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Chromaticity of a family of K_4-homeomorphs ☆

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Abstract

We discuss the chromaticity of one family of K_4-homeomorphs which has exactly 2 adjacent paths of length 1, and give sufficient and necessary condition for the graphs in the family to be chromatically unique.

Keywords: Chromatic polynomial; K_4-homeomorph; Chromatic uniqueness

1. Introduction

In this paper, we consider graphs which are simple. For such a graph G, let $P(G;\lambda)$ denote the chromatic polynomial of G. Two graphs G and H are chromatically equivalent, denoted by $G\sim H$, if $P(G;\lambda)=P(H;\lambda)$. A graph G is chromatically unique if for any graph H such that $H\sim G$, we have $H\cong G$, i.e., H is isomorphic to G.

A K_4-homeomorph is a subdivision of the complete graph K_4. Such a homeomorph is denoted by $K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ if the six edges of K_4 are replaced by the six paths of length $\alpha,\beta,\gamma,\delta,\varepsilon,\eta$, respectively, as shown in Fig. 1. Each of these six paths is called a $*$-path.

So far, the study of the chromaticity of K_4-homeomorphs with at least 3 $*$-paths of length 1 has been fulfiled (see [2,6,3]). In this paper, we study the chromaticity of K_4-homeomorphs $K_4(\alpha,1,1,\delta,\varepsilon,\eta)$ (as Fig. 2(a)) with 2 $*$-path of length 1 are adjacent.

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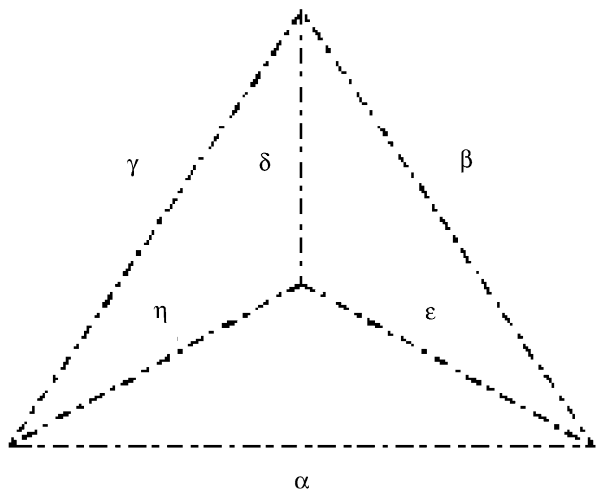

Fig. 1.

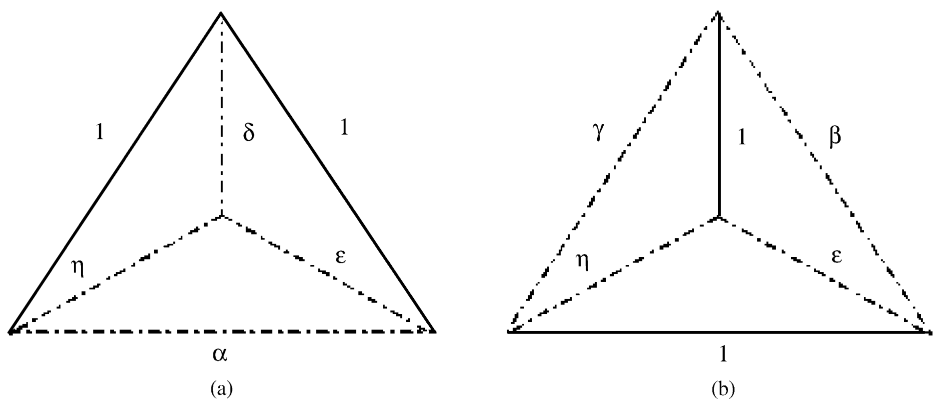

Fig. 2.

2. Auxiliary results

In this section, we cite some known results used in the sequel.

Proposition 1. *Let* $G \sim H$. *Then*

(1) $|V(G)|=|V(H)|$, $|E(G)|=|E(H)|$ (*see* [3]);
(2) *If G is a* K_4*-homeomorph, then H is a* K_4*-homeomorph as well* (*see* [1]);
(3) *If G and H are homeomorphic to* K_4, *then both the minimum values of parameters and the number of parameters equal to this minimum value of the graphs G and H coincide* (*see* [5]).

Proposition 2 (Li [4]). *Suppose that* $G=K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ *and* $H=K_4(\alpha',\beta',\gamma',\delta',\varepsilon',\eta')$ *are chromatically equivalent homeomorphs such that two multisets* $(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ *and* $(\alpha',\beta',\gamma',\delta',\varepsilon',\eta')$ *are the same, then H is isomorphic to G.*

Proposition 3 (Guo and Whitehead Jr. [2] and Xu [6]). $K_4(1,\beta,\gamma,1,\varepsilon,\eta)$ (*see Fig.* 2(b)) *is not chromatically unique if and only if it is* $K_4(1,b+2,b,1,2,2)$ *or* $K_4(1,a+1,a+3,1,2,a)$ *or* $K_4(1,a+2,b,1,2,a)$, *where* $a\geqslant 2$, $b\geqslant 1$, *and*

$$K_4(1,b+2,b,1,2,2)\sim K_4(3,1,1,2,b,b+1),$$

$$K_4(1,a+1,a+3,1,2,a)\sim K_4(a+1,1,1,a,3,a+2),$$

$$K_4(1,a+2,b,1,2,a)\sim K_4(a+1,1,1,b,3,a).$$

3. Main results

Lemma. *If* $G\cong K_4(\alpha,1,1,\delta,\varepsilon,\eta)$ *and* $H\cong K_4(\alpha',1,1,\delta',\varepsilon',\eta')$, *then we have*

(1) $P(G)=(-1)^{n+1}[r/(r-1)^2][-r^{n+1}-r^2+r+2+Q(G)]$, *where*

$$Q(G)=-r^{\alpha}-r^{\delta}-r^{\varepsilon}-r^{\eta}-r^{\alpha+1}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta}$$

$r=1-\lambda$, *n is the number of vertices of G.*

(2) *If* $P(G)=P(H)$, *then* $Q(G)=Q(H)$.

Proof. (1) Let $r=1-\lambda$. From [5], we have the chromatic polynomial of K_4-homeomorph $K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta)$ as follows:

$$\begin{aligned}P(K_4(\alpha,\beta,\gamma,\delta,\varepsilon,\eta))=(-1)^{n+1}[r/(r-1)^2][&(r^2+3r+2)\\&-(r+1)(r^{\alpha}+r^{\beta}+r^{\gamma}+r^{\delta}+r^{\varepsilon}+r^{\eta})\\&+(r^{\alpha+\delta}+r^{\beta+\eta}+r^{\gamma+\varepsilon}+r^{\alpha+\beta+\varepsilon}\\&+r^{\beta+\delta+\gamma}+r^{\alpha+\gamma+\eta}+r^{\delta+\varepsilon+\eta}-r^{n+1})].\end{aligned}$$

Then

$$\begin{aligned}P(G)&=P(K_4(\alpha,1,1,\delta,\varepsilon,\eta))\\&=(-1)^{n+1}[r/(r-1)^2][(r^2+3r+2)-(r+1)(r^{\alpha}+r^{\delta}+r^{\varepsilon}+r^{\eta}+2r)\\&\quad+(r^{\varepsilon+1}+r^{\eta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta}-r^{n+1})]\end{aligned}$$

$$= (-1)^{n+1}[r/(r-1)^2](-r^{n+1} - r^2 + r + 2 - r^{\alpha} - r^{\delta} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1}$$
$$- r^{\delta+1} + r^{\delta+2} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta})$$
$$= (-1)^{n+1}[r/(r-1)^2](-r^{n+1} - r^2 + r + 2 + Q(G))$$

where

$$Q(G) = -r^{\alpha} - r^{\delta} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} - r^{\delta+1} + r^{\delta+2} + r^{\alpha+\delta}$$
$$+ r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta}. \qquad \Box$$

Proof. (2) If $P(G)=P(H)$, then it is easy to see that $Q(G)=Q(H)$. $\Box$

Theorem. *K_4-homeomorphs $K_4(\alpha,1,1,\delta,\varepsilon,\eta)$ $(\min\{\alpha,\delta,\varepsilon,\eta\}\geqslant 2)$ is not chromatically unique if and only if it is $K_4(a,1,1,a+b+1,b,b+1)$, $K_4(a,1,1,b,b+2,a+b)$, $K_4(a+1,1,1,a+3,2,a)$, $K_4(a+2,1,1,a,2,a+2)$, $K_4(3,1,1,2,b,b+1)$, $K_4(a+1,1,1,a,3,a+2)$ or $K_4(a+1,1,1,b,3,a)$, where $a\geqslant 2$, $b\geqslant 2$.*

Proof. Let $G\cong K_4(\alpha,1,1,\delta,\varepsilon,\eta)$ and $\min\{\alpha,\delta,\varepsilon,\eta\}\geqslant 2$ (see Fig. 2(a)). If there is a graph H such that $P(H)=P(G)$, then from Proposition 1, we know that H is a K_4-homeomorph $K_4(\alpha',\beta',\gamma',\delta',\varepsilon',\eta')$ and two of $\alpha',\beta',\gamma',\delta',\varepsilon',\eta'$ must be 1. We can assume that $\alpha'=\delta'=1$ or $\beta'=\gamma'=1$. We now solve the equation $P(G)=P(H)$ to get all solutions.

