moreover, using [Gros] it is possible to have explicit values of these error terms $o(1)$. However, we will show in Section 5 below, that even in the simplest case that $S=\emptyset$ and q is prime, the error terms $o(1)$ in such bounds $L(1,\chi) \leqslant (\frac{1}{4}+o(1))\log q$ are not that small for reasonable values of q, and such asymptotic bounds are worse than the bound $|L(1,\chi)| \leqslant (\log q_\chi + \kappa_{\text{even}})/2$ for reasonable values of q_χ (say $q_\chi \leqslant 10^{16}$). Therefore, contrary to our present bounds (1) used for example in [Lou7], such asymptotic bounds are of no practical use when dealing with class number problems for number fields.

Finally, we mention that at the moment we do not have a satisfactory approach to obtain similar results for odd primitive Dirichlet characters (however, see [Lou3] for the case that $S=\emptyset$).

Here again, as in [Lou1,Lou2], our proof of Theorem 1 stems from the use of θ-functions to obtain integral representations of Dirichlet L-series. However, as we will be using non-primitive Dirichlet L-series, these θ-functions do not satisfy simple functional equations. This will make our proof more complicated than the one given in [Lou2]. Our present proof is a generalization of the proof given in [Lou5] of the previous Corollary.

1.1. Notation

From now on, we let S be a given finite set of pairwise distinct rational primes. We let χ be a primitive even Dirichlet character modulo $q_\chi > 1$, we set

$$d_1 := \prod_{p \in S \text{ and } \gcd(p, q_\chi) = 1} p \geqslant 1 \tag{2}$$

(which is square-free and relatively prime with q_χ), we let $\omega := \omega(d_1) \geqslant 0$ denote the number of distinct prime factors of d_1 (i.e. the number of primes $p \in S$ which do not divide q_χ), we set

$$d_2 := \prod_{p \in S \text{ and } p | q_\chi} p \geqslant 1 \tag{3}$$

(which is a square-free divisor of q_χ), we set $d := d_1 d_2 = \prod_{p \in S} \geqslant 1$ (which is a square-free divisor of q_ψ), and we let ψ be the even Dirichlet character modulo $q_\psi = d_1 q_\chi$ induced by χ. Hence,

$$\left\{ \prod_{p \in S} \left(1 - \frac{\chi(p)}{p} \right) \right\} L(1, \chi) = L(1, \psi). \tag{4}$$

Notice that ψ is not primitive for $d_1 > 1$. We let μ and ϕ denote the Möbius and Euler totient functions. Whenever $D \geqslant 1$ is a positive square-free integer, we set $\tilde{\phi}(D) = 1$ if $D = 1$ and $\tilde{\phi}(D) = \prod_{p|D} (p - 2)$ if $D > 1$.

2. First bound for $L(s, \chi)$

We have

$$(q_\psi / \pi)^{s/2} \Gamma(s/2) L(s, \psi) = \int_0^\infty \theta(x, \psi) x^{s/2} \frac{dx}{x} \quad (\Re(s) > 1),$$

where

$$\theta(x, \psi) = \sum_{n \geqslant 1} \psi(n) e^{-\pi n^2 x / q_\psi} = \frac{1}{2} \sum_{n \in \mathbf{Z}} \psi(n) e^{-\pi n^2 x / q_\psi} \quad (x > 0) \tag{5}$$

(for ψ is even). Hence, for any $a > 0$ (to be suitably chosen later on (see (14) below)) it holds that:

$$(q_\psi / \pi)^{s/2} \Gamma(s/2) L(s, \psi) = \int_{1/a}^\infty \theta(1/x, \psi) x^{-s/2} \frac{dx}{x} + \int_a^\infty \theta(x, \psi) x^{s/2} \frac{dx}{x} \quad (\Re(s) > 1). \tag{6}$$

