J. Math. Anal. Appl. 379 (2011) 272–289

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On a class of J-self-adjoint operators with empty resolvent set

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ARTICLE INFO

Article history:
Received 9 September 2010
Available online 24 December 2010
Submitted by Steven G. Krantz

Keywords:
Krein spaces
J-self-adjoint operators
Empty resolvent set
Stable C-symmetry
Sturm–Liouville operators

ABSTRACT

In the present paper we investigate the set Σ_J of all J-self-adjoint extensions of an operator S which is symmetric in a Hilbert space $\mathfrak{H}$ with deficiency indices $\langle 2, 2\rangle$ and which commutes with a non-trivial fundamental symmetry J of a Krein space $(\mathfrak{H}, [\cdot,\cdot])$,

$$SJ = JS.$$

Our aim is to describe different types of J-self-adjoint extensions of S, which, in general, are non-self-adjoint operators in the Hilbert space $\mathfrak{H}$. One of our main results is the equivalence between the presence of J-self-adjoint extensions of S with empty resolvent set and the commutation of S with a Clifford algebra $Cl_2(J, R)$, where R is an additional fundamental symmetry with $JR = -RJ$. This enables one to parameterize in terms of $Cl_2(J, R)$ the set of all J-self-adjoint extensions of S with stable C-symmetry. Here an extension has stable C-symmetry if it commutes with a fundamental symmetry and, in turn, this fundamental symmetry commutes with S. Such a situation occurs naturally in many applications, here we discuss the case of indefinite Sturm–Liouville operators and the case of a one-dimensional Dirac operator with point interaction.

1. Introduction

Let $(\mathfrak{H}, [\cdot,\cdot])$ be a Krein space with a non-trivial fundamental symmetry J (i.e., $J^2 = I$, $J \neq \pm I$, and $(\mathfrak{H}, [J\cdot,\cdot])$ is a Hilbert space) and corresponding fundamental decomposition

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \tag{1.1}$$

where $\mathfrak{H}_\pm = \frac{1}{2}(I \pm J)\mathfrak{H}$. Let A be a linear operator in $\mathfrak{H}$ which is J-self-adjoint with respect to the Krein space inner product $[\cdot,\cdot]$. In general, J-self-adjoint operators A are non-self-adjoint in the Hilbert space $(\mathfrak{H}, [J\cdot,\cdot])$ and their spectra $\sigma(A)$ are only symmetric with respect to the real axis: $\mu \in \sigma(A)$ if and only if $\overline{\mu} \in \sigma(A)$. Moreover, the situation where $\sigma(A) = \mathbb{C}$ (i.e., A has the empty resolvent set) is also possible.

It is simple to construct infinitely many J-self-adjoint operators with empty resolvent set. For instance, let $\mathcal{K}$ be a Hilbert space and let L be a closed symmetric (non-self-adjoint) operator in $\mathcal{K}$. Consider the operators

$$A := \begin{pmatrix} L & 0 \\ 0 & L^* \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

in the product Hilbert space $\mathfrak{H} = \mathcal{K} \oplus \mathcal{K}$. Then J is a fundamental symmetry in $\mathfrak{H}$ and A is a J-self-adjoint operator. As $\rho(L) = \emptyset$, it is clear that $\rho(A) = \emptyset$.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.12.048

This example shows that the property $\rho(A)=\emptyset$ is a consequence of the special structure of A. It is natural to suppose that this relationship can be made more exact for some special types of J-self-adjoint operators.

In the present paper we investigate a closed symmetric operator S in the Hilbert space $\mathfrak{H}$ with inner product $(\cdot,\cdot)=[J\cdot,\cdot]$. We assume the deficiency indices of S to be $\langle 2,2\rangle$ and we assume that S commutes with the fundamental symmetry J,

$$SJ=JS. \tag{1.2}$$

Hence, S is simultaneously symmetric and J-symmetric.

Our aim is to describe different types of J-self-adjoint extensions of S. For this let Σ_J be the set of all J-self-adjoint extensions of S and let us denote (see Section 2.4 below) by $\mathfrak{U}$ the set of all fundamental symmetries which commute with S, by Σ_J^{st} we denote the set of all J-self-adjoint extensions of S which commute with a fundamental symmetry in $\mathfrak{U}$, by Υ_J the set of all J-self-adjoint extensions of S which commute with J and by $\Upsilon_{\mathfrak{U}}$, the set of all J-self-adjoint extensions which commute with all operators in $\mathfrak{U}$. By definition we have $J\in\mathfrak{U}$ and

$$\Upsilon_{\mathfrak{U}}\subset\Upsilon_J\subset\Sigma_J^{st}. \tag{1.3}$$

Operators from Σ_J^{st} are said to have the property of stable C-symmetry, see [24]. In particular, they are fundamentally reducible and, hence, similar to self-adjoint operators in Hilbert spaces. Therefore, J-self-adjoint operators with stable C-symmetries admit detailed spectral analysis, see also [2,20], and the set Σ_J^{st} may be useful for the explanation of exceptional points phenomenon in $\mathcal{PT}$-symmetric quantum mechanics (see [9,18,29,30] and the references therein).

In the case of a simple symmetric operator S, we show in this paper that the existence of at least one J-self-adjoint extension of S with empty resolvent set leads to the quite specific structure of the underlying symmetric operator S. Namely, we have in (1.3) strict inclusions,

$$\Upsilon_{\mathfrak{U}}\subset\Upsilon_J\subset\Sigma_J^{st}\quad\bigl(\Upsilon_{\mathfrak{U}}\neq\Upsilon_J\neq\Sigma_J^{st}\bigr),$$

and it follows from the definition of the classes $\Upsilon_{\mathfrak{U}}$, Υ_J and Σ_J^{st} that we have a rich structure of different extensions of S. Moreover, in Corollary 4.7 and Theorem 4.8 below we give a full parametrization of the sets $\Upsilon_{\mathfrak{U}}$, Υ_J and Σ_J^{st} in terms of (up to) four real parameters.

If, on the other hand, all J-self-adjoint extension of S have non-empty resolvent set, we show (cf. Theorem 4.1 below) equality in (1.3),

$$\Upsilon_{\mathfrak{U}}=\Upsilon_J=\Sigma_J^{st}.$$

Moreover, we have $\mathfrak{U}=\{J\}$. This is in particular the case, if there exists at least one definitizable extension (see Corollary 4.2 below).

We show that the existence of at least one J-self-adjoint extension of S with empty resolvent set is equivalent to one of the following statements.

- There exists an additional fundamental symmetry R in $\mathfrak{H}$ such that
$$SR=RS,\qquad JR=-RJ.$$
- The operator $S_+:=S\restriction_{\mathfrak{H}_+}$ is unitarily equivalent to $S_-:=S\restriction_{\mathfrak{H}_-}$, where $\mathfrak{H}_\pm$ are from the fundamental decomposition (1.1) corresponding to J.
- The characteristic function s_+ of S_+ (in the sense of A. Straus, see [32]) is equal (up to the multiplication by a unimodular constant) to the characteristic function s_- of S_-.

If, in addition, the characteristic function of S is not identically equal to zero, we provide a complete description of the set $\mathfrak{U}$ in terms of R and J. More precisely (see Theorem 4.6 below), $\mathfrak{U}$ consists of all operators C of the form

$$C=(\cosh\chi)J+(\sinh\chi)JR\bigl[\cos\omega+i(\sin\omega)J\bigr]$$

with $\chi\in\mathbb{R}$ and $\omega\in[0,2\pi)$.

The operators J and R can be interpreted as basis (generating) elements of the complex Clifford algebra $\mathcal{C}l_2(J,R):=\operatorname{span}\{I,J,R,JR\}$ and they give rise to a 'rich' family Σ_J^{st}. It is the consequence of the results of the present paper that the existence of J-self-adjoint extensions with empty resolvent set for a symmetric operator S with property (1.2) and deficiency indices $\langle 2,2\rangle$ is equivalent to the commutation of S with an arbitrary element of the Clifford algebra $\mathcal{C}l_2(J,R)$.

The paper is structured as follows. Section 2 contains auxiliary results related to the Krein space theory and the extension theory of symmetric operators. In the latter case we emphasize the usefulness of the Krein spaces ideology for the description of the set Σ_J of J-self-adjoint extensions of S in terms of unitary 2×2-matrices U and the definition of the characteristic function of S.

In Section 3, we establish a necessary and sufficient condition under which Σ_J contains operators with empty resolvent set (Theorem 3.1 and Corollary 3.3) and explicitly describe these operators in terms of unitary matrices U (Corollary 3.2).

In Section 4 we establish our main result (Theorem 4.3) about the equivalence between the presence of J-self-adjoint extensions of S with empty resolvent set and the commutation of S with a Clifford algebra $\mathcal{C}l_2(J,R)$. This enables one to construct the collection of operators $C_{\chi,\omega}$ realizing the property of stable C-symmetry for extensions $A \in \Sigma_J$ directly in terms of $\mathcal{C}l_2(J,R)$ (Theorem 4.6) and to describe the corresponding subset Σ_J^{st} of extensions $A \in \Sigma_J$ with stable C-symmetry in terms of matrices U (Corollary 4.7 and Theorem 4.8).

Section 5 contains some examples. In the case of a degenerated Sturm–Liouville expression on a finite interval we describe all J-self-adjoint extensions with an empty resolvent set. Moreover, we consider the case of an indefinite Sturm–Liouville expression on the real line. Imposing an additional boundary conditions at zero, the symmetric operator S is obtained as the orthogonal sum of two symmetric operators related to two differential expressions defined on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. Then with the results from Section 3 we are able to prove that all J-self-adjoint extensions of S have non-empty resolvent set. This extends some results from [7,8,12,21]. Finally, we consider a one-dimensional impulse and a Dirac operator with point perturbation.

Throughout the paper, the symbols $\mathcal{D}(A)$ and $\mathcal{R}(A)$ denote the domain and the range of a linear operator A. $A\upharpoonright_{\mathcal{D}}$ is the restriction of A onto a set $\mathcal{D}$. The notation $\sigma(A)$ and $\rho(A)$ are used for the spectrum and the resolvent set of A. The sign $\square$ denotes the end of a proof.

2. Preliminaries

2.1. Elements of the Krein space theory

Let $\mathfrak{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and with non-trivial fundamental symmetry J (i.e., $J = J^*$, $J^2 = I$, and $J \neq \pm I$). The space $\mathfrak{H}$ endowed with the indefinite inner product (indefinite metric) $[\cdot,\cdot] := (J\cdot,\cdot)$ is called a *Krein space* $(\mathfrak{H},[\cdot,\cdot])$. For the basic theory of Krein spaces and operators acting therein we refer to the monographs [4] and [11].

The projectors $P_\pm = \frac{1}{2}(I \pm J)$ determine *a fundamental decomposition* of $\mathfrak{H}$,

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-, \quad \mathfrak{H}_- = P_-\mathfrak{H},\ \mathfrak{H}_+ = P_+\mathfrak{H}, \tag{2.1}$$

where $(\mathfrak{H}_+,[\cdot,\cdot])$ and $(\mathfrak{H}_-,-[\cdot,\cdot])$ are Hilbert spaces. With respect to the fundamental decomposition (2.1), the operator J has the following form

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

A subspace $\mathfrak{L}$ of $\mathfrak{H}$ is called *hypermaximal neutral* if

$$\mathfrak{L} = \mathfrak{L}^{[\perp]} = \{x \in \mathfrak{H}:\ [x,y] = 0,\ \forall y \in \mathfrak{L}\}.$$

A subspace $\mathfrak{L} \subset \mathfrak{H}$ is called *uniformly positive (uniformly negative)* if $[x,x] \geqslant a^2\|x\|^2$ (resp. $-[x,x] \geqslant a^2\|x\|^2$) $a \in \mathbb{R}$, $a \neq 0$, for all $x \in \mathfrak{L}$. The subspaces $\mathfrak{H}_\pm$ in (2.1) are examples of uniformly positive and uniformly negative subspaces and, moreover, they are maximal, i.e., $\mathfrak{H}_+$ ($\mathfrak{H}_-$) is not a proper subspace of a uniformly positive (resp. negative) subspace.