Case A: If $\alpha'=\delta'=1$, then $H\cong K_4(1,\beta',\gamma',1,\varepsilon',\eta')$. From Proposition 3, we know the solutions of the equation $P(G)=P(H)$ are

$$K_4(3,1,1,2,b,b+1)\sim K_4(1,b+2,b,1,2,2),$$
$$K_4(a+1,1,1,a,3,a+2)\sim K_4(1,a+1,a+3,1,2,a),$$
$$K_4(a+1,1,1,b,3,a)\sim K_4(1,a+2,b,1,2,a).$$

Case B: If $\beta'=\gamma'=1$, then $H\cong K_4(\alpha',1,1,\delta',\varepsilon',\eta')$. We solve the equation $Q(G)=Q(H)$. From lemma, we have

$$Q(G) = -r^{\alpha} - r^{\delta} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} - r^{\delta+1} + r^{\delta+2} + r^{\alpha+\delta}$$
$$+ r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$
$$Q(H) = -r^{\alpha'} - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\delta'}$$
$$+ r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

We know that $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$ (from Proposition 1) and we can assume $\varepsilon\leqslant\eta$, $\varepsilon'\leqslant\eta'$, $\min\{\alpha',\delta',\varepsilon',\eta'\}\geqslant 2$. We denote the lowest remaining power by l.r.p. and the highest remaining power by h.r.p.

Case 1: If $\min\{\alpha,\delta,\varepsilon,\eta\}=\alpha$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\alpha'$, then the lowest power in $Q(G)$ is α and the lowest power in $Q(H)$ is α'. Therefore $\alpha=\alpha'$. We obtain the following after simplification:

$$Q(G):\ -r^{\delta}-r^{\varepsilon}-r^{\eta}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\delta'}-r^{\varepsilon'}-r^{\eta'}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}.$$

By considering the h.r.p. in $Q(G)$ and the h.r.p. in $Q(H)$, we have $\alpha+\eta+1=\alpha'+\eta'+1$ or $\alpha+\delta=\alpha'+\delta'$ or $\alpha+\delta=\alpha'+\eta'+1$ or $\alpha+\eta+1=\alpha'+\delta'$.

Case 1.1: If $\alpha+\eta+1=\alpha'+\eta'+1$, then $\eta=\eta'$. After canceling $-r^{\eta}$ in $Q(G)$ with $-r^{\eta'}$ in $Q(H)$, we have the l.r.p. in $Q(G)$ is δ or ε and the l.r.p. in $Q(H)$ is δ' or ε'. Therefore $\delta=\varepsilon'$ or $\varepsilon=\delta'$ or $\delta=\delta'$ or $\varepsilon=\varepsilon'$. From $\alpha=\alpha'$, $\eta=\eta'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we know that the two multisets $(\alpha,1,1,\delta,\varepsilon,\eta)$ and $(\alpha',1,1,\delta',\varepsilon',\eta')$ are the same. Since $G\sim H$, from Proposition 2, we have G is isomorphic to H.

Case 1.2: If $\alpha+\delta=\alpha'+\delta'$, then we can handle this case in the same fashion as case 1.1, so we get $G\cong H$.

Case 1.3: If $\alpha+\delta=\alpha'+\eta'+1$, then $\delta=\eta'+1$ (since $\alpha=\alpha'$) and $\varepsilon+\eta+1=\varepsilon'+\delta'$ (since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$). After simplification, we have

$$Q(G):\ -r^{\delta}-r^{\varepsilon}-r^{\eta}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\delta'}-r^{\varepsilon'}-r^{\eta'}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}.$$

Since $\delta=\eta'+1$ (which implies $\delta>\eta'$) and $\varepsilon'\leqslant\eta'$, we have the l.r.p. in $Q(G)$ is ε and the l.r.p. in $Q(H)$ is ε' or δ'. Then, $\varepsilon=\varepsilon'$ or $\varepsilon=\delta'$. If $\varepsilon=\varepsilon'$, then $\delta'=\eta+1$ since $\varepsilon+\eta+1=\varepsilon'+\delta'$. After canceling $-r^{\varepsilon}$ in $Q(G)$ with $-r^{\varepsilon'}$ in $Q(H)$, we have the l.r.p. in $Q(G)$ is η and the l.r.p. in $Q(H)$ is η'. Therefore $\eta=\eta'$ which yields $\delta=\delta'$. So $G\cong H$. If $\varepsilon=\delta'$, then $\varepsilon'=\eta+1$ since $\varepsilon+\eta+1=\varepsilon'+\delta'$. After simplification, we have

$$Q(G):\ -r^{\delta}-r^{\eta}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\varepsilon'}-r^{\eta'}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}.$$

Since $\varepsilon=\delta'$ and $\varepsilon'=\eta+1$ and $\varepsilon\leqslant\eta$, we have $\delta'+1\leqslant\varepsilon'\leqslant\eta'$. By $\delta=\eta'+1$, we know that no terms in $Q(H)$ is equal to $-r^{\delta}$. So the term $-r^{\delta}$ and the term $-r^{\delta+1}$ must be cancelled by the term $+r^{\alpha+\varepsilon+1}$ and by the term $+r^{\alpha+\eta+1}$, respectively, therefore

$$\delta=\alpha+\varepsilon+1,\quad \delta+1=\alpha+\eta+1.$$

Consider $-r^{\eta}$ in $Q(G)$ (noting $\varepsilon'=\eta+1$ and $\delta'+1\leqslant\varepsilon'\leqslant\eta'$). We have $-r^{\eta}=-r^{\delta'+1}$. So $\varepsilon'=\delta'+2$. Let $\alpha=a$, $\varepsilon=b$, we obtain the solution (noting $\alpha=\alpha'$, $\varepsilon=\delta'$, $\varepsilon'=\delta'+2$, $\delta=\eta'+1$, $\eta=\delta'+1$, $\delta=\alpha+\varepsilon+1$ and $\delta+1=\alpha+\eta+1$) where G is isomorphic to $K_4(a,1,1,a+b+1,b,b+1)$ and H is isomorphic to $K_4(a,1,1,b,b+2,a+b)$.

Case 1.4: If $\alpha+\eta+1=\alpha'+\delta'$, then the results are similar to case 1.3.

Case 2: If $\min\{\alpha,\delta,\varepsilon,\eta\}=\alpha$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\delta'$, then $\alpha=\delta'$. Since the case of $\min\{\alpha,\delta,\varepsilon,\eta\}=\alpha$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\alpha'$ has been discussed in case 1, we can suppose $\delta' \neq \alpha'$ in case 2.

$$Q(G) = -r^{\alpha} - r^{\delta} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} - r^{\delta+1} + r^{\delta+2} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H) = -r^{\alpha'} - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

By considering the highest power in $Q(G)$ and the highest power in $Q(H)$, we have $\delta+\varepsilon+\eta=\alpha'+\eta'+1$ or $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. If $\delta+\varepsilon+\eta=\alpha'+\eta'+1$, then $\alpha+1=\varepsilon'+\delta'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. This is a contradiction since $\alpha=\delta'$ and $\varepsilon'\geqslant 2$. If $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, then $\alpha=\alpha'$. since $\alpha=\delta'$, we have $\alpha'=\delta'$ which contradicts $\delta'\neq\alpha'$.