Lemma 3. 1. (*See* [MV, *Lemma* 5.4]). *Set*

$$\tau(\chi) := \sum_{a=1}^{q_\chi} \chi(a) e^{2\pi i a/q_\chi}.$$

Then, for any $b \in \mathbf{Z}$ *it holds that*

$$\tau_b(\psi) := \sum_{a=1}^{q_\psi} \psi(a) e^{2\pi i ab/q_\psi} = \mu(d_1)\chi(d_1)\mu(\delta_b)\phi(\delta_b)\bar{\chi}(b)\tau_1(\chi),$$

where $\delta_b := \gcd(b, d_1)$. *In particular, it holds that*

$$|\tau_b(\psi)/\sqrt{q_\chi}| = \begin{cases} 0 & \text{if } \gcd(b, q_\chi) > 1, \\ \phi(\delta_b) & \text{if } \gcd(b, q_\chi) = 1. \end{cases} \tag{7}$$

2. *Set*

$$\theta(x) = \sum_{n \geqslant 1} e^{-\pi n^2 x} \quad (x > 0).$$

Then,

$$|\theta(x, \psi)| \leqslant \sum_{\delta | d} \mu(\delta)\theta(\delta^2 x/q_\psi) \quad (x > 0) \tag{8}$$

and

$$|\theta(1/x, \psi)| \leqslant \sqrt{x/d_1} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2)\theta(\delta_1^2 \delta_2^2 x/q_\psi) \quad (x > 0). \tag{9}$$

Proof. Since $\psi(n) = 0$ for $\gcd(n, d) > 1$, we have

$$\begin{aligned} |\theta(x, \psi)| &\leqslant \sum_{\substack{n \geqslant 1 \\ \gcd(n,d)=1}} \exp(-\pi n^2 x/q_\psi) \\ &= \sum_{n \geqslant 1} \sum_{\delta | n \text{ and } \delta | d} \mu(\delta) \exp(-\pi n^2 x/q_\psi) \\ &= \sum_{\delta | d} \mu(\delta) \sum_{m \geqslant 1} \exp(-\pi (m\delta)^2 x/q_\psi), \end{aligned}$$

and (8) follows. Now, set

$$\theta(x, a, q) = \sum_{b \in \mathbf{Z}} e^{-\pi(a + bq)^2 x/q} \quad (x > 0 \text{ and } q > 0).$$

The Poisson summation formula yields

$$\theta(x,a,q)=\frac{1}{\sqrt{qx}}\sum_{b\in\mathbf{Z}}e^{2\pi iab/q}e^{-\pi b^2/qx}\quad(x>0).$$

Using (5), we obtain:

$$\begin{aligned}\theta(x,\psi)&=\frac{1}{2}\sum_{a=1}^{q_\psi}\sum_{b\in\mathbf{Z}}\psi(a+bq_\psi)e^{-\pi(a+bq_\psi)^2x/q_\psi}\\&=\frac{1}{2}\sum_{a=1}^{q_\psi}\psi(a)\theta(x,a,q_\psi)=\frac{1}{2\sqrt{q_\psi x}}\sum_{b\in\mathbf{Z}}\tau_b(\psi)e^{-\pi b^2/q_\psi x}\end{aligned}$$

and

$$\theta(1/x,\psi)=\sqrt{x/d_1}\sum_{b\geqslant 1}\frac{\tau_b(\psi)}{\sqrt{q_\chi}}e^{-\pi b^2x/q_\psi}$$

(for ψ is even). To get (9), we notice that

$$\sum_{\delta_1|d_1}\tilde{\phi}(\delta_1)\sum_{\delta_2|d_2}\mu(\delta_2)\theta(\delta_1^2\delta_2^2x/q_\psi)=\sum_{b\geqslant 1}a_be^{-\pi b^2x/q_\psi}$$

with

$$\begin{aligned}a_b&=\sum\sum_{\substack{\delta_1|d_1,\delta_2|d_2\\ \text{and }\ \delta_1\delta_2|b}}\tilde{\phi}(\delta_1)\mu(\delta_2)\\&=\left(\sum_{\delta_1|\gcd(b,d_1)}\tilde{\phi}(\delta_1)\right)\left(\sum_{\delta_2|\gcd(b,d_2)}\mu(\delta_2)\right)=\begin{cases}0 & \text{if }\ \gcd(b,d_2)>1,\\ \phi(\gcd(b,d_1)) & \text{if }\ \gcd(b,d_2)=1\end{cases}\end{aligned}$$