Let $\mathfrak{L}_+ (\neq \mathfrak{H}_+)$ be an arbitrary maximal uniformly positive subspace. Then its J-orthogonal complement $\mathfrak{L}_- = \mathfrak{L}_+^{[\perp]}$ is maximal uniformly negative and the direct J-orthogonal sum

$$\mathfrak{H} = \mathfrak{L}_+ [\dot{+}] \mathfrak{L}_- \tag{2.2}$$

gives a fundamental decomposition of $\mathfrak{H}$.

With respect to (2.2) we define an operator C via

$$C = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

We have $C^2 = I$ and C is a self-adjoint operator in the Hilbert space $(\mathfrak{H},(\cdot,\cdot)_C)$, where the inner product $(\cdot,\cdot)_C$ is given by

$$(x,y)_C := [Cx,y] = (JCx,y), \quad x,y \in \mathfrak{H}.$$

Note that $(\cdot,\cdot)_C$ and $(\cdot,\cdot)$ are equivalent, see, e.g., [27]. Hence, one can view C as a fundamental symmetry of the Krein space $(\mathfrak{H},[\cdot,\cdot])$ with an underlying Hilbert space $(\mathfrak{H},(\cdot,\cdot)_C)$.

Summing up, there is a one-to-one correspondence between the set of all decompositions (2.2) of the Krein space $(\mathfrak{H},[\cdot,\cdot])$ and the set of all bounded operators C such that

$$C^2 = I, \qquad JC > 0. \tag{2.3}$$

Definition 2.1. An operator A acting in a Krein space $(\mathfrak{H},[\cdot,\cdot])$ has the property of C-symmetry if there exists a bounded linear operator C in $\mathfrak{H}$ such that:

(i) $C^2 = I$;
(ii) $JC > 0$;
(iii) $AC = CA$.

In particular, if A is a J-self-adjoint operator with the property of C-symmetry, then its counterparts

$$A_\pm := A\restriction_{\mathfrak{L}_\pm}, \qquad \mathfrak{L}_\pm = \frac{1}{2}(I \pm C)\mathfrak{H}$$

are self-adjoint operators in the Hilbert spaces $\mathfrak{L}_+$ and $\mathfrak{L}_-$ endowed with the inner products $[\cdot,\cdot]$ and $-[\cdot,\cdot]$, respectively. This simple observation leads to the following statement, which is a direct consequence of the Phillips theorem [4, Chapter 2, Corollary 5.20].

Proposition 2.2. *A J-self-adjoint operator A has the property of C-symmetry if and only if A is similar to a self-adjoint operator in $\mathfrak{H}$.*

In conclusion, we emphasize that the notion of C-symmetry in Definition 2.1 coincides with the notion of fundamentally reducible operator (see, e.g., [19]). However, in the context of this paper and motivated by [2,3,9,10,17,29,30], we prefer to use the notion of C-symmetry.

2.2. Elements of the extension theory in Hilbert spaces

Here and in the following we denote by $\mathbb{C}_+$ ($\mathbb{C}_-$) the open upper (resp. lower) half plane. Let S be a closed symmetric densely defined operator with equal deficiency indices acting in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$.

We denote by $\mathfrak{N}_\mu = \ker(S^* - \mu I)$, $\mu \in \mathbb{C} \setminus \mathbb{R}$, the defect subspaces of S and consider the Hilbert space $\mathfrak{M} = \mathfrak{N}_i \dot{+} \mathfrak{N}_{-i}$ with the inner product

$$(x, y)_{\mathfrak{M}} = 2\big[(x_i, y_i) + (x_{-i}, y_{-i})\big], \tag{2.4}$$

where $x = x_i + x_{-i}$ and $y = y_i + y_{-i}$ with $x_i, y_i \in \mathfrak{N}_i$, $x_{-i}, y_{-i} \in \mathfrak{N}_{-i}$.

The operator Z which acts as identity operator I on $\mathfrak{N}_i$ and minus identity operator $-I$ on $\mathfrak{N}_{-i}$ is an example of a fundamental symmetry in $\mathfrak{M}$.

According to the von-Neumann formulas (see, e.g., [31,23]) any closed intermediate extension A of S (i.e., $S \subset A \subset S^*$) in the Hilbert space $(\mathfrak{H}, (\cdot,\cdot))$ is uniquely determined by the choice of a subspace $M \subset \mathfrak{M}$:

$$A = S^*\restriction_{\mathcal{D}(A)}, \quad \mathcal{D}(A) = \mathcal{D}(S) \dot{+} M. \tag{2.5}$$

Let us set $M = \mathfrak{N}_\mu$ ($\mu \in \mathbb{C}_+$) in (2.5) and denote by

$$A_\mu = S^*\restriction_{\mathcal{D}(A_\mu)}, \quad \mathcal{D}(A_\mu) = \mathcal{D}(S) \dot{+} \mathfrak{N}_\mu, \ \forall \mu \in \mathbb{C}_+ \tag{2.6}$$

the corresponding maximal dissipative extensions of S. The operator-function

$$\mathsf{Sh}(\mu) = (A_\mu - iI)(A_\mu + iI)^{-1}\restriction_{\mathfrak{N}_i} : \mathfrak{N}_i \to \mathfrak{N}_{-i}, \quad \mu \in \mathbb{C}_+ \tag{2.7}$$

is the characteristic function of S defined by A. Straus, see [32].

The characteristic function $\mathsf{Sh}(\cdot)$ is connected with the Weyl function of the symmetric operator S constructed in terms of boundary triplets (see [13, p. 12], [16, p. 1123]). For instance, if $M(\cdot)$ is the Weyl function of S associated with the boundary triplet $(\mathfrak{N}_i, \Gamma_0, \Gamma_1)$, where

$$\Gamma_0 f = f_i + V f_{-i}, \qquad \Gamma_1 f = i f_i - i V f_{-i}, \quad f = u + f_i + f_{-i} \in \mathcal{D}\big(S^*\big) \tag{2.8}$$

and $V : \mathfrak{N}_{-i} \to \mathfrak{N}_i$ is an arbitrary unitary mapping, then

$$M(\mu) = i\big(I + V\mathsf{Sh}(\mu)\big)\big(I - V\mathsf{Sh}(\mu)\big)^{-1}, \quad \mu \in \mathbb{C}_+. \tag{2.9}$$

The function $V\mathsf{Sh}(\cdot)$ in (2.9) coincides with the characteristic function of S associated with the boundary triplet $(\mathfrak{N}_i, \Gamma_0, \Gamma_1)$, cf. [25].

Another (equivalent) definition of $\mathsf{Sh}(\cdot)$ (see [32]) is based on the relation

$$\mathcal{D}(A_\mu) = \mathcal{D}(S) \dot{+} \mathfrak{N}_\mu = \mathcal{D}(S) \dot{+} \big(I - \mathsf{Sh}(\mu)\big)\mathfrak{N}_i, \quad \mu \in \mathbb{C}_+, \tag{2.10}$$

which also allows one to uniquely determine $\mathsf{Sh}(\cdot)$.

The characteristic function $\mathsf{Sh}(\cdot)$ can be easily interpreted in the Krein space setting. Indeed, according to the von-Neumann formulas, $\mathcal{D}(A_\mu) = \mathcal{D}(S) \dot{+} L_\mu$, where $L_\mu \subset \mathfrak{M}$ is a maximal uniformly positive subspace in the Krein space $(\mathfrak{M}, [\cdot,\cdot]_Z)$. Using (2.10), we conclude that $L_\mu = (I - \mathsf{Sh}(\mu))\mathfrak{N}_i$ and hence, $-\mathsf{Sh}(\mu)$ is the angular operator of L_μ with respect to the maximal uniformly positive subspace $\mathfrak{N}_i$ of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_Z)$ (see [4] for the concept of angular operators).

2.3. Elements of the extension theory in Krein spaces

In what follows we assume that S satisfies (1.2), where J is a fundamental symmetry in $(\mathfrak{H}, (\cdot,\cdot))$.

The condition (1.2) immediately leads to the special structure of S with respect to the fundamental decomposition (2.1):

$$S = \begin{pmatrix} S_+ & 0 \\ 0 & S_- \end{pmatrix}, \quad S_+ = S\restriction_{\mathfrak{H}_+}, \ S_- = S\restriction_{\mathfrak{H}_-}, \tag{2.11}$$

where $S_\pm$ are closed symmetric densely defined operators in $\mathfrak{H}_\pm$.

Denote by Σ_J *the collection of all J-self-adjoint extensions of S* and set

$$\Upsilon_J = \{A \in \Sigma_J \mid AJ = JA\}. \tag{2.12}$$

It is clear that $\Upsilon_J \subset \Sigma_J$ and an arbitrary $A \in \Upsilon_J$ is a simultaneously self-adjoint and J-self-adjoint extension of S. The set Υ_J *is non-empty if and only if each symmetric operator $S_\pm$ in* (2.11) *has equal deficiency indices*. We always suppose that $\Upsilon_J \neq \emptyset$.

Since S satisfies (1.2) *the subspaces $\mathfrak{N}_{\pm i}$ reduce J* and the restriction $J \restriction \mathfrak{M}$ gives rise to a fundamental symmetry in the Hilbert space $\mathfrak{M}$. Moreover, according to the properties of Z mentioned above, $JZ = ZJ$ and JZ is a fundamental symmetry in $\mathfrak{M}$. Therefore, the sesquilinear form

$$[x, y]_{JZ} = (JZx, y)_{\mathfrak{M}} = 2\big[(Jx_i, y_i) - (Jx_{-i}, y_{-i})\big] \tag{2.13}$$

defines an indefinite metric on $\mathfrak{M}$.

It is known (see, e.g., [2, Proposition 3.1]) that an arbitrary J-self-adjoint extension A of S is uniquely determined by (2.5), where M is a *hypermaximal neutral subspace* of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$.

In comparison with self-adjoint extensions in the sense of Hilbert spaces, we remark that *self-adjoint extensions* of S in $(\mathfrak{H}, (\cdot,\cdot))$ are also determined by (2.5) but then subspaces M are assumed to be hypermaximal neutral in the Krein space $(\mathfrak{M}, [\cdot,\cdot]_Z)$ with the indefinite metric (cf. (2.13))

$$[x, y]_Z = (Zx, y)_{\mathfrak{M}} = 2\big[(x_i, y_i) - (x_{-i}, y_{-i})\big].$$

2.4. J-self-adjoint operators with stable C-symmetries

Denote by $\mathfrak{U}$ the set of *all possible C-symmetries* of the closed symmetric operator S. By Definition 2.1, this means that

$$C \in \mathfrak{U} \iff C^2 = I, \qquad JC > 0, \qquad SC = CS.$$

The next result follows directly from [2]. We repeat principal stages for the reader's convenience.

Lemma 2.3. *The set $\mathfrak{U}$ is non-empty and $C \in \mathfrak{U}$ if and only if $C^* \in \mathfrak{U}$.*

Proof. It follows from (1.2) that $J \in \mathfrak{U}$. Therefore, $\mathfrak{U} \neq \emptyset$.