Case 3: If $\min\{\alpha,\delta,\varepsilon,\eta\}=\alpha$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\varepsilon'$, then $\alpha=\varepsilon'$. Since the case of $\min\{\alpha,\delta,\varepsilon,\eta\}=\alpha$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\alpha'$ has been discussed in case 1, we can suppose $\varepsilon'\neq\alpha'$ in case 3.

$$Q(G) = -r^{\alpha} - r^{\delta} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} - r^{\delta+1} + r^{\delta+2} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H) = -r^{\alpha'} - r^{\delta'} - r^{\varepsilon'} - r^{\eta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

By considering the highest power in $Q(G)$ and the highest power in $Q(H)$, we have $\delta+\varepsilon+\eta=\alpha'+\delta'$ or $\delta+\varepsilon+\eta=\alpha'+\eta'+1$ or $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. If $\delta+\varepsilon+\eta=\alpha'+\delta'$, then $\alpha=\varepsilon'+\eta'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. This is a contradiction since $\alpha=\varepsilon'$. If $\delta+\varepsilon+\eta=\alpha'+\eta'+1$, then $\alpha+1=\varepsilon'+\delta'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. This is a contradiction since $\alpha=\varepsilon'$ and $\delta'\geqslant 2$. If $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, then $\alpha=\alpha'$. Since $\alpha=\varepsilon'$, we have $\alpha'=\varepsilon'$ which contradicts $\varepsilon'\neq\alpha'$.

Case 4: If $\min\{\alpha,\delta,\varepsilon,\eta\}=\delta$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\delta'$, then $\delta=\delta'$. After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G):\ -r^{\alpha} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\varepsilon'} - r^{\eta'} - r^{\alpha'+1} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

By considering the l.r.p. in $Q(G)$ and the l.r.p. in $Q(H)$, we have $\alpha=\alpha'$ or $\varepsilon=\varepsilon'$ or $\alpha=\varepsilon'$ or $\varepsilon=\alpha'$.

Case 4.1: If $\alpha=\alpha'$, then $\varepsilon+\eta=\varepsilon'+\eta'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. After simplifying $Q(G)$ and $Q(H)$, we have the l.r.p. in $Q(G)$ is ε and the l.r.p. in $Q(H)$ is ε'. Then $\varepsilon=\varepsilon'$. Therefore, $\eta=\eta'$ which implies that G is isomorphic to H.

Case 4.2: If $\varepsilon=\varepsilon'$, then $\alpha+\eta=\alpha'+\eta'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. After canceling $-r^{\varepsilon}$ in $Q(G)$ with $-r^{\varepsilon'}$ in $Q(H)$, we have the l.r.p. in $Q(G)$ is $\min\{\alpha,\eta\}$ and the l.r.p. in $Q(H)$ is $\min\{\alpha',\eta'\}$. Therefore $\alpha=\alpha'$ or $\eta=\eta'$ or $\alpha=\eta'$ or $\eta=\alpha'$. From

$\delta=\delta'$, $\varepsilon=\varepsilon'$ and $\alpha+\eta=\alpha'+\eta'$, we know that the two multisets $(\alpha,1,1,\delta,\varepsilon,\eta)$ and $(\alpha',1,1,\delta',\varepsilon',\eta')$ are the same. Since $G\sim H$, by Proposition 2, we have G is isomorphic to H.

Case 4.3: If $\alpha=\varepsilon'$ which implies that $\alpha\leqslant\varepsilon$, then we obtain the following after simplification:

$$Q(G):\quad -r^{\varepsilon}-r^{\eta}-r^{\alpha+1}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\quad -r^{\alpha'}-r^{\eta'}-r^{\alpha'+1}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

By considering the h.r.p. in $Q(G)$ and the h.r.p. in $Q(H)$, we have the h.r.p. in $Q(G)$ is $\delta+\varepsilon+\eta$ (since $\alpha\leqslant\varepsilon$) and the h.r.p. in $Q(H)$ is $\alpha'+\eta'+1$ (since $\min\{\alpha',\delta',\varepsilon',\eta'\}=\delta'$) or $\delta'+\varepsilon'+\eta'$. Then $\delta+\varepsilon+\eta=\alpha'+\eta'+1$ or $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. If $\delta+\varepsilon+\eta=\alpha'+\eta'+1$, then $\alpha+1=\varepsilon'+\delta'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. This is a contradiction since $\alpha=\varepsilon'$ and $\delta'\geqslant 2$. If $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, then $\alpha=\alpha'$. Since $\alpha=\varepsilon'$, we have $\alpha'=\varepsilon'$. After simplification, we have

$$Q(G):\quad -r^{\varepsilon}-r^{\eta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\quad -r^{\varepsilon'}-r^{\eta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}.$$

By considering the l.r.p. in $Q(G)$ and the l.r.p. in $Q(H)$, we have $\varepsilon=\varepsilon'$. Since $\alpha=\alpha'$ and $\delta=\delta'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have $\eta=\eta'$ which implies that G is isomorphic to H.

Case 4.4: If $\varepsilon=\alpha'$, then we can handle this case in the same fashion as Case 4.3. The results are similar to case 4.3.

Case 5: If $\min\{\alpha,\delta,\varepsilon,\eta\}=\varepsilon$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\varepsilon'$, then $\varepsilon=\varepsilon'$. After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G):\quad -r^{\alpha}-r^{\delta}-r^{\eta}-r^{\alpha+1}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\quad -r^{\alpha'}-r^{\delta'}-r^{\eta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

By considering the l.r.p. in $Q(G)$ and the l.r.p. in $Q(H)$, we have $\min\{\alpha,\delta,\eta\}=\min\{\alpha',\delta',\eta'\}$. There are six cases to consider.

Case 5.1: If $\min\{\alpha,\delta,\eta\}=\alpha$ and $\min\{\alpha',\delta',\eta'\}=\alpha'$, then $\alpha=\alpha'$. After simplification, we have the l.r.p. in $Q(G)$ is $\min\{\delta,\eta\}$ and the l.r.p. in $Q(H)$ is $\min\{\delta',\eta'\}$. Then $\min\{\delta,\eta\}=\min\{\delta',\eta'\}$. From $\alpha=\alpha'$, $\varepsilon=\varepsilon'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we know that the two multisets $(\alpha,1,1,\delta,\varepsilon,\eta)$ and $(\alpha',1,1,\delta',\varepsilon',\eta')$ are the same. Since $G\sim H$, from Proposition 2, we have G is isomorphic to H.

Case 5.2: If $\min\{\alpha,\delta,\eta\}=\alpha$ and $\min\{\alpha',\delta',\eta'\}=\delta'$, then $\alpha=\delta'$. From $\varepsilon=\varepsilon'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have

$$\delta+\eta=\alpha'+\eta'.$$

Since $\min\{\alpha,\delta,\eta\}=\alpha$ and $\min\{\alpha',\delta',\eta'\}=\delta'$, we know that the h.r.p. in $Q(G)$ is $\delta+\varepsilon+\eta$ and the h.r.p. in $Q(H)$ is $\alpha'+\eta'+1$ or $\delta'+\varepsilon'+\eta'$. Therefore $\delta+\varepsilon+\eta=\alpha'+\eta'+1$ or $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. If $\delta+\varepsilon+\eta=\alpha'+\eta'+1$, then $\varepsilon=1$ since $\delta+\eta=\alpha'+\eta'$. This is a contradiction since $\varepsilon\geqslant 2$. If $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, then $\alpha=\alpha'$. Thus, we can prove $G\cong H$ in the same fashion as case 5.1.