(for $\gcd(d_1,d_2)=1$) and use (7) to obtain $0\leqslant|\tau_b(\psi)/\sqrt{q_\chi}|\leqslant a_b$ (for d_2 divides q_χ). □

According to (8) and (9) of Lemma 3, the integral representation (6) is valid for all s in the complex plane. For $s=1$, using (4) we obtain:

$$\left|\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi)\right|\leqslant I_1+I_2,$$

where

$$I_1:=\frac{1}{\sqrt{q_\psi}}\int_{1/a}^{\infty}|\theta(1/x,\psi)|\,\frac{dx}{x\sqrt{x}}$$

and

$$I_2 := \frac{1}{\sqrt{q_\psi}} \int_a^\infty |\theta(x,\psi)| \frac{dx}{\sqrt{x}}$$

and $a > 0$ is to be suitably chosen in due course (see (14) below).

3. Proof of Theorem 1

To compute bounds on I_1 and I_2 we now use (8) and (9) of Lemma 3 and the following Lemma 4 which will enable us to get simple formulae:

Lemma 4. *Let $D \geqslant 1$ be a positive square-free rational integer. Let $t \geqslant 0$ denote its number of prime divisors. It holds that*

$$\sum_{\delta|D} \mu(\delta) = \varepsilon_D := \begin{cases} 1 & \textit{if } D = 1, \\ 0 & \textit{if } D > 1, \end{cases}$$

$$\sum_{\delta|D} \mu(\delta)\log\delta = \eta_D := \begin{cases} -\log p & \textit{if } D = p \textit{ is prime}, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\sum_{\delta|D} \frac{\mu(\delta)}{\delta} = \prod_{p|D}\left(1 - \frac{1}{p}\right) = \frac{\phi(D)}{D},$$

$$\sum_{\delta|D} \frac{\mu(\delta)}{\delta}\log\delta = -\frac{\phi(D)}{D}\sum_{p|D}\frac{\log p}{p-1},$$

$$\sum_{\delta|D} \frac{\tilde{\phi}(\delta)}{\delta} = 2^t \prod_{p|D}\left(1 - \frac{1}{p}\right) = 2^t\frac{\phi(D)}{D},$$

$$\sum_{\delta|D} \tilde{\phi}(\delta)\log\delta = \phi(D)\left(\log D - \sum_{p|D}\frac{\log p}{p-1}\right).$$

Proof. Use induction on the number t of distinct prime factors of D. □

1. Let us first give an upper bound on I_2. Using (8) and the functional equation

$$\theta(1/x) = \sqrt{x}\theta(x) + (\sqrt{x} - 1)/2 \quad (x > 0) \tag{10}$$

(see [Lan, Chapter XIII]) and setting

$$\kappa_{\mathrm{even}} := 2\int_1^\infty \theta(x)\frac{dx}{\sqrt{x}} + 2\int_1^\infty \theta(x)\frac{dx}{x} = 2+\gamma-\log(4\pi) \tag{11}$$

(see [Lou2, (3) and Lemme p. 12]), we obtain:

$$\begin{aligned}
I_2 &= \frac{1}{\sqrt{q_\psi}}\int_a^\infty |\theta(x,\psi)|\frac{dx}{\sqrt{x}}\\
&\leqslant \sum_{\delta|d}\mu(\delta)\frac{1}{\sqrt{q_\psi}}\int_a^\infty \theta(\delta^2x/q_\psi)\frac{dx}{\sqrt{x}}\\
&= \sum_{\delta|d}\frac{\mu(\delta)}{\delta}\int_{a\delta^2/q_\psi}^\infty \theta(x)\frac{dx}{\sqrt{x}}\\
&= \sum_{\delta|d}\frac{\mu(\delta)}{\delta}\left(\int_1^\infty \theta(x)\frac{dx}{\sqrt{x}} + \int_1^{q_\psi/a\delta^2}\theta(1/x)\frac{dx}{x\sqrt{x}}\right)\\
&= \sum_{\delta|d}\frac{\mu(\delta)}{\delta}\left(\frac{\kappa_{\mathrm{even}}}{2} - \int_{q_\psi/a\delta^2}^\infty \theta(x)\frac{dx}{x} + \int_1^{q_\psi/a\delta^2}\frac{\sqrt{x}-1}{2x\sqrt{x}}dx\right)\\
&= \sum_{\delta|d}\frac{\mu(\delta)}{\delta}\left(\frac{\kappa_{\mathrm{even}}}{2} - \int_{d_1q_\chi/a\delta^2}^\infty \theta(x)\frac{dx}{x} + \frac{1}{2}\log\left(\frac{d_1q_\chi}{a\delta^2}\right) - 1 + \sqrt{\frac{a\delta^2}{d_1q_\chi}}\right).
\end{aligned}$$

Hence, using Lemma 4, we obtain

$$I_2 \leqslant \frac{\phi(d)}{2d}\left(\log(d_1q_\chi/a) + \kappa_{\mathrm{even}} - 2 + 2\sum_{p|d}\frac{\log p}{p-1}\right) + R_2, \tag{12}$$

where

$$R_2 = -\sum_{\delta|d}\frac{\mu(\delta)}{\delta}\int_{d_1q_\chi/a\delta^2}^\infty \theta(x)\frac{dx}{x} + \varepsilon_d\sqrt{\frac{a}{d_1q_\chi}}$$

is an explicit error term which tends to zero when q_χ goes to infinity.

2. In the same way, using (9) and the functional equation (10), we obtain:

$$I_{LL1} = \frac{1}{\sqrt{q_\psi}}\int_{1/a}^\infty |\theta(1/x,\psi)|\frac{dx}{x\sqrt{x}}$$

$$\leqslant \frac{1}{\sqrt{d_1 q_\psi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \int_{1/a}^{\infty} \theta(\delta_1^2 \delta_2^2 x / q_\psi) \frac{dx}{x}$$
$$= \frac{1}{\sqrt{d_1 q_\psi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \int_{\delta_1^2 \delta_2^2 / a q_\psi}^{\infty} \theta(x) \frac{dx}{x}$$
$$= \frac{1}{\sqrt{d_1 q_\psi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \left(\int_1^{\infty} \theta(x) \frac{dx}{x} + \int_1^{a q_\psi / \delta_1^2 \delta_2^2} \theta(1/x) \frac{dx}{x} \right)$$
$$= \frac{1}{\sqrt{d_1 q_\psi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \left(\frac{\kappa_{\mathrm{even}}}{2} - \int_{a q_\psi / \delta_1^2 \delta_2^2}^{\infty} \theta(x) \frac{dx}{\sqrt{x}} + \int_1^{a q_\psi / \delta_1^2 \delta_2^2} \frac{\sqrt{x} - 1}{2x} dx \right)$$
$$= \frac{1}{d_1 \sqrt{q_\chi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \left(\frac{\kappa_{\mathrm{even}}}{2} - \int_{a d_1 q_\chi / \delta_1^2 \delta_2^2}^{\infty} \theta(x) \frac{dx}{\sqrt{x}} + \sqrt{\frac{a d_1 q_\chi}{\delta_1^2 \delta_2^2}} - 1 - \frac{1}{2} \log \left(\frac{a d_1 q_\chi}{\delta_1^2 \delta_2^2} \right) \right).$$