Let $C \in \mathfrak{U}$. The conditions $C^2 = I$ and $JC > 0$ are equivalent to the presentation $C = Je^Y$, where Y is a bounded self-adjoint operator in $\mathfrak{H}$ such that $JY = -YJ$, see [2, Remark 2.1]. In that case $C^* = Je^{-Y}$ and, obviously, C^* satisfies the relations $C^{*2} = I$ and $JC^* > 0$.

Since S commutes with J and C one gets $Se^Y = e^Y S$. But then $SC^* = Se^Y J = e^Y JS = C^*S$. Hence, $C^* \in \mathfrak{U}$. □

Definition 2.4. (See [24].) An operator $A \in \Sigma_J$ has the property of stable C-symmetry if A and S have the property of C-symmetry realized by the *same* operator C, i.e., there exists $C \in \mathfrak{U}$ with $AC = CA$.

Denote

$$\Sigma_J^{st} = \{A \in \Sigma_J \mid \exists C \in \mathfrak{U} \text{ such that } AC = CA\}. \tag{2.14}$$

Due to Definition 2.4, Σ_J^{st} consists of J-self-adjoint extensions A of S with the property of stable C-symmetry. It follows from (2.12) and (2.14) that $\Sigma_J^{st} \supset \Upsilon_J$. Hence, Σ_J^{st} is non-empty.

Denote

$$\Upsilon_{\mathfrak{U}} = \{A \in \Sigma_J \mid AC = CA, \ \forall C \in \mathfrak{U}\}. \tag{2.15}$$

It is clear that

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st} \subset \Sigma_J. \tag{2.16}$$

The next theorem gives a condition for the non-emptiness of the left-hand side of the chain (2.16).

Theorem 2.5. *If the characteristic function* $\mathrm{Sh}(\cdot)$ *of* S *is boundedly invertible for at least one* $\mu \in \mathbb{C}_+$*, then* $\Upsilon_{\mathfrak{U}} \neq \emptyset$*.*

Proof. Let $C \in \mathfrak{U}$. Then $S^*C = CS^*$ (see the proof of Lemma 2.3) and, hence,

$$C : \mathfrak{N}_\mu \to \mathfrak{N}_\mu, \quad \forall \mu \in \mathbb{C} \setminus \mathbb{R}. \tag{2.17}$$

Therefore, $A_\mu C = CA_\mu$ for maximal dissipative extensions A_μ of S (see (2.6)). This means that the characteristic function $\mathrm{Sh}(\cdot)$ defined by (2.7) commutes with an arbitrary $C \in \mathfrak{U}$, i.e.,

$$\mathrm{Sh}(\mu)C = C\mathrm{Sh}(\mu), \quad \forall \mu \in \mathbb{C}_+, \ \forall C \in \mathfrak{U}. \tag{2.18}$$

It follows from Lemma 2.3 and (2.18) that $\mathrm{Sh}(\mu)C^* = C^*\mathrm{Sh}(\mu)$. Therefore,

$$\mathrm{Sh}^*(\mu)C = C\mathrm{Sh}^*(\mu), \quad \forall \mu \in \mathbb{C}_+, \ \forall C \in \mathfrak{U}. \tag{2.19}$$

Let $\mathrm{Sh}(\mu)$ be boundedly invertible for a certain $\mu \in \mathbb{C}_+$ and let $V : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ be the isometric factor in the polar decomposition of $\mathrm{Sh}(\mu)$. Then $VC = CV$ for all $C \in \mathfrak{U}$ (since (2.18) and (2.19)). This means that the operator

$$A = S^* \restriction_{\mathcal{D}(A)}, \quad \mathcal{D}(A) = \mathcal{D}(S) \dot{+} \{(I+V)\mathfrak{N}_i\}$$

belongs to $\Upsilon_{\mathfrak{U}}$. □

Remark 2.6. A similar result was established by Kochubei [26, Theorem 1] for the collection of unitary operators $\mathfrak{U} = \{U\}$ with the property that $U \in \mathfrak{U}$ implies $U^* \in \mathfrak{U}$.

According to (2.17), an arbitrary $C \in \mathfrak{U}$ determines two operators $C \restriction_{\mathfrak{N}_{\pm i}}$ acting in $\mathfrak{N}_{\pm i}$.

Lemma 2.7. *If* S *is a simple closed symmetric operator, then the correspondence* $C \in \mathfrak{U} \to \{C \restriction_{\mathfrak{N}_i}, C \restriction_{\mathfrak{N}_{-i}}\}$ *is injective.*

Proof. Assume the existence of an operator pair $\{C \restriction_{\mathfrak{N}_i}, C \restriction_{\mathfrak{N}_{-i}}\}$ for two different operators $C, \widetilde{C} \in \mathfrak{U}$. Then $(C - \widetilde{C})\mathcal{D}(S^*) \subset \mathcal{D}(S)$. Therefore, $(C - \widetilde{C})\mathfrak{N}_\mu \subset \mathcal{D}(S)$. On the other hand, $(C - \widetilde{C})\mathfrak{N}_\mu \subset \mathfrak{N}_\mu$ by (2.17). The obtained relations yield $Cf_\mu = \widetilde{C}f_\mu$ for any $f_\mu \in \mathfrak{N}_\mu$ and $\mu \in \mathbb{C} \setminus \mathbb{R}$. This means that $C = \widetilde{C}$ (since the symmetric operator S is simple). □

3. Necessary and sufficient condition under which Σ_J contains elements with empty resolvent set

In what follows we assume that the deficiency indices of the operators $S_\pm$ in (2.11) are $\langle 1, 1 \rangle$. In that case, the defect subspaces $\mathfrak{N}_{\pm i}(S_\pm)$ of $S_\pm$ are one-dimensional and

$$\mathfrak{N}_i(S_+) = (I+Z)(I+J)\mathfrak{M}; \qquad \mathfrak{N}_{-i}(S_+) = (I-Z)(I+J)\mathfrak{M};$$
$$\mathfrak{N}_i(S_-) = (I+Z)(I-J)\mathfrak{M}; \qquad \mathfrak{N}_{-i}(S_-) = (I-Z)(I-J)\mathfrak{M}.$$

Hence, $\mathfrak{N}_{\pm i}(S_\pm)$ are orthogonal in the Hilbert space $(\mathfrak{M}, (\cdot,\cdot)_{\mathfrak{M}})$ (see (2.4)).

Let $\{e_{++}, e_{+-}, e_{-+}, e_{--}\}$ be an orthogonal basis of $\mathfrak{M}$ such that

$$\begin{aligned} \mathfrak{N}_i(S_+) &= \ker\bigl(S_+^* - iI\bigr) = \operatorname{span}\{e_{++}\}, \\ \mathfrak{N}_i(S_-) &= \ker\bigl(S_-^* - iI\bigr) = \operatorname{span}\{e_{+-}\}, \\ \mathfrak{N}_{-i}(S_+) &= \ker\bigl(S_+^* + iI\bigr) = \operatorname{span}\{e_{-+}\}, \\ \mathfrak{N}_{-i}(S_-) &= \ker\bigl(S_-^* + iI\bigr) = \operatorname{span}\{e_{--}\}, \end{aligned} \tag{3.1}$$

and the elements e_{++}, e_{+-}, e_{-+}, e_{--} have *equal* norms in $\mathfrak{M}$. It follows from the definition of $e_{\pm\pm}$ that

$$\begin{aligned} &Ze_{++} = e_{++}, \qquad Ze_{+-} = e_{+-}, \qquad Ze_{-+} = -e_{-+}, \qquad Ze_{--} = -e_{--}, \\ &Je_{++} = e_{++}, \qquad Je_{+-} = -e_{+-}, \qquad Je_{-+} = e_{-+}, \qquad Je_{--} = -e_{--}. \end{aligned} \tag{3.2}$$

Relations (3.2) mean that the fundamental decomposition of the Krein space $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ has the form

$$\mathfrak{M} = \mathfrak{M}_- \oplus \mathfrak{M}_+, \qquad \mathfrak{M}_- = \operatorname{span}\{e_{+-}, e_{-+}\}, \qquad \mathfrak{M}_+ = \operatorname{span}\{e_{++}, e_{--}\}. \tag{3.3}$$

According to the general theory of Krein spaces [4, Chapter 1, Theorem 8.10], an arbitrary hypermaximal neutral subspace M of $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ is uniquely determined by a unitary mapping of $\mathfrak{M}_-$ onto $\mathfrak{M}_+$. Since $\dim \mathfrak{M}_\pm = 2$ the set of unitary mappings $\mathfrak{M}_- \to \mathfrak{M}_+$ is in one-to-one correspondence with the set of unitary matrices

$$U = e^{i\phi}\begin{pmatrix} qe^{i\gamma} & re^{i\xi} \\ -re^{-i\xi} & qe^{-i\gamma} \end{pmatrix}, \qquad q^2 + r^2 = 1, \quad q, r \in \mathbb{R}_+,\ \phi, \gamma, \xi \in [0, 2\pi). \tag{3.4}$$

In other words, formulas (3.3), (3.4) allow one to describe a hypermaximal neutral subspace M of $(\mathfrak{M}, [\cdot,\cdot]_{JZ})$ as a linear span

$$M = \operatorname{span}\{d_1, d_2\} \tag{3.5}$$

of elements

$$\begin{aligned} d_1 &= e_{++} + qe^{i(\phi+\gamma)}e_{+-} + re^{i(\phi+\xi)}e_{-+}, \\ d_2 &= e_{--} - re^{i(\phi-\xi)}e_{+-} + qe^{i(\phi-\gamma)}e_{-+}. \end{aligned} \tag{3.6}$$

This means that (3.4)–(3.6) establish a one-to-one correspondence between domains $\mathcal{D}(A) = \mathcal{D}(S)\dot{+}M$ of J-self-adjoint extensions A of S and unitary matrices U. To underline this relationship *we will use the notation A_U for the corresponding J-self-adjoint extension A*.

It follows from (2.18) (with $C = J$) that the characteristic function $\mathsf{Sh}(\cdot) : \mathfrak{N}_i \to \mathfrak{N}_{-i}$ commutes with J. Combining this fact with the obvious presentations

$$\begin{aligned} \mathfrak{N}_i &= \mathfrak{N}_i(S_+) \oplus \mathfrak{N}_i(S_-) = \operatorname{span}\{e_{++}, e_{+-}\}, \\ \mathfrak{N}_{-i} &= \mathfrak{N}_{-i}(S_+) \oplus \mathfrak{N}_{-i}(S_-) = \operatorname{span}\{e_{-+}, e_{--}\} \end{aligned} \tag{3.7}$$

and relations (2.10), (3.2), we arrive at the conclusion that

$$\mathsf{Sh}(\mu)e_{++} = s_+(\mu)e_{-+}, \qquad \mathsf{Sh}(\mu)e_{+-} = s_-(\mu)e_{--}, \tag{3.8}$$

where s_j are holomorphic functions in $\mathbb{C}_+$. Moreover, it is easy to see that relations in (3.8) determine the characteristic functions

$$\mathsf{Sh}_+(\mu) : \mathfrak{N}_i(S_+) \to \mathfrak{N}_{-i}(S_+), \qquad \mathsf{Sh}_-(\mu) : \mathfrak{N}_i(S_-) \to \mathfrak{N}_{-i}(S_-) \tag{3.9}$$

of the symmetric operators S_+ and S_-, respectively.