Case 5.3: If $\min\{\alpha,\delta,\eta\}=\alpha$ and $\min\{\alpha',\delta',\eta'\}=\eta'$, then $\alpha=\eta'$. Since $\varepsilon=\varepsilon'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have

$$\delta+\eta=\alpha'+\delta'.$$

By considering the h.r.p. in $Q(G)$ and the h.r.p. in $Q(H)$, we have $\delta+\varepsilon+\eta=\alpha'+\delta'$ or $\delta+\varepsilon+\eta=\alpha'+\eta'+1$ or $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. If $\delta+\varepsilon+\eta=\alpha'+\delta'$, from $\delta+\eta=\alpha'+\delta'$, we have $\varepsilon=0$ which contradicts $\varepsilon\geqslant 2$. If $\delta+\varepsilon+\eta=\alpha'+\eta'+1$, from $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have $\alpha+1=\delta'+\varepsilon'$. Since $\alpha=\eta'$, we have $\eta'+1=\delta'+\varepsilon'$ which implies $\delta'<\eta'$ since $\varepsilon'\geqslant 2$. This is a contradiction since $\min\{\alpha',\delta',\eta'\}=\eta'$. If $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, then $\alpha=\alpha'$. Thus, we can prove $G\cong H$ in the same fashion as case 5.1.

Case 5.4: If $\min\{\alpha,\delta,\eta\}=\delta$ and $\min\{\alpha',\delta',\eta'\}=\delta'$, then $\delta=\delta'$. After canceling $-r^{\delta}$ in $Q(G)$ with $-r^{\delta'}$ in $Q(H)$, and canceling $-r^{\delta+1}$ in $Q(G)$ with $-r^{\delta'+1}$ in $Q(H)$, we have the l.r.p. in $Q(G)$ is $\min\{\alpha,\eta\}$ and the l.r.p. in $Q(H)$ is $\min\{\alpha',\eta'\}$. Then, $\min\{\alpha,\eta\}=\min\{\alpha',\eta'\}$. From $\varepsilon=\varepsilon'$ and $\delta=\delta'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we know that the two multisets $(\alpha,1,1,\delta,\varepsilon,\eta)$ and $(\alpha',1,1,\delta',\varepsilon',\eta')$ are the same. Since $G\sim H$, from Proposition 2, we have G is isomorphic to H.

Case 5.5: If $\min\{\alpha,\delta,\eta\}=\eta$ and $\min\{\alpha',\delta',\eta'\}=\eta'$, then $\eta=\eta'$. Thus we can prove $G\cong H$ in the same fashion as case 5.4.

Case 5.6: If $\min\{\alpha,\delta,\eta\}=\delta$ and $\min\{\alpha',\delta',\eta'\}=\eta'$, then $\delta=\eta'$. From $\varepsilon=\varepsilon'$ and $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have

$$\alpha+\eta=\alpha'+\delta'. \tag{1}$$

After simplification, we have

$$\begin{aligned} Q(G):\quad & -r^{\alpha}-r^{\eta}-r^{\alpha+1}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}\\ & +r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},\\ Q(H):\quad & -r^{\alpha'}-r^{\delta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}\\ & +r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}. \end{aligned}$$

By considering the l.r.p. in $Q(G)$ and the l.r.p. in $Q(H)$, we have $\min\{\alpha,\eta,\delta+1\}=\min\{\alpha',\delta'\}$. If $\min\{\alpha,\eta\}=\min\{\alpha',\delta'\}$, from (1) and $\delta=\eta'$ and $\varepsilon=\varepsilon'$, we know that the two multisets $(\alpha,1,1,\delta,\varepsilon,\eta)$ and $(\alpha',1,1,\delta',\varepsilon',\eta')$ are the same. Since $G\sim H$, from

Proposition 2, we have G is isomorphic to H. If $\min\{\alpha', \delta'\} = \delta + 1$. There are two cases to consider.

Case 5.6.1: If $\delta' \leqslant \alpha'$, then $\delta' = \delta + 1$. Consider $r^{\delta+2}$ in $Q(G)$ and $-r^{\delta'+1}$ in $Q(H)$. It is due to $\delta' \leqslant \alpha'$ that $-r^{\delta'+1}$ can cancel none of the positive terms in $Q(H)$. Thus, no term in $Q(H)$ is equal to $r^{\delta+2}$. Therefore, $\delta + 2$ must equal one of $\alpha, \eta, \alpha + 1$ and $\delta' + 1$ must equal one of $\alpha, \eta, \alpha + 1$. So $\delta + 2 = \delta' + 1 = \alpha = \eta$ or $\delta + 2 = \delta' + 1 = \alpha + 1 = \eta$. If

$$\delta + 2 = \delta' + 1 = \alpha = \eta \tag{2}$$

then we obtain the following after canceling $-r^{\alpha}$ with $r^{\delta+2}$, canceling $-r^{\eta}$ with $-r^{\delta'+1}$, and canceling $-r^{\delta+1}$ with $-r^{\delta'}$:

$$Q(G):\ -r^{\alpha+1} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\alpha'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

Since $\eta = \delta' + 1$ (from (2)) and $\alpha + \eta = \alpha' + \delta'$ (from (1)), we have $\alpha' = \alpha + 1$. By (2), we have $\alpha' + 1 = \delta' + 3$. This is a contradiction since nothing in $Q(H)$ can cancel $-r^{\alpha'+1}$. If

$$\delta + 2 = \delta' + 1 = \alpha + 1 = \eta \tag{3}$$

then we obtain the following after canceling $-r^{\alpha+1}$ with $r^{\delta+2}$, canceling $-r^{\eta}$ with $-r^{\delta'+1}$, and canceling $-r^{\delta+1}$ with $-r^{\delta'}$:

$$Q(G):\ -r^{\alpha} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\alpha'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

Since $\eta = \delta' + 1$ (from (3)), by (1), we have $\alpha' = \alpha + 1$. This is a contradiction since no term in $Q(H)$ is equal to $-r^{\alpha}$ and nothing in $Q(G)$ can cancel $-r^{\alpha}$ (noting $\alpha = \delta + 1$ (from (3))).

Case 5.6.2: If $\alpha' \leqslant \delta'$, then $\alpha' = \delta + 1$. Consider $r^{\delta+2}$ in $Q(G)$ and $-r^{\alpha'+1}$ in $Q(H)$. It is due to $\alpha' \leqslant \delta'$ that $-r^{\alpha'+1}$ can cancel none of the positive terms in $Q(H)$. Thus, no term in $Q(H)$ is equal to $r^{\delta+2}$. Therefore, $\delta + 2$ must equal one of $\alpha, \eta, \alpha + 1$ and $\alpha' + 1$ must equal one of $\alpha, \eta, \alpha + 1$. So $\delta + 2 = \alpha' + 1 = \alpha + 1 = \eta$ or $\delta + 2 = \alpha' + 1 = \alpha = \eta$.

If $\delta + 2 = \alpha' + 1 = \alpha + 1 = \eta$, then we obtain the following after canceling $-r^{\eta}$ with $r^{\delta+2}$, canceling $-r^{\alpha+1}$ with $-r^{\alpha'+1}$, and canceling $-r^{\delta+1}$ with $-r^{\alpha'}$:

$$Q(G):\ -r^{\alpha} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\delta'} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

Since $\alpha' + 1 = \eta$, by (1), we have $\delta' = \alpha + 1$. This is a contradiction since no term in $Q(H)$ is equal to $-r^{\alpha}$ and nothing in $Q(G)$ can cancel $-r^{\alpha}$ (noting $\alpha = \delta + 1$).

If $\delta+2=\alpha'+1=\alpha=\eta$, then we obtain the following after canceling $-r^{\eta}$ with $r^{\delta+2}$, canceling $-r^{\alpha}$ with $-r^{\alpha'+1}$, and canceling $-r^{\delta+1}$ with $-r^{\alpha'}$:

$$Q(G):\ -r^{\alpha+1}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\delta'}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Since $\alpha'+1=\eta$, from (1), we have $\delta'=\alpha+1$. Consider $-r^{\delta'+1}$ in $Q(H)$. We have $\delta'+1=\alpha'+\varepsilon'+1$ or $\delta'+1=\alpha'+\eta'+1$. If $\delta'+1=\alpha'+\varepsilon'+1$, from $\delta'=\alpha+1=\alpha'+2$, we have $\varepsilon'=2$. So far, we have had $\delta=\eta'$, $\delta+2=\alpha'+1=\alpha=\eta$, $\delta'=\alpha+1$, $\varepsilon=\varepsilon'=2$. Let $\delta=a$, we obtain the solution where G is isomorphic to $k_4(a+2,1,1,a,2,a+2)$ and H is isomorphic to $K_4(a+1,1,1,a+3,2,a)$. If $\delta'+1=\alpha'+\eta'+1$, from $\delta'=\alpha+1=\alpha'+2$, we have $\eta'=2$. From $\varepsilon'\leqslant\eta'$ and $\varepsilon'\geqslant 2$, we have $\varepsilon'=2$. Since $\delta=\eta'$, $\delta+2=\alpha'+1=\alpha=\eta$, $\delta'=\alpha+1$, $\varepsilon=\varepsilon'$, we have $\delta=2$, $\alpha=4$, $\eta=4$, $\alpha'=3$, $\delta'=5$, $\varepsilon=\varepsilon'=2$. Then we obtain the solution where G is isomorphic to $k_4(4,1,1,2,2,4)$ and H is isomorphic to $K_4(3,1,1,5,2,2)$.