Using Lemma 4, we obtain

$$I_1 \leqslant 2^{\omega} \frac{\phi(d)}{d} \sqrt{\frac{a}{d_1}} + R_1 \tag{13}$$

where

$$R_1 = \frac{1}{d_1 \sqrt{q_\chi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \left(\frac{\kappa_{\mathrm{even}}}{2} - \int_{a d_1 q_\chi / \delta_1^2 \delta_2^2}^{\infty} \theta(x) \frac{dx}{\sqrt{x}} - 1 - \frac{1}{2} \log \left(\frac{a d_1 q_\chi}{\delta_1^2 \delta_2^2} \right) \right)$$
$$= -\frac{1}{d_1 \sqrt{q_\chi}} \sum_{\delta_1 | d_1} \tilde{\phi}(\delta_1) \sum_{\delta_2 | d_2} \mu(\delta_2) \int_{a d_1 q_\chi / \delta_1^2 \delta_2^2}^{\infty} \theta(x) \frac{dx}{\sqrt{x}}$$
$$+ \frac{\varepsilon_{d_2}}{2 \sqrt{q_\chi}} \frac{\phi(d_1)}{d_1} \left(\kappa_{\mathrm{even}} - 2 - \log (a d_1 q_\chi) + 2 \log d_1 - 2 \sum_{p | d_1} \frac{\log p}{p - 1} \right) + \frac{\eta_{d_2}}{d_1 \sqrt{q_\chi}} \phi(d_1)$$

is an explicit error term which tends to zero when q_χ goes to infinity.

3. Hence, using (12) and (13), we finally obtain

$$\left| \left\{ \prod_{p \in S} \left(1 - \frac{\chi(p)}{p} \right) \right\} L(1, \chi) \right|$$
$$\leqslant \frac{1}{2} \left\{ \prod_{p \in S} \left(1 - \frac{1}{p} \right) \right\} \left(\log \left(\frac{d_1 q_\chi}{a} \right) + 2^{\omega + 1} \sqrt{\frac{a}{d_1}} + \kappa_{\mathrm{even}} - 2 + 2 \sum_{p \in S} \frac{\log p}{p - 1} \right) + R,$$

where $R = R_1 + R_2$ is an explicit error term which tends to zero when q_χ goes to infinity. To get the term $-\log a + 2^{\omega+1} \sqrt{a/d_1}$ as small as possible, we now choose

$$a = d_1 / 4^{\omega} \tag{14}$$

and obtain the following explicit result which proves Theorem 1:

Theorem 5. *It holds that*

$$\left|\left\{\prod_{p\in S}\left(1-\frac{\chi(p)}{p}\right)\right\}L(1,\chi)\right|\leqslant\frac{1}{2}\left\{\prod_{p\in S}\left(1-\frac{1}{p}\right)\right\}\left(\log(4^{\omega}q_{\chi})+\kappa_{\text{even}}+2\sum_{p\in S}\frac{\log p}{p-1}\right)+R,$$

where R is the following explicit error term which tends to zero when q_{χ} goes to infinity:

$$R=-\sum_{\delta\mid d}\frac{\mu(\delta)}{\delta}\int_1^{\infty}\theta(4^{\omega}q_{\chi}x/\delta^2)\frac{dx}{x}-\frac{1}{2^{\omega}}\sum_{\delta_1\mid d_1}\frac{\tilde{\phi}(\delta_1)}{\delta_1}\sum_{\delta_2\mid d_2}\frac{\mu(\delta_2)}{\delta_2}\int_1^{\infty}\theta(d_1^2q_{\chi}x/4^{\omega}\delta_1^2\delta_2^2)\frac{dx}{\sqrt{x}}$$
$$+\frac{\varepsilon_d}{\sqrt{4^{\omega}q_{\chi}}}+\frac{\varepsilon_{d_2}}{2\sqrt{q_{\chi}}}\frac{\phi(d_1)}{d_1}\left(\kappa_{\text{even}}-2-\log(q_{\chi}/4^{\omega})-2\sum_{p\mid d_1}\frac{\log p}{p-1}\right)+\frac{\eta_{d_2}}{\sqrt{q_{\chi}}}\frac{\phi(d_1)}{d_1}.$$