We will use the notation

$$s_+ \approx s_-$$

if the identity $e^{i\alpha}s_+(\mu) = s_-(\mu)$ holds for all $\mu \in \mathbb{C}_+$ and for a certain choice of a unimodular constant $e^{i\alpha}$, i.e., the sign $\approx$ means equality up to the multiplication by a unimodular constant.

Theorem 3.1. *Assume that the deficiency indices of operators $S_\pm$ in the presentation* (2.11) *of S are $\langle 1, 1\rangle$. Then J-self-adjoint extensions of S with empty resolvent set exist if and only if $s_+ \approx s_-$.*

Proof. It follows from (2.10) that a J-self-adjoint extension A_U of S with the domain $\mathcal{D}(A_U) = \mathcal{D}(S)\dot{+}M$ has a non-real eigenvalue $\mu \in \mathbb{C}_+$ if and only if U has a non-trivial intersection with the subspace $L_\mu = (I - \mathsf{Sh}(\mu))\mathfrak{N}_i$. Therefore,

$$\sigma(A_U) \supset \mathbb{C}_+ \quad \text{if and only if} \quad M \cap L_\mu \neq \{0\} \quad \forall \mu \in \mathbb{C}_+.$$

Since A_U is a J-self-adjoint operator, the inclusion $\sigma(A_U) \supset \mathbb{C}_+$ is equivalent to $\sigma(A_U) = \mathbb{C}$.

In view of (3.7) and (3.8), $L_\mu = (I - \mathsf{Sh}(\mu))\mathfrak{N}_i = \operatorname{span}\{c_1(\mu), c_2(\mu)\}$, where

$$c_1(\mu) = e_{++} - s_+(\mu)e_{-+}, \qquad c_2(\mu) = e_{+-} - s_-(\mu)e_{--}. \tag{3.10}$$

Therefore, the relation $M \cap L_\mu \neq \{0\}$ holds if and only if the equation

$$x_1d_1 + x_2d_2 = y_1c_1(\mu) + y_2c_2(\mu) \tag{3.11}$$

has a non-trivial solution $x_1, x_2, y_1, y_2 \in \mathbb{C}$ for all $\mu \in \mathbb{C}_+$. Substituting (3.6) and (3.10) into (3.11) and combining the corresponding coefficients for $e_{\pm\pm}$ we obtain four relations

$$\begin{aligned} &x_1 = y_1, \qquad x_1qe^{i(\phi+\gamma)} - x_2re^{i(\phi-\xi)} = y_2, \\ &x_2 = -y_2s_-(\mu), \qquad x_1re^{i(\phi+\xi)} + x_2qe^{i(\phi-\gamma)} = -s_+(\mu)y_1 \end{aligned}$$

or

$$\begin{aligned} &qe^{i(\phi+\gamma)}y_1 - \bigl(1 - re^{i(\phi-\xi)}s_-(\mu)\bigr)y_2 = 0, \\ &\bigl(re^{i(\phi+\xi)} + s_+(\mu)\bigr)y_1 - qe^{i(\phi-\gamma)}s_-(\mu)y_2 = 0. \end{aligned}$$

The last system has a non-trivial solution y_1, y_2 for all $\mu \in \mathbb{C}_+$ if and only if its determinant

$$\begin{vmatrix} qe^{i(\phi+\gamma)} & -1 + re^{i(\phi-\xi)}s_-(\mu) \\ re^{i(\phi+\xi)} + s_+(\mu) & -qe^{i(\phi-\gamma)}s_-(\mu) \end{vmatrix} = 0, \quad \forall \mu \in \mathbb{C}_+.$$

This is the case if and only if

$$e^{2i\phi}s_-(\mu) = re^{i(\phi+\xi)} + s_+(\mu) - re^{i(\phi-\xi)}s_-(\mu)s_+(\mu), \quad \forall \mu \in \mathbb{C}_+. \tag{3.12}$$

Further, $\mathsf{Sh}(i) = 0$ by the construction (see (2.7) or (2.10)). Hence $s_+(i) = s_-(i) = 0$ and relation (3.12) takes the form $re^{i(\phi+\xi)} = 0$ (for $\mu = i$) which means that $r = 0$. Therefore, an operator $A_U \in \Sigma_J$ has empty resolvent set if and only

$$e^{2i\phi}s_-(\mu) = s_+(\mu), \quad \forall \mu \in \mathbb{C}_+. \qquad \square \tag{3.13}$$

Corollary 3.2. *If $s_+ \approx s_-$, then the operators $A_U \in \Sigma_J$ with empty resolvent set are determined by the matrices*:

$$U = e^{i\phi}\begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}, \quad \gamma \in [0, 2\pi), \tag{3.14}$$

where $\phi \in [0, 2\pi)$ is uniquely determined by (3.13) *if $\mathsf{Sh} \not\equiv 0$ and ϕ is an arbitrary parameter if $\mathsf{Sh} \equiv 0$.*

Corollary 3.3. *Let S be a simple closed symmetric operator. Then Σ_J contains operators with empty resolvent set if and only if the operators $S_\pm$ in* (2.11) *are unitarily equivalent.*

Proof. Assume that Σ_J contains operators with empty resolvent set and $\mathsf{Sh} \not\equiv 0$. Then $s_+ \not\equiv 0$ and (3.13) holds for a certain $\phi \in [0, 2\pi)$. Consider unitary mappings $V_\pm : \mathfrak{N}_{-i}(S_\pm) \to \mathfrak{N}_i(S_\pm)$ defined by the relations

$$V_+e_{-+} = e_{++}, \qquad V_-e_{--} = e^{2i\phi}e_{+-}.$$

By virtue of (3.8) and (3.9), we get

$$\begin{aligned} &V_+\mathsf{Sh}_+(\mu)e_{++} = s_+(\mu)e_{++}, \\ &V_-\mathsf{Sh}_-(\mu)e_{+-} = e^{2i\phi}s_-(\mu)e_{+-} = s_+(\mu)e_{+-}. \end{aligned} \tag{3.15}$$

Then $V_+\mathsf{Sh}_+(\cdot)$ and $V_-\mathsf{Sh}_-(\cdot)$ are the characteristic functions (in the sense of [25]) of $S_\pm$ associated with the boundary triplets $(\mathfrak{N}_i(S_\pm), \Gamma_0, \Gamma_1)$ of $S_\pm^*$ defined by (2.8). Identifying the defect subspaces $\mathfrak{N}_i(S_+) = \operatorname{span}\{e_{++}\}$ and $\mathfrak{N}_i(S_-) = \operatorname{span}\{e_{+-}\}$ with $\mathbb{C}$ and using (3.15) we arrive at the conclusion that the characteristic functions of $S_\pm$ associated with the boundary triplets $(\mathbb{C}, \Gamma_0, \Gamma_1)$ coincide.

The same is true when $\mathsf{Sh} \equiv 0$. In that case, $s_+ \equiv s_- \equiv 0$ and the characteristic functions $\mathsf{Sh}_\pm$ of $S_\pm$ are equal to zero.

Since S is a simple symmetric operator, $S_\pm$ are also simple symmetric operators. In that case, the equality of characteristic functions of $S_\pm$ implies the unitary equivalence of $S_\pm$, see, e.g., [16,25].

Conversely, if $S_\pm$ are unitarily equivalent then $S_+ = W^{-1}S_-W$, where W is a unitary mapping of $\mathfrak{H}_+$ onto $\mathfrak{H}_-$. Therefore,

$$W : \mathfrak{N}_\mu(S_+) \to \mathfrak{N}_\mu(S_-) \quad \text{and} \quad W\mathsf{Sh}_+(\mu) = \mathsf{Sh}_-(\mu)W. \tag{3.16}$$

Assuming $\mu = \pm i$ in the first identity of (3.16) and using (3.7), we find $w_1, w_2 \in \mathbb{C}$ with

$$We_{++} = w_1e_{+-}, \qquad We_{-+} = w_2e_{--}, \qquad |w_1| = |w_2| = 1. \tag{3.17}$$

It follows from (3.8) and (3.17) that $W\mathsf{Sh}_+(\mu)e_{++} = s_+(\mu)We_{-+} = w_2s_+(\mu)e_{--}$ and $\mathsf{Sh}_-(\mu)We_{++} = w_1\mathsf{Sh}_-(\mu)e_{+-} = w_1s_-(\mu)e_{--}$. Combining the last two identities with the second relation in (3.16) we obtain $e^{2i\phi}s_-(\mu) = s_+(\mu)$, where $e^{2i\phi} = w_1/w_2$. The statement of Corollary 3.3 follows now from Theorem 3.1. $\square$

4. *J*-self-adjoint extensions with empty resolvent set

As above the deficiency indices of operators $S_\pm$ in the presentation (2.11) of S are supposed to be $\langle 1, 1\rangle$. In the following we discuss the different situations which can occur:

- no member of Σ_J has non-empty resolvent set;
- there are members of Σ_J with empty resolvent set. We discuss the cases $\mathsf{Sh}(\cdot) \not\equiv 0$ and $\mathsf{Sh}(\cdot) \equiv 0$ separately in the Sections 4.2.1 and 4.2.2 below.

4.1. The set Σ_J contains no operators with empty resolvent set

Theorem 4.1. *If Σ_J contains no operators with empty resolvent set, then*

$$\Upsilon_{\mathfrak{U}} = \Upsilon_J = \Sigma_J^{st}$$

in (2.16)*. Moreover, if S is a simple closed symmetric operator, then $\mathfrak{U} = \{J\}$.*

Proof. Let $C \in \mathfrak{U}$. It follows from (2.17) that the operator $C\restriction_{\mathfrak{N}_{\pm i}}$ acts in $\mathfrak{N}_{\pm i}$ and satisfies the relations

$$(C\restriction_{\mathfrak{N}_{\pm i}})^2 = I, \qquad JC\restriction_{\mathfrak{N}_{\pm i}} > 0. \tag{4.1}$$

Denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the 2×2-matrix representations of $C\restriction_{\mathfrak{N}_i}$ and $C\restriction_{\mathfrak{N}_{-i}}$ with respect to the orthogonal bases e_{++}, e_{+-} and e_{-+}, e_{--} of $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively. Then (4.1) takes the form

$$\mathcal{C}_j^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathcal{C}_j > 0, \quad j = 1, 2 \tag{4.2}$$

(since $J\restriction_{\mathfrak{N}_{\pm i}}$ are determined by (3.2)). The Hermiticity of the matrix in the second relation of (4.2) enables one to deduce that a matrix $\mathcal{C}_j$ satisfy (4.2) if and only if

$$\mathcal{C}_j = \mathcal{C}_{\chi_j,\omega_j} := \begin{pmatrix} \cosh\chi_j & (\sinh\chi_j)e^{-i\omega_j} \\ -(\sinh\chi_j)e^{i\omega_j} & -\cosh\chi_j \end{pmatrix}, \quad \chi_j \in \mathbb{R},\ \omega_j \in [0, 2\pi). \tag{4.3}$$

Combining (2.18) with (3.8) and (4.3) we get

$$\begin{aligned} &\begin{pmatrix} s_+(\mu) & 0 \\ 0 & s_-(\mu) \end{pmatrix}\begin{pmatrix} \cosh\chi_1 & (\sinh\chi_1)e^{-i\omega_1} \\ -(\sinh\chi_1)e^{i\omega_1} & -\cosh\chi_1 \end{pmatrix} \\ &\quad = \begin{pmatrix} \cosh\chi_2 & (\sinh\chi_2)e^{-i\omega_2} \\ -(\sinh\chi_2)e^{i\omega_2} & -\cosh\chi_2 \end{pmatrix}\begin{pmatrix} s_+(\mu) & 0 \\ 0 & s_-(\mu) \end{pmatrix} \end{aligned} \tag{4.4}$$

for matrix representations $\mathcal{C}_{\chi_j,\omega_j}$ of the operators $C\restriction_{\mathfrak{N}_{\pm i}}$.