Case 6: If $\min\{\alpha,\delta,\varepsilon,\eta\}=\delta$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\varepsilon'$, then

$$\delta=\varepsilon'. \tag{4}$$

After simplifying $Q(G)$ and $Q(H)$, we have

$$Q(G):\ -r^{\alpha}-r^{\varepsilon}-r^{\eta}-r^{\alpha+1}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'}-r^{\delta'}-r^{\eta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Since the case of $\min\{\alpha,\delta,\varepsilon,\eta\}=\alpha$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\varepsilon'$ has been discussed in case 3, we can suppose $\delta\neq\alpha$ in case 6. Thus $\delta<\alpha$. Since the case of $\min\{\alpha,\delta,\varepsilon,\eta\}=\varepsilon$ and $\min\{\alpha',\delta',\varepsilon',\eta'\}=\varepsilon'$ has been discussed in case 5, we can suppose $\delta\neq\varepsilon$ in case 6. Thus

$$\delta<\varepsilon. \tag{5}$$

Therefore, the l.r.p. in $Q(G)$ is $\delta+1$ and the l.r.p. in $Q(H)$ is α' or δ' or η'. So, we have $\delta+1=\alpha'$ or $\delta+1=\delta'$ or $\delta+1=\eta'$. There are three cases to consider.

Case 6.1: If $\delta+1=\alpha'$, then the h.r.p. in $Q(G)$ is $\alpha+\eta+1$ or $\delta+\varepsilon+\eta$ and the h.r.p. in $Q(H)$ is $\delta'+\varepsilon'+\eta'$ (since $\min\{\alpha',\delta',\eta'\}=\alpha'$). Therefore, $\alpha+\eta+1=\delta'+\varepsilon'+\eta'$ or $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. If $\alpha+\eta+1=\delta'+\varepsilon'+\eta'$, from $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have $\varepsilon+\delta=\alpha'+1$. Since $\delta+1=\alpha'$, we have $\varepsilon=2$ which contradicts $\varepsilon>\delta\geqslant 2$. If $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$, then $\alpha=\alpha'$. After canceling $-r^{\alpha+1}$ with $-r^{\alpha'+1}$, canceling $-r^{\alpha}$ with $-r^{\alpha'}$, and canceling $-r^{\delta+\varepsilon+\eta}$ with $-r^{\delta'+\varepsilon'+\eta'}$, we have

$$Q(G):\ -r^{\varepsilon}-r^{\eta}-r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\delta'}-r^{\eta'}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}.$$

Since $\delta<\varepsilon$ (from (5)) and $\varepsilon\leqslant\eta$, we have the h.r.p. in $Q(G)$ is $\alpha+\eta+1$. The h.r.p. in $Q(H)$ is $\alpha'+\delta'$ or $\alpha'+\eta'+1$ (noting $\varepsilon'\leqslant\eta'$). Therefore, $\alpha+\eta+1=\alpha'+\delta'$ or

$\alpha+\eta+1=\alpha'+\eta'+1$. If $\alpha+\eta+1=\alpha'+\delta'$, by $\alpha=\alpha'$, we have $\delta'=\eta+1$. Thus, no terms in $Q(G)$ are equal to $-r^{\delta'}$ and $-r^{\delta'+1}$. Then $-r^{\delta'}$ must be cancelled by $r^{\alpha'+\varepsilon'+1}$ and $-r^{\delta'+1}$ must be cancelled by $r^{\alpha'+\eta'+1}$. Since nothing in $Q(G)$ can cancel $-r^{\delta+1}$ (noting $\delta<\varepsilon$ and $\varepsilon\leqslant\eta$), we have $-r^{\delta+1}=-r^{\eta'}$. Since no terms in $Q(H)$ are equal to $-r^{\varepsilon}$ and $-r^{\eta}$, $-r^{\varepsilon}$ must be canceled by $r^{\delta+2}$ and $-r^{\eta}$ must be canceled by $r^{\alpha+\delta}$. So far, we have had $\alpha=\alpha'$, $\delta'=\alpha'+\varepsilon'+1$ and $\delta'+1=\alpha'+\eta'+1$ (which implies $\eta'=\varepsilon'+1$), $\delta+1=\eta'$, $\varepsilon=\delta+2$, $\eta=\alpha+\delta$. Let $\alpha=a$, $\delta=b$. We obtain the solution where G is isomorphic to $K_4(a,1,1,b,b+2,a+b)$ and H is isomorphic to $K_4(a,1,1,a+b+1,b,b+1)$. If $\alpha+\eta+1=\alpha'+\eta'+1$, then $\eta=\eta'$ since $\alpha=\alpha'$. From $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$ and $\delta=\varepsilon'$ (from (4)), we have $\delta'=\varepsilon$. After simplification, we have

$$Q(G):\ -r^{\delta+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1},$$

$$Q(H):\ -r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}.$$

This is a contradiction since no term in $Q(H)$ is equal to $-r^{\delta+1}$ (by noting $\delta<\varepsilon=\delta'$).

Case 6.2: If $\delta+1=\delta'$, then we can suppose $\delta'\neq\alpha'$ in case 6.2 since the case of $\delta+1=\alpha'$ has been discussed in case 6.1. Thus

$$\delta'<\alpha'. \tag{6}$$

After canceling $-r^{\delta+1}$ with $-r^{\delta'}$, we have

$$Q(G):\ -r^{\alpha}-r^{\varepsilon}-r^{\eta}-r^{\alpha+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'}-r^{\eta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}$$
$$+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Consider $r^{\delta+2}$ in $Q(G)$ and $-r^{\delta'+1}$ in $Q(H)$. It is due to (6) that $-r^{\delta'+1}$ can cancel none of the positive terms in $Q(H)$. Thus, no terms in $Q(H)$ is equal to $r^{\delta+2}$. Therefore, $-r^{\delta'+1}$ and $r^{\delta+2}$ must equal one of $-r^{\alpha}$, $-r^{\varepsilon}$, $-r^{\eta}$, $-r^{\alpha+1}$. So, $\delta'+1=\delta+2=\alpha=\varepsilon$ or $\delta'+1=\delta+2=\alpha=\eta$ or $\delta'+1=\delta+2=\alpha+1=\varepsilon$ or $\delta'+1=\delta+2=\alpha+1=\eta$ or $\delta'+1=\delta+2=\varepsilon=\eta$. Without loss of generality, only the following three cases need to be considered.