If $d_2=1$ or if d_2 is prime, then $R\leqslant 0$ for q_{χ} large enough. If $d_2>1$ is not prime, then

$$R=-\sum_{\delta\mid d}\frac{\mu(\delta)}{\delta}\int_1^{\infty}\theta(4^{\omega}q_{\chi}x/\delta^2)\frac{dx}{x}-\frac{1}{2^{\omega}}\sum_{\delta_1\mid d_1}\frac{\tilde{\phi}(\delta_1)}{\delta_1}\sum_{\delta_2\mid d_2}\frac{\mu(\delta_2)}{\delta_2}\int_1^{\infty}\theta(d_1^2q_{\chi}x/4^{\omega}\delta_1^2\delta_2^2)\frac{dx}{\sqrt{x}}$$

tends rapidly to zero when q_{χ} goes to infinity, for $R=O_d(\frac{1}{q_{\chi}}e^{-c\pi q_{\chi}})$ with $c=\min(4^{\omega}/d^2,1/4^{\omega}d_2^2)$.

Proof. Use

$$\int_1^{\infty}\theta(Ax)\tfrac{dx}{x}\leqslant\int_1^{\infty}\theta(Ax)\tfrac{dx}{\sqrt{x}}\leqslant\int_1^{\infty}\theta(Ax)dx=\sum_{n\geqslant 1}\frac{1}{\pi n^2A}e^{-\pi n^2A}\leqslant\frac{\pi}{6A}e^{-\pi A}.\qquad\square$$

4. Some explicit special cases

1. Assume that $d_1=1$ and $d=d_2\geqslant 1$. Then $\omega=0$ and

$$R=-\sum_{\delta\mid d}\frac{\mu(\delta)}{\delta}\int_1^{\infty}(1+\sqrt{x})\theta(q_{\chi}x/\delta^2)\frac{dx}{x}+\frac{\varepsilon_d}{2\sqrt{q_{\chi}}}(\kappa_{\text{even}}-\log q_{\chi})+\frac{\eta_d}{\sqrt{q_{\chi}}}.$$

In particular, for $d=1$ we have $\varepsilon_d=1$, $\eta_d=0$ and $q_{\chi}\geqslant 5>e^{\kappa_{\text{even}}}$ yields $R\leqslant 0$, which proves the first bound in (1) of Corollary 2.

In the same way, if $d=2$ we have $\varepsilon_d=0$, $\eta_d=-\log 2$ and $q_{\chi}\geqslant 5>e^{\kappa_{\text{even}}}$ yields

$$R\leqslant\frac{1}{2}\int_1^{\infty}(1+\sqrt{x})\theta(5x/4)\frac{dx}{x}-\frac{\log 2}{\sqrt{5}}\leqslant\int_1^{\infty}\theta(x)\,dx-\frac{\log 2}{\sqrt{5}}$$
$$=-\frac{\log 2}{\sqrt{5}}+\sum_{n\geqslant 1}\frac{e^{-\pi n^2}}{\pi n^2}\leqslant 0,$$

which proves the second bound in (1) of Corollary 2.

2. Assume that $d = d_1 \geqslant 1$ and $d_2 = 1$. Then

$$R = -\sum_{\delta|d} \frac{\mu(\delta)}{\delta} \int_1^\infty \theta(4^\omega q_\chi x/\delta^2)\,\frac{dx}{x} - 2^\omega \sum_{\delta|d} \frac{\tilde{\phi}(\delta)}{\delta} \int_1^\infty \theta(d^2 q_\chi x/4^\omega \delta^2)\,\frac{dx}{\sqrt{x}} + \frac{\varepsilon_d}{\sqrt{4^\omega q_\chi}} + \frac{\phi(d)}{2d\sqrt{q_\chi}}\left(\kappa_{\text{even}} - 2 - \log(q_\chi/4^\omega) - 2\sum_{p|d} \frac{\log p}{p-1}\right).$$