If Σ_J has no operators with empty resolvent set, then $s_+ \not\approx s_-$ (Theorem 3.1). In that case identity (4.4) holds only in the case $\chi_1 = \chi_2 = 0$, i.e., $\mathcal{C}_{0,\omega_1} = \mathcal{C}_{0,\omega_2} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Therefore, if $s_+ \not\approx s_-$ then

$$C\restriction_{\mathfrak{N}_{\pm i}} = J\restriction_{\mathfrak{N}_{\pm i}}, \quad \forall C \in \mathfrak{U}. \tag{4.5}$$

Let us consider an arbitrary $A_U \in \Sigma_J^{st}$. Then $A_U C = CA_U$ for some choice of $C \in \mathfrak{U}$. It is known that $A_U C = CA_U$ if and only if $CM = M$, where M is defined by (3.5) and (3.6), cf. [2, Theorem 3.1]. This and (4.5) give $CM = M$ if and only if $JM = M$. Therefore, $A_U J = JA_U$ and $A_U \in \Upsilon_J$. Thus $\Upsilon_J = \Sigma_J^{st}$. The identity $\Upsilon_{\mathfrak{U}} = \Upsilon_J$ is verified in a similar manner.

If S is a simple symmetric operator, then $\mathfrak{U} = \{J\}$ due to Lemma 2.7 and relation (4.5). □

Recall, that a J-self-adjoint operator A in a Krein space $(\mathfrak{H}, [\cdot,\cdot])$ is called definitizable (see [27]) if $\rho(A) \neq \emptyset$ and there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geqslant 0$ for all $x \in \mathfrak{H}$.

Corollary 4.2. *If Σ_J contains at least one definitizable operator, then $\Upsilon_{\mathfrak{U}} = \Upsilon_J = \Sigma_J^{st}$.*

Proof. If $A \in \Sigma_J$ is definitizable then an arbitrary operator from Σ_J is also definitizable, see [5,6]. Therefore, Σ_J has no operators with empty resolvent sets. □

4.2. The set Σ_J contains operators with empty resolvent set

In that case two quite different arrangements for the sets $\Upsilon_{\mathfrak{U}}$, Υ_J, and Σ_J^{st} are possible and they will be discussed in Sections 4.2.1 and 4.2.2 below.

We recall that Σ_J contains operators with empty resolvent set if and only if $e^{i\alpha}s_+(\mu) = s_-(\mu)$, $\mu \in \mathbb{C}_+$, for a certain parameter $e^{i\alpha}$ (Theorem 3.1). Here, the functions $s_\pm(\cdot)$ are defined in (3.8) with the help of the elements $\{e_{\pm\pm}\}$ which are determined up to the multiplication with a unimodular constant. Therefore, without loss of generality, we may assume

$$s_+ = s_-. \tag{4.6}$$

Theorem 4.3. *Let S be a simple closed symmetric operator. Then the set Σ_J contains operators with empty resolvent set if and only if there exists a fundamental symmetry R (i.e., $R^2 = I$ and $R = R^*$) in $\mathfrak{H}$ such that*

$$SR = RS, \qquad JR = -RJ. \tag{4.7}$$

Proof. By virtue of Corollary 3.3, the existence of J-self-adjoint extensions of S with empty resolvent set implies that the symmetric operators $S_\pm$ in (2.11) are unitarily equivalent. Hence, $S_+ = W^{-1}S_-W$, where W is an isometric mapping of $\mathfrak{H}_+$ onto $\mathfrak{H}_-$. It is clear that the operator

$$R = \begin{pmatrix} 0 & W^{-1} \\ W & 0 \end{pmatrix} \tag{4.8}$$

determined with respect to the fundamental decomposition (2.1) is a fundamental symmetry in $\mathfrak{H}$ and satisfies (4.7).

Conversely, if (4.7) hold, then $S_+ = RS_-R$. Therefore $S_\pm$ are unitarily equivalent and Σ_J contains elements with empty resolvent set (Corollary 3.3). □

Remark 4.4. If the relations in (4.7) hold then the existence of J-self-adjoint extensions of S with empty resolvent set can be established without the assumption of *simplicity* of S in Theorem 4.3. Indeed, the operator S is reduced by the decomposition

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1, \quad \mathfrak{H}_1 = \bigcap_{\forall \mu \in \mathbb{C}\setminus\mathbb{R}} \mathcal{R}(S - \mu I), \tag{4.9}$$

where $\mathfrak{H}_1$ is the maximal subspace invariant for S on which the operator $S_1 = S\restriction_{\mathfrak{H}_1}$ is self-adjoint; the subspace $\mathfrak{H}_0$ coincides with the closed linear span of all $\ker(S^* - \mu I)$ and the restriction $S_0 := S\restriction_{\mathfrak{H}_0}$ is a simple closed symmetric operator in $\mathfrak{H}_0$, see, e.g., [15, p. 9].

If (4.7) hold, then the restrictions $J_0 = J\restriction_{\mathfrak{H}_0}$ and $R_0 = R\restriction_{\mathfrak{H}_0}$ are fundamental symmetries in $\mathfrak{H}_0$ and they satisfy (4.7) for S_0. Applying Theorem 4.3, we establish the existence of J_0-self-adjoint extensions of S_0 with empty resolvent set. Since an operator $A \in \Sigma_J$ has the decomposition $A = A_0 \oplus S_1$ with respect to (4.9), where A_0 is a J_0-self-adjoint extension of S_0, the set Σ_J contains J-self-adjoint operators with empty resolvent set.

However, we cannot drop the condition of simplicity of S in Theorem 4.3 for the inverse implication. In that case, the existence of a fundamental symmetry R_0 satisfying (4.7) for S_0 in $\mathfrak{H}_0$ is easily deduced from Theorem 4.3 but it is not clear how to extend R_0 to $\mathfrak{H}$ with preservation of the relations in (4.7).

From (4.7) one concludes that the four operators I, J, R, and JR are linearly independent. Hence, the operators J and R can be interpreted as basis (generating) elements of the complex Clifford algebra

$$\mathcal{C}l_2 = \operatorname{span}\{I, J, R, JR\}.$$

Corollary 4.5. *Let S satisfy (4.7) and let $\widetilde{J} \in \mathcal{C}l_2$ be a non-trivial fundamental symmetry in $\mathfrak{H}$. Then there exists $\widetilde{J}$-self-adjoint extensions of S with empty resolvent set.*

Proof. It is easy to see that an operator $\widetilde{J} \in \mathcal{C}l_2$ is a non-trivial fundamental symmetry in $\mathfrak{H}$ (i.e., $\widetilde{J}^2 = I$, $\widetilde{J} = \widetilde{J}^*$, and $\widetilde{J} \neq I$) if and only if

$$\widetilde{J} = \alpha_1 J + \alpha_2 R + \alpha_3 iJR, \qquad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1, \quad \alpha_j \in \mathbb{R}. \tag{4.10}$$

Denote $\widetilde{R} = \beta_1 J + \beta_2 R + \beta_3 iJR$, where $\sum \beta_j^2 = 1$, $\beta_j \in \mathbb{R}$. By virtue of (4.10), $\widetilde{R}$ is a fundamental symmetry in $\mathfrak{H}$ which commutes with S. Assuming $\sum \alpha_j\beta_j = 0$, we obtain $\widetilde{J}\widetilde{R} = -\widetilde{R}\widetilde{J}$. Since $\widetilde{J}$ is a fundamental symmetry in $\mathfrak{H}$ which commutes with S, the statement follows from Theorem 4.3. □

4.2.1. The case $\mathsf{Sh} \not\equiv 0$

Theorem 4.6. *Let S be a simple closed symmetric operator with non-zero characteristic function $\mathsf{Sh}(\cdot)$ and let the set Σ_J contains operators with empty resolvent set. Then all operators $C \in \mathfrak{U}$ have the form*

$$C := C_{\chi,\omega} = J\big[(\cosh\chi)I + (\sinh\chi)R_\omega\big], \tag{4.11}$$

where R satisfies (4.7), $R_\omega = Re^{i\omega J} = R[\cos\omega + i(\sin\omega)J]$, and $\chi \in \mathbb{R}$, $\omega \in [0, 2\pi)$.

Proof. First, we will show $C_{\chi,\omega} \in \mathfrak{U}$. Since Σ_J contains operators with empty resolvent set, there exists a unitary mapping $W : \mathfrak{H}_+ \to \mathfrak{H}_-$ such that $S_+ = W^{-1}S_-W$ (Corollary 3.3). This allows one to determine a fundamental symmetry R in $\mathfrak{H}$ with the help of formula (4.8).

By construction, the operator R satisfies (4.7). Therefore, the subspaces $\mathfrak{N}_{\pm i}$ reduce R. Let $\mathcal{R}_1 = (r_{ij}^1)_{i,j=1}^2$ and $\mathcal{R}_2 = (r_{ij}^2)_{i,j=1}^2$ be the matrix representations of $R\restriction_{\mathfrak{N}_i}$ and $R\restriction_{\mathfrak{N}_{-i}}$ with respect to the bases e_{++}, e_{+-} and e_{-+}, e_{--} of $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively. It follows from (3.17) and (4.8) that $\mathcal{R}_j = \begin{pmatrix} 0 & w_j^{-1} \\ w_j & 0 \end{pmatrix}$, where $|w_1| = |w_2| = 1$. Moreover, since we assume (4.6), the parameter ϕ in the proof of Corollary 3.3 is equal to zero and, hence, $w := w_1 = w_2$. The exact value of the unimodular constant w depends on the choice of W. Without loss of generality we may assume (multiplying W by a unimodular constant if it is necessarily) that $w = 1$. Then

$$\mathcal{R} := \mathcal{R}_1 = \mathcal{R}_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.12}$$

Let us consider the collections of all operators $C_{\chi,\omega}$ determined by (4.11) It is known that $C_{\chi,\omega} = Je^{\chi R_\omega}$, where $R_\omega = Re^{i\omega J} = R[\cos\omega + i(\sin\omega)J]$ is a fundamental symmetry in $\mathfrak{H}$, which anticommutes with J (i.e., $R_\omega J = -JR_\omega$), see [2]. Such a representation leads to the conclusion that $C^2_{\chi,\omega} = I$ and $JC_{\chi,\omega} > 0$. Moreover $SC_{\chi,\omega} = SC_{\chi,\omega}$ due to (1.2) and (4.7). Therefore, an arbitrary $C_{\chi,\omega}$ belongs to $\mathfrak{U}$.

Rewriting (4.11) as follows

$$C_{\chi,\omega} = (\cosh\chi)J + (\sinh\chi)(\cos\omega)JR - i(\sinh\chi)(\sin\omega)R$$

and using (4.12) we obtain that both matrix representations of $C_{\chi,\omega}\restriction\mathfrak{N}_i$ and of $C_{\chi,\omega}\restriction\mathfrak{N}_{-i}$ coincide with

$$\mathcal{C}_{\chi,\omega} = \begin{pmatrix} \cosh\chi & (\sinh\chi)e^{-i\omega} \\ -(\sinh\chi)e^{i\omega} & -\cosh\chi \end{pmatrix}.$$

Let $C \in \mathfrak{U}$. Then the matrix representations of its restrictions $C\restriction\mathfrak{N}_i$ and $C\restriction\mathfrak{N}_i$ coincide with $\mathcal{C}_{\chi_1,\omega_1}$ and $\mathcal{C}_{\chi_2,\omega_2}$ defined by (4.3). Furthermore, since $\mathsf{Sh}(\mu)C = C\mathsf{Sh}(\mu)$ (see (2.18)), the identity (4.4) holds. That is equivalent to the relations $\chi_1 = \chi_2$ and $e^{-i\omega_1} = e^{-i\omega_2}$ (since (4.6) is true and $s_+ \not\equiv 0$).