Case 6.2.1: If $\delta'+1=\delta+2=\alpha$, we consider $r^{\delta'+2}$ in $Q(H)$ and $-r^{\alpha+1}$ in $Q(G)$. It is due to $\alpha=\delta+2$ that $-r^{\alpha+1}$ can cancel none of the positive terms in $Q(G)$. Thus, no terms in $Q(G)$ is equal to $r^{\delta'+2}$. Therefore, $\alpha+1=\delta'+2=\alpha'=\eta'$ or $\alpha+1=\delta'+2=\alpha'+1=\eta'$. If

$$\alpha+1=\delta'+2=\alpha'=\eta' \tag{7}$$

then we obtain the following after canceling $-r^{\alpha}$ with $r^{\delta+2}$, canceling $-r^{\alpha+1}$ with $-r^{\alpha'}$, and canceling $-r^{\eta'}$ with $r^{\delta'+2}$:

$$Q(G):\ -r^{\varepsilon}-r^{\eta}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'+1}-r^{\delta'+1}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

By considering the l.r.p. in $Q(G)$ and the l.r.p. in $Q(H)$, we have $\varepsilon = \delta' + 1$. Since $\alpha + \delta + \varepsilon + \eta = \alpha' + \delta' + \varepsilon' + \eta'$ and $\delta = \varepsilon'$ (from (4)) and $\alpha' = \alpha + 1$ (from (7)), we have $\eta = \eta'$. Therefore $\eta = \alpha'$ (noting (7)). This is a contradiction since no term in $Q(G)$ is equal to $-r^{\alpha'+1}$ and nothing in $Q(H)$ can cancel $-r^{\alpha'+1}$ (noting (7)). If $\alpha + 1 = \delta' + 2 = \alpha' + 1 = \eta'$, then we obtain the following after canceling $-r^{\alpha}$ with $r^{\delta+2}$, canceling $-r^{\alpha+1}$ with $-r^{\alpha'+1}$, canceling $-r^{\eta'}$ with $r^{\delta'+2}$, and canceling $-r^{\delta+\varepsilon+\eta}$ with $-r^{\delta'+\varepsilon'+\eta'}$

$$Q(G):\ -r^{\varepsilon} - r^{\eta} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\alpha'} - r^{\delta'+1} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1}.$$

Consider $r^{\alpha+\delta}$ in $Q(G)$. It is due to $\alpha = \alpha'$, and $\delta' = \delta + 1$ and $\delta = \varepsilon'$ (from (4)) that no term in $Q(H)$ is equal to $r^{\alpha+\delta}$. This is a contradiction since nothing in $Q(G)$ can cancel $r^{\alpha+\delta}$ (by noting $-r^{\alpha'} = -r^{\delta'+1} = -r^{\varepsilon} = -r^{\eta}$).

Case 6.2.2: If $\delta' + 1 = \delta + 2 = \alpha + 1$, then $\alpha = \delta'$. After canceling $-r^{\alpha+1}$ with $r^{\delta+2}$, we have

$$Q(G):\ -r^{\alpha} - r^{\varepsilon} - r^{\eta} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\eta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

It is due to $\delta' < \alpha'$ (from (6)) and $\alpha = \delta'$ that $\alpha < \alpha'$. Then $-r^{\alpha} = -r^{\eta'}$. Since $\delta = \varepsilon'$ (from (4)) and $\alpha = \eta'$ and $\alpha + \delta + \varepsilon + \eta = \alpha' + \delta' + \varepsilon' + \eta'$, we have

$$\delta' + \alpha' = \varepsilon + \eta.$$

Consider $-r^{\delta'+1}$ in $Q(H)$. It is due to (6) that nothing in $Q(H)$ can cancel $-r^{\delta'+1}$. Therefore $-r^{\delta'+1} = -r^{\varepsilon}$ or $-r^{\delta'+1} = -r^{\eta}$. If $\delta' + 1 = \varepsilon$, then $\alpha' = \eta + 1$ since $\delta' + \alpha' = \varepsilon + \eta$. This is a contradiction since no terms in $Q(G)$ are equal to $-r^{\alpha'}$ and $-r^{\alpha'+1}$ (noting $\alpha < \alpha'$). If $\delta' + 1 = \eta$, then $\alpha' = \varepsilon + 1$ since $\delta' + \alpha' = \varepsilon + \eta$. After canceling $-r^{\alpha}$ with $-r^{\eta'}$, canceling $-r^{\delta'+1}$ with $-r^{\eta}$, we have

$$Q(G):\ -r^{\varepsilon} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\alpha+\eta+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\alpha'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

This is a contradiction since no terms in $Q(G)$ are equal to $-r^{\alpha'}$ and $-r^{\alpha'+1}$.

Case 6.2.3: If

$$\delta' + 1 = \delta + 2 = \varepsilon = \eta. \tag{8}$$

Then, from (4) and $\delta' + 1 = \varepsilon$ and $\alpha + \delta + \varepsilon + \eta = \alpha' + \delta' + \varepsilon' + \eta'$, we have

$$\alpha + \eta + 1 = \alpha' + \eta'. \tag{9}$$

After canceling $-r^{\eta}$ with $r^{\delta+2}$, canceling $-r^{\varepsilon}$ with $-r^{\delta'+1}$, we have

$$Q(G):\ -r^{\alpha}-r^{\alpha+1}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'}-r^{\eta'}-r^{\alpha'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}$$
$$+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Consider $-r^{\alpha}$ and $-r^{\alpha+1}$. One of $-r^{\alpha}$ and $-r^{\alpha+1}$ must equal $-r^{\alpha'}$ or $-r^{\alpha'+1}$.

If $-r^{\alpha+1}=-r^{\alpha'}$, then $\alpha+1=\alpha'$. Therefore, $\eta=\eta'$ since $\alpha+\eta+1=\alpha'+\eta'$ (from (9)). By $\eta=\delta'+1$ (from (8)), we have $\eta'=\delta'+1$. After canceling $-r^{\alpha'}$ with $-r^{\alpha+1}$, we have

$$Q(G):\ -r^{\alpha}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'+1}-r^{\eta'}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Consider $-r^{\eta'}$ in $Q(H)$. It is due to $\eta'=\delta'+1<\alpha'+1$ (noting (6)) that nothing in $Q(H)$ can cancel $-r^{\eta'}$. So $-r^{\eta'}=-r^{\alpha}$. Then $\alpha'+1=\alpha+2=\eta'+2=\delta'+3$. This is a contradiction since nothing in $Q(H)$ can cancel $-r^{\alpha'+1}$ and no term in $Q(G)$ is equal to $-r^{\alpha'+1}$.

If $-r^{\alpha+1}=-r^{\alpha'+1}$, then $\alpha=\alpha'$. Therefore, $\eta+1=\eta'$ since $\alpha+\eta+1=\alpha'+\eta'$. By $\eta=\delta'+1$ (from (8)), we have $\eta'=\delta'+2$. After canceling $-r^{\alpha}$ with $-r^{\alpha'}$, canceling $-r^{\alpha+1}$ with $-r^{\alpha'+1}$, canceling $-r^{\eta'}$ with $r^{\delta'+2}$, and canceling $-r^{\delta+\varepsilon+\eta}$ with $-r^{\delta'+\varepsilon'+\eta'}$, we have

$$Q(G):\ r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}.$$

This is a contradiction since no term in $Q(H)$ is equal to $r^{\alpha+\delta}$ (noting $\alpha=\alpha'$ and $\delta=\varepsilon'$ (from (4)) and $\delta+1=\delta'$ (from (8))).

If $-r^{\alpha}=-r^{\alpha'}$, by the same reason as in case $-r^{\alpha+1}=-r^{\alpha'+1}$, we have a contradiction.

If $-r^{\alpha}=-r^{\alpha'+1}$, then $\alpha=\alpha'+1$. Therefore, $\eta'=\eta+2$ since $\alpha+\eta+1=\alpha'+\eta'$. By $\eta=\delta'+1$ (from (8)), we have $\eta'=\delta'+3$. After simplification, we have

$$Q(G):\ -r^{\alpha+1}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'}-r^{\eta'}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Consider $-r^{\alpha'}$ in $Q(H)$. It is due to $\alpha=\alpha'+1$ that $-r^{\alpha'}$ must be canceled by $r^{\delta'+2}$ or by $r^{\delta'+\varepsilon'+\eta'}$. If $\alpha'=\delta'+2$, by $\eta'=\delta'+3$, we have $\alpha'+1=\eta'$. So $\alpha=\eta'$. This is a contradiction since nothing in $Q(H)$ can cancel $-r^{\eta'}$ and no term in $Q(G)$ is equal to $-r^{\eta'}$. If $\alpha'=\delta'+\varepsilon'+\eta'$ (which implies $\eta'<\alpha'=\alpha-1$), then we obtain a contradiction since nothing in $Q(H)$ can cancel $-r^{\eta'}$ and no term in $Q(G)$ is equal to $-r^{\eta'}$.