In particular, for $d = 2$ we have $\varepsilon_d = 0$ and $\omega = 1$, and $q_\chi \geqslant 5$ yields

$$\begin{aligned} R &\leqslant \frac{1}{2}\int_1^\infty \theta(q_\chi x)\,\frac{dx}{x} + \frac{1}{4\sqrt{q_\chi}}(\kappa_{\text{even}} - 2 - \log q_\chi) \\ &\leqslant \frac{1}{2}\int_1^\infty \theta(q_\chi x) + \frac{1}{4\sqrt{q_\chi}}(\kappa_{\text{even}} - 2 - \log q_\chi) \\ &= \frac{1}{4\sqrt{q_\chi}}(\kappa_{\text{even}} - 2 - \log q_\chi) + \frac{1}{2q_\chi}\sum_{n\geqslant 1} \frac{e^{-\pi q_\chi n^2}}{\pi n^2} \leqslant 0, \end{aligned}$$

which proves the third bound in (1) of Corollary 2.

3. Assume that $d_1 = 3$ and $d_2 = 2$. Then $\omega = 1$, $d = 6$, $\varepsilon_d = 0$, $\varepsilon_{d_2} = 1$ and $\eta_{d_2} = -\log 2$, and $q_\chi \geqslant 5$ yields

$$\begin{aligned} R &\leqslant \frac{1}{2}\int_1^\infty \theta(q_\chi x)\,\frac{dx}{x} + \frac{1}{3}\int_1^\infty \theta(4q_\chi x/9)\,\frac{dx}{x} + \frac{1}{3}\int_1^\infty \theta(q_\chi x/16)\,\frac{dx}{\sqrt{x}} \\ &\quad + \frac{1}{3\sqrt{q_\chi}}(\kappa_{\text{even}} - 2 - \log q_\chi - \log 3) \\ &\leqslant \frac{7}{6}\int_1^\infty \theta(q_\chi x/16)dx + \frac{1}{3\sqrt{q_\chi}}(\kappa_{\text{even}} - 2 - \log q_\chi - \log 3) \\ &= \frac{1}{3\sqrt{q_\chi}}(\kappa_{\text{even}} - 2 - \log q_\chi - \log 3) + \frac{56}{3q_\chi}\sum_{n\geqslant 1} \frac{e^{-\pi q_\chi n^2/16}}{\pi n^2} \leqslant 0. \end{aligned}$$

Hence, we obtain:

Corollary 6. *Set* $\kappa := 2 + \gamma - \log(\pi/12) = 3.91739\cdots$. *Let* χ *be an even primitive Dirichlet character of conductor* $q_\chi > 1$, *and assume that* $\chi(2) = 0$ *but* $\chi(3) = -1$. *Then,* $|L(1,\chi)| \leqslant (\log q_\chi + \kappa)/8$.

Remark 7. This result should be compared with the previously known two bounds on $|L(1,\chi)|$ for even primitive Dirichlet characters of conductors $q_\chi > 1$ and such that $\chi(2) = 0$ but $\chi(3) = -1$: the first one quoted in [Le, Lemma 3] (whose proof stems from the use of [Lou1] (see the proof of [Lou6, (1)])) according to which $|L(1,\chi)| \leqslant (\log q_\chi + \kappa')/8$ where $\kappa' = \log(216) + 8 = 13.37527\cdots$, and the second one given in [Lou6, (11)] according to which $|L(1,\chi)| \leqslant (\log q_\chi + \kappa'')/8$ where $\kappa'' = 6$.