Setting $\chi = \chi_1 = \chi_2$ and $\omega = \omega_1$, one concludes that the matrix representations $\mathcal{C}_{\chi_j,\omega_j}$ coincides with $\mathcal{C}_{\chi,\omega}$. Therefore, $C = C_{\chi,\omega}$ due to Lemma 2.7. Thus, the collection of operators $\{C_{\chi,\omega}\}$ defined by (4.11) coincides with $\mathfrak{U}$. □

Combining Theorem 4.6 with [2, Theorem 3.2, Proposition 3.3], we immediately derive the following statement.

Corollary 4.7. *Let S and Σ_J satisfy the condition of Theorem* 4.6 *and let $A_U \in \Sigma_J$ be defined by* (3.4)–(3.6)*. Then the strict inclusions*

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st}$$

hold and the following relations are true.

(i) *A_U belongs to $\Upsilon_{\mathfrak{U}}$ if and only if*

$$U = e^{i\frac{\pi}{2}} \begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix}, \quad \xi \in [0, 2\pi);$$

(ii) *A_U belongs to Υ_J if and only if*

$$U = e^{i\phi} \begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix}, \quad \phi, \xi \in [0, 2\pi);$$

(iii) *A_U belongs to $\Sigma_J^{st} \setminus \Upsilon_J$ if and only if*

$$U = e^{i\phi} \begin{pmatrix} qe^{i\gamma} & re^{i\xi} \\ -re^{-i\xi} & qe^{-i\gamma} \end{pmatrix}, \quad \gamma, \xi \in [0, 2\pi),\ q, r > 0, \qquad q^2 + r^2 = 1,$$

where $0 < q < |\cos\phi|$. In that case the operator A_U has $C_{\chi,\omega}$-symmetry, where $\omega = \gamma$ and χ is determined by the relation $q = -\tanh\chi\cos\phi$.

4.2.2. The case $\mathsf{Sh} \equiv 0$

If $\mathsf{Sh} \equiv 0$, then $s_+(\mu) = s_-(\mu) = 0$ for all $\mu \in \mathbb{C}_+$. Therefore, by Theorem 3.1, Σ_J contains operators with empty resolvent set and Theorem 4.3 and Corollary 4.5 hold. However Theorem 4.6 is not true due to the fact that the set of all stable C-symmetries $\mathfrak{U}$ is much more greater then the formula (4.11) provides. That is why the commutation condition (4.4) is vanished for $s_\pm \equiv 0$ and we cannot establish the relationship between parameters χ_1, ω_1 and χ_2, ω_2 of matrices $\mathcal{C}_{\chi_j,\omega_j}$ (see the proof of Theorem 4.6).

Theorem 4.8. *Let S be a simple closed symmetric operator with zero characteristic function and let $A_U \in \Sigma_J$ be defined by* (3.4)–(3.6)*. Then $\Upsilon_{\mathfrak{U}} = \emptyset$ and the strict inclusions*

$$\Upsilon_{\mathfrak{U}} \subset \Upsilon_J \subset \Sigma_J^{st}$$

hold.

(i) *A_U belongs to Υ_J if and only if*

$$U = e^{i\phi}\begin{pmatrix} 0 & e^{i\xi} \\ -e^{-i\xi} & 0 \end{pmatrix}, \quad \phi, \xi \in [0, 2\pi);$$

(ii) *A_U belongs to $\Sigma_J^{st} \setminus \Upsilon_J$ if and only if*

$$U = e^{i\phi}\begin{pmatrix} qe^{i\gamma} & re^{i\xi} \\ -re^{-i\xi} & qe^{-i\gamma} \end{pmatrix}, \quad \phi, \gamma, \xi \in [0, 2\pi),\ q, r > 0, \qquad q^2 + r^2 = 1.$$

Proof. (i) follows from [2, Proposition 3.3].

In order to show (ii) let $A_U \in \Sigma_J^{st}$. Then $A_U C = C A_U$ for some choice of $C \in \mathfrak{U}$. This is equivalent to the relation $CM = M$, where $M = \operatorname{span}\{d_1, d_2\}$ is defined by (3.5) and (3.6) (see the proof of Theorem 4.1). Moreover, it follows from the proof of Theorem 4.1 that the operators $C\restriction_{\mathfrak{N}_i}$ and $C\restriction_{\mathfrak{N}_{-i}}$ acts in $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$, respectively and they have the matrix representations $\mathcal{C}_{\chi_1,\omega_1}$ and $\mathcal{C}_{\chi_2,\omega_2}$ defined by formula (4.3).

Combining [24, Lemma 3.3] with Lemma 2.7 we conclude that the correspondence

$$C \in \mathfrak{U} \to \{\mathcal{C}_{\chi_1,\omega_1}, \mathcal{C}_{\chi_2,\omega_2}\}, \qquad \chi_j \in \mathbb{R}, \quad \omega_j \in [0, 2\pi) \tag{4.13}$$

is *bijective* for the case of a zero characteristic function ($\mathsf{Sh} \equiv 0$).

It follows from (3.6) and (4.3) that

$$\begin{aligned} Cd_1 &= \mathcal{C}_{\chi_1,\omega_1} e_{++} + qe^{i(\phi+\gamma)}\mathcal{C}_{\chi_1,\omega_1} e_{+-} + re^{i(\phi+\xi)}\mathcal{C}_{\chi_2,\omega_2} e_{-+} \\ &= k_1 e_{++} - \left[\sinh \chi_1 e^{i\omega_1} + qe^{i(\phi+\gamma)} \cosh \chi_1\right] e_{+-} + \left[re^{i(\phi+\xi)} \cosh \chi_2\right] e_{-+} + k_2 e_{--}, \end{aligned}$$

where

$$k_1 = \cosh \chi_1 + qe^{i(\phi+\gamma)} \sinh \chi_1 e^{-i\omega_1}, \qquad k_2 = -re^{i(\phi+\xi)} \sinh \chi_2 e^{i\omega_2}. \tag{4.14}$$

Taking the definition (3.6) of d_j into account we conclude that $Cd_1 \in M$ if and only if $Cd_1 = k_1 d_1 + k_2 d_2$, where k_j are defined by (4.14). A direct calculation shows that the last identity holds if we set

$$\chi = \chi_1 = \chi_2 = -\tanh^{-1} q, \qquad \omega_1 = \frac{\gamma + \phi}{2}, \qquad \omega_2 = \frac{\gamma - \phi}{2}. \tag{4.15}$$

A similar reasoning shows that $Cd_2 \in M$ if we choose parameters χ_j and ω_j according to (4.15). Note that χ can be defined in (4.15) only in the case $0 \leqslant q < 1$.

Thus, if $A_U \in \Sigma_J$ is defined by (3.4)–(3.6) with $0 \leqslant q < 1$, then choosing parameters χ_j, ω_j due to (4.15) and using the bijection (4.13), we establish the existence of $C \in \mathcal{U}$ such that $A_U C = C A_U$. Therefore $A_U \in \Sigma_J^{st}$. Since $A_U \in \Upsilon_J$ when $q = 0$ (see (i)) and the spectrum of A_U coincides with $\mathbb{C}$ when $q = 1$ (which follows from Corollary 3.2), we prove (ii).

Let us assume that $A_U \in \Upsilon_{\mathfrak{U}}$. In that case $A_U C = C A_U$ for all $C \in \mathfrak{U}$. Taking (4.13) into account, we conclude that the element $Cd_1 = \mathcal{C}_{\chi_1,\omega_1} e_{++} + qe^{i(\phi+\gamma)}\mathcal{C}_{\chi_1,\omega_1} e_{+-} + re^{i(\phi+\xi)}\mathcal{C}_{\chi_2,\omega_2} e_{-+}$ belongs to M (i.e., $Cd_1 = k_1 d_1 + k_2 d_2$, where k_j are defined by (4.14)) for *all values* of parameters χ_j and ω_j. This is impossible. Hence, $\Upsilon_{\mathfrak{U}} = \emptyset$. □

The next statement is a direct consequence of Proposition 2.2 and Theorem 4.8.

Corollary 4.9. *(See [24].) If S is a simple closed symmetric operator with zero characteristic function, then an operator $A_U \in \Sigma_J$ has real spectrum if and only if A_U has stable C-symmetry and, hence, A_U is similar to a self-adjoint operator. Otherwise, the spectrum of A_U coincides with $\mathbb{C}$.*

5. Examples

5.1. Degenerate Sturm–Liouville problems on a finite interval

The necessary and sufficient conditions for the Dirichlet eigenvalue problem associated with the Sturm–Liouville equation

$$-\big(p(x)y'\big)' = \lambda r(x)y, \qquad -\infty < a \leqslant x \leqslant b < \infty \tag{5.1}$$

to be degenerate (i.e., the spectrum of this eigenvalue problem fills the whole complex plane) were established in [28]. We consider one of the simplest cases where

$$p(x) = r(x) = (\operatorname{sgn} x) \quad \text{and} \quad [a, b] = [-1, 1].$$

Define the closed symmetric operator S associated with the expression $-(\operatorname{sgn} x)((\operatorname{sgn} x)y')'$ and boundary conditions $y(-1) = y(1) = 0$ via

$$Sy = -y'',$$

with domain

$$\mathcal{D}(S) = \{ y \in W_2^2(-1,0) \oplus W_2^2(0,1) \mid y(0\pm) = y'(0\pm) = y(\pm 1) = 0 \}. \tag{5.2}$$

Then (5.1) takes the form $Sy = \lambda y$.

The operator S has deficiency indices $\langle 2, 2\rangle$ and it commutes with the fundamental symmetry $Jy(x) = (\operatorname{sgn} x)y(x)$ in $\mathfrak{H} = L_2(-1,1)$. The corresponding closed symmetric operators $S_\pm y = -y''$ (see (2.11)) with the domains

$$\mathcal{D}(S_+) = \{ y \in W_2^2(0,1) \mid y(0+) = y'(0+) = y(1) = 0 \},$$
$$\mathcal{D}(S_-) = \{ y \in W_2^2(-1,0) \mid y(0-) = y'(0-) = y(-1) = 0 \}$$

act in $\mathfrak{H}_+ = L_2(0,1)$ and $\mathfrak{H}_- = L_2(-1,0)$, respectively.

Consider the parity operator $\mathcal{P}y(x) = y(-x)$ and set $R := \mathcal{P}$. It is clear that R is a fundamental symmetry in $L_2(-1,1)$ and it satisfies (4.7). To describe these operators we observe that solutions $y_\mu^\pm(x)$ of the equations

$$S_\pm^* y - \mu y = -y''(x) - \mu y(x) = 0, \qquad y(\pm 1) = 0, \quad \mu \in \mathbb{C}_+$$

have the form

$$y_\mu^+(x) = \begin{cases} \sin\sqrt{\mu}(x-1), & x \in [0,1], \\ 0, & x \in [-1,0], \end{cases}$$
$$y_\mu^-(x) = \begin{cases} 0, & x \in [0,1], \\ -\sin\sqrt{\mu}(x+1), & x \in [-1,0]. \end{cases}$$

Here $\sqrt{\cdot}$ denotes the branch of the square root defined in $\mathbb{C}$ with a cut along $[0,\infty)$ and fixed by $\operatorname{Im}\sqrt{\lambda} > 0$ if $\lambda \notin [0,\infty)$. Moreover, $\sqrt{\cdot}$ is continued to $[0,\infty)$ via $\lambda \mapsto \sqrt{\lambda} \geqslant 0$ for $\lambda \in [0,\infty)$. According to (3.1), the elements $e_{\pm\pm}$ can be chosen as follows:

$$e_{++} = y_i^+, \qquad e_{+-} = y_i^-, \qquad e_{-+} = y_{-i}^+, \qquad e_{--} = y_{-i}^-$$

and the functions $s_\pm(\mu)$ in (3.8) can be calculated immediately by repeating the arguments in [32]. For completeness we outline the method.