Case 6.3: If $\delta+1=\eta'$, then we can suppose $\eta'\neq\delta'$ in case 6.3 since the case of $\delta+1=\delta'$ has been discussed in Case 6.2. Thus $\eta'<\delta'$ which implies

$$\eta'+1\leqslant\delta'. \tag{10}$$

After simplification, we have

$$Q(G):\ -r^{\alpha}-r^{\varepsilon}-r^{\eta}-r^{\alpha+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'}-r^{\delta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

By considering the h.r.p. in $Q(G)$ and the h.r.p. in $Q(H)$, we have the h.r.p. in $Q(G)$ is $\alpha+\eta+1$ (since $\min\{\alpha,\delta,\varepsilon,\eta\}=\delta$) or $\delta+\varepsilon+\eta$, the h.r.p. in $Q(H)$ is $\alpha'+\delta'$ (since $\eta'+1\leqslant\delta'$) or $\delta'+\varepsilon'+\eta'$. There are four cases to consider.

Case 6.3.1: When $\alpha+\eta+1>\delta+\varepsilon+\eta$ and $\alpha'+\delta'>\delta'+\varepsilon'+\eta'$, we have

$$\alpha+\eta+1=\alpha'+\delta'. \tag{11}$$

Since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$ and $\delta+1=\eta'$, we have $\varepsilon=\varepsilon'+2$. From (4), we have $\varepsilon=\delta+2$. After simplification, we have

$$Q(G):\ -r^{\alpha}-r^{\eta}-r^{\alpha+1}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'}-r^{\delta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

By considering the l.r.p. in $Q(G)$ and the l.r.p. in $Q(H)$, we have $\alpha=\alpha'$ or $\alpha=\delta'$ or $\eta=\alpha'$ or $\eta=\delta'$

If $\alpha=\alpha'$, then $\delta'=\eta+1$ since $\alpha+\eta+1=\alpha'+\delta'$ (from (11)). After canceling $-r^{\alpha}$ with $-r^{\alpha'}$, and canceling $-r^{\alpha+1}$ with $-r^{\alpha'+1}$, we know that the terms $-r^{\delta'}$ and $-r^{\delta'+1}$ must be canceled by the terms in $Q(H)$. Then $\delta'=\alpha'+\varepsilon'+1$ and $\delta'+1=\alpha'+\eta'+1$. Let $\alpha'=a$ and $\varepsilon'=b$. Then we obtain the solution (noting $\alpha=\alpha'$, $\delta=\varepsilon'$ (from (4)), $\varepsilon=\varepsilon'+2$, $\eta'=\delta+1$, $\delta'=\eta+1$, $\delta'=\alpha'+\varepsilon'+1$, $\delta'+1=\alpha'+\eta'+1$) where G is isomorphic to $K_4(a,1,1,b,b+2,a+b)$ and H is isomorphic to $K_4(a,1,1,a+b+1,b,b+1)$.

If $\alpha=\delta'$, then $\alpha'=\eta+1$ since $\alpha+\eta+1=\alpha'+\delta'$. After canceling $-r^{\alpha}$ with $-r^{\delta'}$, and canceling $-r^{\alpha+1}$ with $-r^{\delta'+1}$, we know that no term in $Q(G)$ is equal to $-r^{\alpha'}$ or $-r^{\alpha'+1}$. This is a contradiction since only one of $-r^{\alpha'}$ and $-r^{\alpha'+1}$ can be canceled in $Q(H)$.

If $\eta=\alpha'$, then $\alpha+1=\delta'$ since $\alpha+\eta+1=\alpha'+\delta'$. After simplification, we have

$$Q(G):\ -r^{\alpha}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Consider $-r^{\delta'+1}$ in $Q(H)$. It is due to $\delta'=\alpha+1$ that $-r^{\delta'+1}$ must be canceled by the term in $Q(H)$. Then $\delta'+1=\alpha'+\varepsilon'+1$ or $\delta'+1=\alpha'+\eta'+1$. Thus $-r^{\alpha'+1}$ cannot be canceled by the term in $Q(H)$. So $\alpha'+1=\alpha$. Turn to the term $r^{\alpha'+\varepsilon'+1}$, we have $r^{\alpha'+\varepsilon'+1}=r^{\alpha+\delta}$ (noting $\alpha'+1=\alpha$ and $\delta=\varepsilon'$ (from (4))). Therefore $-r^{\delta'+1}$ must be canceled by $r^{\alpha'+\eta'+1}$. From $\delta'=\alpha+1$ and $\alpha=\alpha'+1$, we have $\delta'=\alpha'+2$. By $\delta'+1=\alpha'+\eta'+1$, we have $\eta'=2$. From $\delta+1=\eta'$, we have $\delta=1$ which contradicts $\delta\geqslant 2$.

If $\eta = \delta'$, then $\alpha + 1 = \alpha'$ since $\alpha + \eta + 1 = \alpha' + \delta'$. After simplification, we have

$$Q(G):\ -r^{\alpha} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1} + r^{\delta'+\varepsilon'+\eta'}.$$

Consider $-r^{\alpha'+1}$ in $Q(H)$. It is due to $\alpha' = \alpha + 1$ that $-r^{\alpha'+1}$ must be canceled by $r^{\delta'+2}$ or $r^{\delta'+\varepsilon'+\eta'}$. Thus we have a contradiction since no term in $Q(G)$ is equal to $-r^{\delta'+1}$.

Case 6.3.2: When $\alpha + \eta + 1 > \delta + \varepsilon + \eta$ and $\alpha' + \delta' \leqslant \delta' + \varepsilon' + \eta'$, we have $\alpha + \eta + 1 = \delta' + \varepsilon' + \eta'$. Then, by $\alpha + \delta + \varepsilon + \eta = \alpha' + \delta' + \varepsilon' + \eta'$, we have

$$\alpha' + 1 = \delta + \varepsilon. \tag{12}$$

From $\alpha + \eta + 1 > \delta + \varepsilon + \eta (\alpha + 1 > \delta + \varepsilon)$, we have

$$\alpha > \alpha'. \tag{13}$$

After simplification, we have

$$Q(G):\ -r^{\alpha} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} + r^{\delta+2} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\delta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\delta'} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1}.$$

Since the h.r.p. in $Q(G)$ is $\alpha + \varepsilon + 1$ (since $\min\{\alpha, \delta, \varepsilon, \eta\} = \delta$) or $\delta + \varepsilon + \eta$ and the h.r.p. in $Q(H)$ is $\alpha' + \delta'$ (since $\eta' + 1 \leqslant \delta'$ (from (10))), we have $\alpha + \varepsilon + 1 = \alpha' + \delta'$ or $\delta + \varepsilon + \eta = \alpha' + \delta'$.

If $\alpha + \varepsilon + 1 = \alpha' + \delta'$, by $\alpha + \delta + \varepsilon + \eta = \alpha' + \delta' + \varepsilon' + \eta'$, we have $\varepsilon' + \eta' + 1 = \delta + \eta$. Then, from $\delta + 1 = \eta'$ and $\delta = \varepsilon'$ (from (4)), we have $\eta = \delta + 2$. Therefore, $-r^{\eta}$ is canceled by $r^{\delta+2}$. After simplification, we have

$$Q(G):\ -r^{\alpha} - r^{\varepsilon} - r^{\alpha+1} + r^{\alpha+\delta} + r^{\delta+\varepsilon+\eta},$$

$$Q(H):\ -r^{\alpha'} - r^{\delta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1}.$$

Consider $-r^{\alpha'}$ in $Q(H)$. It is due to $\alpha > \alpha'$ (from (13)) and $\alpha' + 1 = \delta + \varepsilon$ (which implies $\alpha' > \varepsilon$) that $-r^{\alpha'}$ must be canceled by the term in $Q(H)$. Thus $\alpha' = \delta' + 2$. This is a contradiction since $\alpha > \alpha' = \delta' + 2$ and none of the terms $-r^{\delta'}$ and $-r^{\delta'+1}$ can be canceled by terms in $Q(H)$.