5. Better asymptotic bounds

Throughout this section, we let χ denote a primitive Dirichlet character modulo $p\geqslant 8731$ a prime. We set $S_\chi(n)=\sum_{k=1}^{n}\chi(k)$. Then, for any arbitrary positive integer r it holds that

$$|S_\chi(n)|\leqslant\begin{cases} n & \text{(trivial bound)},\\ 35n^{1-1/(r+1)}p^{1/4r}(\log p)^{1/(2r+2)} & \text{(Burgess' bound (see[Gros, Theorem 1, p. 1177]))}.\end{cases}$$

Hence, for $X\geqslant 1$ real, it holds that

$$|L(1,\chi)|=\left|\sum_{n\geqslant 1}\frac{S_\chi(n)}{n(n+1)}\right|\leqslant\left(\sum_{1\leqslant n\leqslant X+1}\frac{1}{n+1}\right)+\left(\sum_{n>X+1}\frac{|S_\chi(n)|}{n(n+1)}\right).$$

Since

$$\sum_{1\leqslant n\leqslant X+1}\frac{1}{n+1}\leqslant -1+\gamma+\log X+\frac{2}{X}$$

(where $\gamma=0.577215\cdots$ denotes Euler constant), and since

$$\sum_{n>X+1}\frac{|S_\chi(n)|}{n^2}\leqslant\int_X^\infty 35n^{-1-1/(r+1)}p^{1/4r}(\log p)^{1/(2r+2)}dn=35(r+1)\frac{p^{1/4r}(\log p)^{1/(2r+2)}}{X^{1/(r+1)}},$$

we obtain

$$|L(1,\chi)|\leqslant -1+\gamma+\log X+\frac{2}{X}+35(r+1)\frac{p^{1/4r}(\log p)^{1/(2r+2)}}{X^{1/(r+1)}}. \tag{15}$$

Since the Burgess bound is better than the trivial bound for

$$n\geqslant X_{p,r}\coloneqq(35)^{r+1}p^{(r+1)/4r}\sqrt{\log p}\geqslant 1225\cdot p^{1/4}\sqrt{\log p},$$

we choose $X=X_{p,r}$ for which $35(r+1)p^{1/4r}(\log p)^{1/(2r+2)}X^{-1/(r+1)}=r+1$. Using (15) with these values of X, we obtain

$$|L(1,\chi)|\leqslant\log X_{p,r}+r+\gamma+\frac{2}{X_{p,r}}=\left(\frac{1}{4}+\frac{1}{4r}+o_r(1)\right)\log p.$$

It remains to choose the integer $r \geqslant 1$ to obtain the best possible bound. Set $c_1 = \log(35) + 1$, $c_2 = \log(35) + \gamma$ and

$$h(r) := \log X_{p,r} + r + \gamma = \left(\frac{1}{4} + \frac{1}{4r}\right)\log p + \frac{1}{2}\log\log p + c_1 r + c_2.$$

Then $h(r) \leqslant h(r_p) = \frac{1}{4}\log p + \sqrt{c_1 \log p} + \frac{1}{2}\log\log p + c_2$ where $r = r_p := \sqrt{\frac{\log p}{4c_1}}$, for which $2/X_{p,r_p} = o(1)$. Choosing r equal to the greatest integer less than or equal to r_p, we finally obtain:

Theorem 8. *Set* $c_1 = \log(35) + 1 = 4.555348\cdots$ *and* $c_2 = \log(35) + \gamma = 4.132563\cdots$. *If* χ *is a primitive Dirichlet character of prime conductor* $p \geqslant 8731$, *then*

$$|L(1,\chi)| \leqslant \frac{1}{4}\log p + \sqrt{c_1 \log p} + \frac{1}{2}\log\log p + c_2 + o(1) \tag{16}$$

where the error term $o(1)$ *is explicit and tends to zero as p goes to infinity.*

Now, since in the range $9731 \leqslant p \leqslant 5 \times 10^{51}$ the main term $\frac{1}{4}\log p + \sqrt{c_1 \log p} + \frac{1}{2}\log\log p + c_2$ of this bound (16) is greater than the worst of our explicit bounds $(\log p + \kappa_{\mathrm{even}})/2$ on $|L(1,\chi)|$, this asymptotic bound (16) is useless when it comes to practical application of bounds for $L(1,\chi)$ (as in [Le,MP,Mos,SSW]).

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