The characteristic function $\mathsf{Sh}_+(\mu)$ of S_+ is determined by the first relations in (3.8) and (3.9). Employing here (2.10) we get

$$y_\mu^+(x) = u(x) + ce_{++} - cs_+(\mu)e_{-+}, \quad u \in \mathcal{D}(S_+),\ x \in [0,1], \tag{5.3}$$

where c is a constant which is easily determined by setting $x = 0$ and taking into account the relevant boundary conditions:

$$c = \frac{\sin\sqrt{\mu}}{\sin\sqrt{i} - s_+(\mu)\sin\sqrt{-i}}.$$

Differentiating (5.3) with a subsequent setting $x = 0$ we obtain

$$\sqrt{\mu}\cos\sqrt{\mu} = c\sqrt{i}\cos\sqrt{i} - cs_+(\mu)\sqrt{-i}\cos\sqrt{-i}.$$

The last two relations leads to the conclusion:

$$s_+(\mu) = \frac{\sqrt{i}\sin\sqrt{\mu}\cos\sqrt{i} - \sqrt{\mu}\cos\sqrt{\mu}\sin\sqrt{i}}{\sqrt{-i}\sin\sqrt{\mu}\cos\sqrt{-i} - \sqrt{\mu}\cos\sqrt{\mu}\sin\sqrt{-i}}.$$

Considering the characteristic function Sh_- of S_- we obtain the same expression for $s_-(\mu)$. Thus $s_+ = s_- \not\equiv 0$. By Theorem 3.1, the set Σ_J of J-self-adjoint extensions of S contains operators with empty resolvent set. Applying Corollary 3.2 and taking the explicit form of elements $e_{\pm\pm}$ into account we derive the following description of J-self-adjoint extensions of S with empty resolvent set.

Proposition 5.1. *Let S be a symmetric operator in $L_2(-1,1)$ defined by (5.2) and let $Jy(x) = (\operatorname{sgn} x)y(x)$ for $y \in L_2(-1,1)$. Then the collection of all possible J-self-adjoint extensions A_γ of S with empty resolvent set is determined by the formulas*

$$A_\gamma y = -y'',$$
$$\mathcal{D}(A_\gamma) = \left\{ y \in W_2^2(-1,0) \oplus W_2^2(0,1) \;\middle|\; \begin{array}{c} e^{i\gamma} y(0+) = y(0-) \\ e^{i\gamma} y'(0+) = -y'(0-) \\ y(\pm 1) = 0 \end{array} \right\},$$

where $\gamma \in [0, 2\pi)$ is an arbitrary parameter.

Remark 5.2. Since the symmetric operator S satisfies (4.7) with fundamental symmetries $J = (\operatorname{sgn} x)I$ and $R = \mathcal{P}$, Corollary 4.5 implies the existence of $\widetilde{J}$-self-adjoint extensions of S with empty resolvent set for any non-trivial fundamental symmetry $\widetilde{J}$ which belongs to the Clifford algebra $\mathcal{C}l_2 = \operatorname{span}\{I, J, R, JR\}$.

5.2. Indefinite singular Sturm–Liouville operators

Consider the indefinite Sturm–Liouville differential expression

$$a(y)(x) = (\operatorname{sgn} x)\big(-y''(x) + q(x)y(x)\big), \quad x \in \mathbb{R}$$

with a real potential $q \in L^1_{loc}(\mathbb{R})$ and denote by $\mathfrak{D}$ the set of all functions $y \in L_2(\mathbb{R})$ such that y and y' are absolutely continuous and $a(y) \in L_2(\mathbb{R})$. On $\mathfrak{D}$ we define the operator A as follows:

$$Ay = a(y), \qquad \mathcal{D}(A) = \mathfrak{D}. \tag{5.4}$$

Assume in what follows the limit point case of $a(y)$ at both $-\infty$ and $+\infty$. Then the operator A is J-self-adjoint in the Krein space $(L_2(\mathbb{R}), [\cdot,\cdot]_J)$, where $J = (\operatorname{sgn} x)I$, see, e.g., [12].

The operator A is a J-self-adjoint extension of the symmetric operator S,

$$S = (\operatorname{sgn} x)\left(-\frac{d^2}{dx^2} + q\right), \qquad \mathcal{D}(S) = \big\{y \in \mathfrak{D} \bigm| y(0) = y'(0) = 0\big\}. \tag{5.5}$$

The operator S commutes with J and has deficiency indices $\langle 2, 2\rangle$. Its restrictions onto the subspaces $L_2(\mathbb{R}_\pm)$ of the fundamental decomposition $L_2(\mathbb{R}) = L_2(\mathbb{R}_+) \oplus L_2(\mathbb{R}_-)$ coincides with the symmetric operators

$$S_+ = -\frac{d^2}{dx^2} + q_+, \qquad S_- = \frac{d^2}{dx^2} - q_-, \qquad \mathcal{D}(S_\pm) = P_\pm \mathcal{D}(S), \qquad q_\pm = q\restriction_{\mathbb{R}_\pm}$$

with deficiency indices $\langle 1, 1\rangle$ acting in the Hilbert spaces $\mathfrak{H}_+ = L_2(\mathbb{R}_+)$ and $\mathfrak{H}_- = L_2(\mathbb{R}_-)$, respectively. Here $P_\pm$ are the orthogonal projectors onto $L_2(\mathbb{R}_\pm)$ in $L_2(\mathbb{R})$.

Denote by $c_\mu(\cdot)$, $s_\mu(\cdot)$ the solutions of the equation

$$-f''(x) + q(x)f(x) = \mu f(x), \quad x \in \mathbb{R},\ \mu \in \mathbb{C}$$

with boundary conditions

$$c_\mu(0) = s'_\mu(0) = 1, \qquad c'_\mu(0) = s_\mu(0) = 0. \tag{5.6}$$

Due to the limit point case at $\pm\infty$ there exist unique holomorphic functions $M_\pm(\mu)$ ($\mu \in \mathbb{C} \setminus \mathbb{R}$) such that the functions

$$\psi^\pm_\mu(x) = \begin{cases} s_{\pm\mu}(x) - M_\pm(\mu)c_{\pm\mu}(x), & x \in \mathbb{R}_\pm, \\ 0, & x \in \mathbb{R}_\mp \end{cases} \tag{5.7}$$

belongs to $L_2(\mathbb{R})$. The functions $M_\pm(\cdot)$ are called the Titchmarsh–Weyl coefficients of the differential expression $a(\cdot)$ (see, e.g., [22, Definition 2.2]). They are Nevanlinna functions and they satisfy the following asymptotic behavior

$$M_\pm(\mu) = \pm\frac{i}{\sqrt{\pm\mu}} + O\left(\frac{1}{|\mu|}\right) \quad (\mu \to \infty,\ 0 < \delta < \arg\mu < \pi - \delta) \tag{5.8}$$

for $\delta \in (0, \frac{\pi}{2})$, see [14].

The asymptotic behavior (5.8) was used for justifying the property $\rho(A) \neq \emptyset$ for the concrete J-self-adjoint extension A of S defined by (5.4), cf. [21]. We extend this result to all operators in Σ_J.

Theorem 5.3. *Let the symmetric operator S be defined by* (5.5) *and* $J = (\operatorname{sgn} x)I$. *Then the set Σ_J of J-self-adjoint extensions of S does not contain operators with empty resolvent set.*

Proof. The proof is divided into two steps. In the first one we calculate the characteristic function of S. In the second step we apply Theorem 3.1.

Step 1. It follows from the definition of $S_\pm$ and (5.7) that the defect subspaces $\mathfrak{N}_{\pm i}(S_+)$ coincides with $\operatorname{span}\{\psi^+_{\pm i}\}$ and the defect subspaces $\mathfrak{N}_{\pm i}(S_-)$ coincides with $\operatorname{span}\{\psi^-_{\pm i}\}$. Therefore, we can choose basis elements $\{e_{\pm\pm}\}$ as follows:

$$e_{++} = \psi^+_i, \qquad e_{-+} = \psi^+_{-i}, \qquad e_{+-} = c\psi^-_i, \qquad e_{--} = c\psi^-_{-i},$$

where an auxiliary constant $c > 0$ is determined by the condition $\|\psi^+_i\| = \|\psi^-_i\|$ (or, what is equivalent, by the condition $\|\psi^+_{-i}\| = \|\psi^-_{-i}\|$). This ensures the equality of the norms $\|e_{++}\| = \|e_{+-}\| = \|e_{-+}\| = \|e_{--}\|$.

By virtue of (5.6) and (5.7) we have

$$e_{\pm+}(0) = -M_+(\pm i), \qquad e'_{\pm+}(0) = 1, \qquad e_{\pm-}(0) = -cM_-(\pm i), \qquad e'_{\pm-}(0) = c. \tag{5.9}$$

Using these boundary conditions and repeating the arguments of Subsection 5.1, we arrive at the conclusion that the characteristic function Sh of S is defined by the following functions $s_+(\cdot)$ and $s_-(\cdot)$ in (3.8):

$$s_+(\mu) = \frac{M_+(\mu) - M_+(i)}{M_+(\mu) - M_+(-i)}, \qquad s_-(\mu) = \frac{M_-(\mu) - M_-(i)}{M_-(\mu) - M_-(-i)}. \tag{5.10}$$

Step 2. By Theorem 3.1 the set Σ_J contains operators with empty resolvent set if and only if $e^{2i\phi}s_-(\mu) = s_+(\mu)$, $\mu \in \mathbb{C}_+$, for a certain choice of $\phi \in [0, 2\pi)$. Tending $\mu \to \infty$ in this identity and taking (5.8) and (5.10) into account, we obtain that

$$e^{2i\phi} = \frac{M_+(i)M_-(-i)}{M_+(-i)M_-(i)}. \tag{5.11}$$

Rewriting $e^{2i\phi}s_-(\mu) = s_+(\mu)$ with the use of (5.10) and (5.11) we get

$$M_+(\mu)M_-(\mu)\big[e^{2i\phi} - 1\big] + M_+(\mu)\big[M_-(-i) - e^{2i\phi}M_-(i)\big] + M_-(\mu)\big[M_+(i) - e^{2i\phi}M_+(-i)\big] = 0. \tag{5.12}$$

Denote $M_\pm(i) = e^{i\theta_\pm}|M_\pm(i)|$, where $\theta_\pm \in (0, \pi)$ (since Im $M_\pm(i) > 0$). Then (5.11) takes the form $e^{2i\phi} = e^{2i(\theta_+ - \theta_-)}$ and relation (5.12) can rewriting (after routine transformations) as follows:

$$M_+(\mu)M_-(\mu)\sin(\theta_+ - \theta_-) - M_+(\mu)\big|M_-(i)\big|\sin\theta_+ + M_-(\mu)\big|M_+(i)\big|\sin\theta_- = 0 \quad \forall \mu \in C_+. \tag{5.13}$$

Since the coefficients $|M_\mp(i)|\sin\theta_\pm$ of $M_\pm(\mu)$ are real, identity (5.13) cannot be true for the whole $\mathbb{C}_+$ (due to the asymptotic behavior (5.8)). Therefore, Σ_J does not contain operators with empty resolvent set. □

Remark 5.4. For the concrete operator A defined by (5.4), the relation (5.13) is reduced by a simple calculation using Corollary 3.2 to $M_+ \equiv M_-$. However, in that particular case, the property $\rho(A) \neq \emptyset$ is already contained in [21, Proposition 2.5].