If $\delta + \varepsilon + \eta = \alpha' + \delta'$, then $\delta' = \eta + 1$ since $\alpha' + 1 = \delta + \varepsilon$ (from (12)). After simplification, we have

$$Q(G):\ -r^{\alpha} - r^{\varepsilon} - r^{\eta} - r^{\alpha+1} + r^{\delta+2} + r^{\alpha+\delta} + r^{\alpha+\varepsilon+1},$$

$$Q(H):\ -r^{\alpha'} - r^{\delta'} - r^{\alpha'+1} - r^{\delta'+1} + r^{\delta'+2} + r^{\alpha'+\varepsilon'+1} + r^{\alpha'+\eta'+1}.$$

If $\delta' \leqslant \alpha'$, then we have a contradiction since no term in $Q(G)$ is equal to $-r^{\delta'}$ (by noting $\alpha > \alpha'$ and $\delta' = \eta + 1$). If $\delta' > \alpha'$. Consider $-r^{\alpha'}$ in $Q(H)$. It is due to $\alpha > \alpha'$ (from (13)) and $\alpha' + 1 = \delta + \varepsilon$ (which implies $\alpha' > \varepsilon$) that $-r^{\alpha'} = -r^{\eta}$. From $\delta' = \eta + 1$, we have $\delta'=\alpha'+1$. Thus $-r^{\delta'}=-r^{\alpha'+1}$. This is a contradiction since no pair of terms in $Q(G)$ are equal to $-r^{\delta'}$ and $-r^{\alpha'+1}$ (by noting $\alpha'=\eta\geqslant\varepsilon$).

Case 6.3.3: When $\alpha+\eta+1\leqslant\delta+\varepsilon+\eta$ and $\alpha'+\delta'\leqslant\delta'+\varepsilon'+\eta'$, we have $\delta+\varepsilon+\eta=\delta'+\varepsilon'+\eta'$. Then $\alpha=\alpha'$ since $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$. After simplification, we have

$$Q(G):\ -r^{\varepsilon}-r^{\eta}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\delta'}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\delta'}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}.$$

Since the h.r.p. in $Q(G)$ is $\alpha+\eta+1$ (since $\min\{\alpha,\delta,\varepsilon,\eta\}=\delta$ and $\varepsilon\leqslant\eta$) and the h.r.p. in $Q(H)$ is $\alpha'+\delta'$ (since $\eta'+1\leqslant\delta'$ (from (10))), we have $\alpha+\eta+1=\alpha'+\delta'$. Then $\delta'=\eta+1$ since $\alpha=\alpha'$. Thus $-r^{\delta'}$ and $-r^{\delta'+1}$ must be canceled by terms in $Q(H)$, and $-r^{\varepsilon}$, $-r^{\eta}$ must be canceled by terms in $Q(G)$. So, we have $\delta'=\alpha'+\varepsilon'+1$, $\delta'+1=\alpha'+\eta'+1$, $\varepsilon=\delta+2$ and $\eta=\alpha+\delta$. Let $\alpha'=a$ and $\varepsilon'=b$. Then we obtain the solution (noting $\alpha=\alpha'$, $\delta=\varepsilon'$ (from (4)), $\varepsilon=\delta+2$, $\eta'=\delta+1$, $\delta'=\eta+1$, $\delta'=\alpha'+\varepsilon'+1$, $\delta'+1=\alpha'+\eta'+1$, $\eta=\alpha+\delta$) where G is isomorphic to $K_4(a,1,1,b,b+2,a+b)$ and H is isomorphic to $K_4(a,1,1,a+b+1,b,b+1)$.

Case 6.3.4: When $\alpha+\eta+1\leqslant\delta+\varepsilon+\eta$ and $\alpha'+\delta'>\delta'+\varepsilon'+\eta'$, we have

$$\delta+\varepsilon+\eta=\alpha'+\delta' \tag{14}$$

by $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$, we have

$$\alpha=\varepsilon'+\eta'. \tag{15}$$

Then, from $\alpha'+\delta'>\delta'+\varepsilon'+\eta'$ ($\alpha'>\varepsilon'+\eta'$), we have

$$\alpha'>\alpha. \tag{16}$$

After simplification, we have

$$Q(G):\ -r^{\alpha}-r^{\varepsilon}-r^{\eta}-r^{\alpha+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1}+r^{\alpha+\eta+1},$$

$$Q(H):\ -r^{\alpha'}-r^{\delta'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\varepsilon'+1}+r^{\alpha'+\eta'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Since the h.r.p. in $Q(G)$ is $\alpha+\eta+1$ (since $\min\{\alpha,\delta,\varepsilon,\eta\}=\delta$ and $\varepsilon\leqslant\eta$) and the h.r.p. in $Q(H)$ is $\delta'+\varepsilon'+\eta'$ or $\alpha'+\eta'+1$, we have $\alpha+\eta+1=\delta'+\varepsilon'+\eta'$ or $\alpha+\eta+1=\alpha'+\eta'+1$.

If $\alpha+\eta+1=\delta'+\varepsilon'+\eta'$, then, by $\alpha=\varepsilon'+\eta'$ (from (15)), we have

$$\delta'=\eta+1.$$

Consider $-r^{\alpha'+1}$ in $Q(H)$. It is due to $\alpha'>\alpha$ (from (16)) that $\alpha'+1=\varepsilon$ or $\alpha'+1=\eta$. If $\alpha'+1=\varepsilon$, then $\delta'=\delta+\eta+1$ since $\delta+\varepsilon+\eta=\delta'+\alpha'$ (from (14)). This is a contradiction since $\delta'=\eta+1$. If $\alpha'+1=\eta$, then $\delta'=\alpha'+2$ since $\delta'=\eta+1$. From $\alpha'>\alpha$, we have $\delta'>\alpha+2$. So we have a contradiction since no term in $Q(G)$ is equal to $-r^{\delta'}$ (noting $\delta'=\eta+1$ and $\varepsilon\leqslant\eta$) and nothing in $Q(H)$ can cancel $-r^{\delta'}$ (by noting $\delta'=\alpha'+2$).

If $\alpha+\eta+1=\alpha'+\eta'+1$, then, by $\alpha+\delta+\varepsilon+\eta=\alpha'+\delta'+\varepsilon'+\eta'$ and $\delta=\varepsilon'$ (from (4)), we have $\varepsilon=\delta'$. Since $\delta+\varepsilon+\eta=\alpha'+\delta'$ (from (14)), we have $\alpha'=\delta+\eta$ which implies

$$\eta<\alpha'. \tag{17}$$

After canceling $-r^{\varepsilon}$ with $-r^{\delta'}$, canceling $r^{\alpha+\eta+1}$ with $r^{\alpha'+\eta'+1}$, we have

$$Q(G):\ -r^{\alpha}-r^{\eta}-r^{\alpha+1}+r^{\delta+2}+r^{\alpha+\delta}+r^{\alpha+\varepsilon+1},$$

$$Q(H):\ -r^{\alpha'}-r^{\alpha'+1}-r^{\delta'+1}+r^{\delta'+2}+r^{\alpha'+\varepsilon'+1}+r^{\delta'+\varepsilon'+\eta'}.$$

Consider $-r^{\alpha'+1}$ in $Q(H)$. It is due to $\alpha'>\alpha$ (from (16)) and $\eta<\alpha'$ (from (17)) that $-r^{\alpha'+1}$ must be canceled by $r^{\delta'+2}$ or $r^{\delta'+\varepsilon'+\eta'}$. Thus, $\alpha'+1=\delta'+2$ or $\alpha'+1=\delta'+\varepsilon'+\eta'$. If $\alpha'+1=\delta'+2$, then we have a contradiction since no pair of terms in $Q(G)$ are equal to $-r^{\alpha'}$ and $-r^{\delta'+1}$ (by noting $\eta<\alpha'$). If $\alpha'+1=\delta'+\varepsilon'+\eta'$, then $\alpha'=\delta'+\alpha-1$ since $\alpha=\varepsilon'+\eta'$ (from (15)). Since $\delta'\geqslant\eta'+1$ (from (10)) and $\eta'\geqslant 2$, we have $\delta'\geqslant 3$. Then $\alpha'=\delta'+\alpha-1\geqslant\alpha+2$. This is a contradiction since no term in $Q(G)$ is equal to $-r^{\alpha'}$ (noting $\eta<\alpha'$) and nothing in $Q(H)$ can cancel $-r^{\alpha'}$ (by noting $\alpha'+1=\delta'+\varepsilon'+\eta'\geqslant\delta'+4$).

So far, we have solved the equation $P(G)=P(H)$ and got the solution as follows:

$$k_4(a,1,1,a+b+1,b,b+1)\sim K_4(a,1,1,b,b+2,a+b),$$

$$k_4(a+1,1,1,a+3,2,a)\sim K_4(a+2,1,1,a,2,a+2).$$

The proof is completed. □

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