By virtue of Theorems 4.1, 5.3 the set Σ_J^{st} of J-self-adjoint operators with stable C-symmetry is reduced to the set Υ_J of self-adjoint extensions of S which commute with J in the case of indefinite Sturm–Liouville operators. The set Υ_J consists of all self-adjoint extensions of S with *separated boundary conditions on* 0, i.e.,

$$A \in \Upsilon_J \iff Ay = a(y), \qquad \mathcal{D}(A) = \big\{y \in \mathfrak{D} \;\big|\; a_\pm f(0\pm) - b_\pm f'(0\pm) = 0\big\}.$$

5.3. One-dimensional impulse operator with point perturbation

Consider the closed symmetric operator

$$S = -i\frac{d}{dx}, \qquad \mathcal{D}(S) = \big\{y \in W_2^1\big(\mathbb{R}, \mathbb{C}^2\big) \;\big|\; y(0) = 0\big\}$$

in the Hilbert space $L_2(\mathbb{R}, \mathbb{C}^2) := L_2(\mathbb{R}) \otimes \mathbb{C}^2$.

Lemma 5.5. *The operator S has deficiency indices $\langle 2, 2\rangle$ and its characteristic function* Sh *is equal to zero.*

Proof. The operator S can be presented as $S = S_1 + S_2$ with respect to the decomposition $L_2(\mathbb{R}, \mathbb{C}^2) = L_2(\mathbb{R}_-, \mathbb{C}^2) \oplus L_2(\mathbb{R}_+, \mathbb{C}^2)$. The restrictions $S_1 = S\restriction_{L_2(\mathbb{R}_-, \mathbb{C}^2)}$ and $S_2 = S\restriction_{L_2(\mathbb{R}_+, \mathbb{C}^2)}$ are maximal symmetric operators in the Hilbert spaces $L_2(\mathbb{R}_-, \mathbb{C}^2)$ and $L_2(\mathbb{R}_+, \mathbb{C}^2)$, respectively, with deficiency indices $\langle 0, 2\rangle$ and $\langle 2, 0\rangle$, respectively. Therefore S has deficiency indices $\langle 2, 2\rangle$ and $\mathfrak{N}_\mu(S) = \mathfrak{N}_\mu(S_2)$ for all $\mu \in \mathbb{C}_+$ (since S_2 has deficiency indices $\langle 2, 0\rangle$). An arbitrary $f_\mu \in \mathfrak{N}_\mu(S)$ admits the representation

$$f_\mu = u + f_i, \quad u \in \mathcal{D}(S_2), \;\; f_i \in \mathfrak{N}_i(S_2).$$

Comparing the obtained formula with (2.10) we obtain $\mathrm{Sh}(\mu) = 0$. □

Remark 5.6. The operator $J = (\operatorname{sgn} x)I$ is a fundamental symmetry in $L_2(\mathbb{R}, \mathbb{C}^2)$ and S commutes with J. The symmetric operators S_- and S_+ in (2.11) coincides with S_1 and S_2, respectively, and hence their deficiency indices are $\langle 0, 2\rangle$ and $\langle 2, 0\rangle$. Hence, there are no J-self-adjoint extensions of S and the sets Σ_J and Υ_J are empty.

To achieve a non-empty set Σ_J, we have to choose a fundamental symmetry J in such a way that the deficiency indices of $S_\pm$ in (2.11) are $\langle 1,1\rangle$. To this end, we write an arbitrary element $y \in L_2(\mathbb{R},\mathbb{C}^2)$ as follows

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = y_1 \otimes h_+ + y_2 \otimes h_-, \quad h_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ h_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and consider the fundamental symmetry $Jy = \left(\begin{smallmatrix} y_1 \\ -y_2 \end{smallmatrix}\right)$ in $L_2(\mathbb{R},\mathbb{C}^2)$. In that case, the operators $S_\pm$ in (2.11) act in the Hilbert spaces $L_2(\mathbb{R},\mathcal{H}_\pm)$, where $\mathcal{H}_\pm = \operatorname{span}\{h_\pm\}$ and they are determined by the formulas

$$S_\pm = -i\frac{d}{dx}, \qquad \mathcal{D}(S_\pm) = \bigl\{ y \in W_2^1(\mathbb{R},\mathcal{H}_\pm) \bigm| y(0) = 0 \bigr\}. \tag{5.14}$$

Obviously, $S_\pm$ have deficiency indices $\langle 1,1\rangle$. This means that the set Σ_J is non-empty and its elements can be parameterized by unitary matrices U in (3.4).

In order to describe the subset of J-self-adjoint extensions with empty resolvent set in Σ_J we have to calculate basis elements $\{e_{\pm\pm}\}$ (see (3.1)) and to apply Corollary 3.2.

Denote by

$$y_i(x) = \begin{cases} e^{-x}, & x \geqslant 0, \\ 0, & x < 0, \end{cases} \qquad y_{-i}(x) = \begin{cases} 0, & x \geqslant 0, \\ e^{x}, & x < 0, \end{cases}$$

the solutions of the equation $-iy' - \mu y = 0$ ($\mu \in \{i,-i\}$). Using the definition of $S_\pm$ and (3.1) we obtain

$$e_{++} = y_i \otimes h_+, \qquad e_{+-} = y_i \otimes h_-, \qquad e_{-+} = y_{-i} \otimes h_+, \qquad e_{--} = y_{-i} \otimes h_-.$$

Corollary 3.2 and equalities (3.5), (3.6) imply that an arbitrary J-self-adjoint extension A_U with empty resolvent set has the domain $\mathcal{D}(A_U) = \mathcal{D}(S) \dot{+} M$, where M is a linear span of elements

$$d_1 = e_{++} + e^{i(\phi+\gamma)} e_{+-}, \qquad d_2 = e_{--} + e^{i(\phi-\gamma)} e_{-+}, \quad \phi,\gamma \in [0,2\pi).$$

The obtained expression leads (after some trivial calculations) to the following description of J-self-adjoint extensions $A_U (= A_{\phi\gamma})$ of S with empty resolvent set:

$$A_{\phi\gamma} y = -iy', \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{D}(A_{\phi\gamma}),$$

where $\phi,\gamma \in [0,2\pi)$ are arbitrary parameters and

$$\mathcal{D}(A_{\phi\gamma}) = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W_2^1\bigl(\mathbb{R}\setminus\{0\}\bigr) \otimes \mathbb{C}^2 \;\middle|\; \begin{matrix} y_2(0+) = e^{i(\gamma+\phi)} y_1(0+) \\ y_2(0-) = e^{i(\gamma-\phi)} y_1(0-) \end{matrix} \right\}.$$

5.4. One-dimensional Dirac operator with point perturbation

Let us consider the free Dirac operator D in the space $L_2(\mathbb{R}) \otimes \mathbb{C}^2$:

$$D = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3, \qquad \mathcal{D}(D) = W_2^1(\mathbb{R}) \otimes \mathbb{C}^2,$$

where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ are Pauli matrices and $c > 0$.

The closed symmetric Dirac operator

$$S = D \restriction \bigl\{ u \in W_2^1(\mathbb{R}) \otimes \mathbb{C}^2 \bigm| u(0) = 0 \bigr\}$$

has deficiency indices $\langle 2,2\rangle$, see [1], and it commutes with the fundamental symmetry $J = \mathcal{P} \otimes \sigma_3$ in $L_2(\mathbb{R}) \otimes \mathbb{C}^2$, where $\mathcal{P}$ is the parity operator $\mathcal{P}y(x) = y(-x)$. In that case, the operators $S_\pm$ in (2.11) are restrictions of S onto the Hilbert spaces

$$\bigl[L_2^{even}(\mathbb{R}) \otimes \mathcal{H}_+\bigr] \oplus \bigl[L_2^{odd}(\mathbb{R}) \otimes \mathcal{H}_-\bigr], \qquad \bigl[L_2^{odd}(\mathbb{R}) \otimes \mathcal{H}_+\bigr] \oplus \bigl[L_2^{even}(\mathbb{R}) \otimes \mathcal{H}_-\bigr],$$

respectively, where $\mathcal{H}_\pm$ are as in Section 5.3 and the closed symmetric operators $S_\pm$ have deficiency indices $\langle 1,1\rangle$.

The defect subspaces $\mathfrak{N}_i$ and $\mathfrak{N}_{-i}$ of S coincide, respectively, with the linear spans of the functions $\{y_{1+}, y_{2+}\}$ and $\{y_{1-}, y_{2-}\}$, where

$$y_{1\pm}(x) = \begin{pmatrix} -ie^{\mp it} \\ (\operatorname{sgn} x) \end{pmatrix} e^{i\tau|x|}, \qquad y_{2\pm}(x) = (\operatorname{sgn} x)\, y_{1\pm}(x), \tag{5.15}$$

$\tau = \frac{i}{c}\sqrt{\frac{c^4}{4}+1}$, and $e^{it} := (\frac{c^2}{2} - i)(\sqrt{\frac{c^4}{4}+1})^{-1}$, see, e.g., [1].

Using the definition of $S_\pm$ and (3.1) we obtain

$$e_{++} = y_{1+}, \qquad e_{+-} = y_{2+}, \qquad e_{-+} = y_{1-}, \qquad e_{--} = y_{2-}. \tag{5.16}$$

The adjoint operator

$$S^* = -ic\frac{d}{dx} \otimes \sigma_1 + \frac{c^2}{2} \otimes \sigma_3$$

is defined on the domain $\mathcal{D}(S^*) = W_2^1(\mathbb{R} \setminus \{0\}) \otimes \mathbb{C}^2$ and an arbitrary J-self-adjoint extension $A_U \in \Sigma_J$ is the restriction of S^* onto $\mathcal{D}(A_U) = \mathcal{D}(S) \dot{+} M$, where M is defined by (3.5) and (3.6) with $e_{\pm\pm}$ determined by (5.16).

It is easy to see that the fundamental symmetry $R = (\operatorname{sgn} x)I$ in $L_2(\mathbb{R}) \otimes \mathbb{C}^2$ also commutes with S and $JR = -RJ$. Taking into account Remark 4.4 we establish the existence of J-self-adjoint extensions of S with empty resolvent set.

A routine calculation with the use of Corollary 3.2 gives that $A_U \in \Sigma_J$ has empty resolvent set if and only if $A_U (= A_\gamma)$ is the restriction of S^* onto the set

$$\mathcal{D}(A_\gamma) = \left\{ y \in W_2^1\big(\mathbb{R} \setminus \{0\}\big) \otimes \mathbb{C}^2 \;\middle|\; \begin{array}{l} \Lambda_\gamma[y(0+) + y(0-)] = y(0+) - y(0-) \\ y'(0+) + y'(0-) = \Lambda_\gamma[y'(0+) - y'(0-)] \end{array} \right\},$$

where $\Lambda_\gamma = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$.

Acknowledgments

The first author (S.K.) expresses his gratitude to the DFG (project TR 903/11-1) for the support and the Fakultät für Mathematik und Naturwissenschaften of the Technische Universität Ilmenau for the warm hospitality.

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