# Interference of conversion and bremsstrahlung amplitudes in the decay $K_L \to \mu^+\mu^-\gamma$

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## Abstract

In the region of large $\mu^+\mu^-$ invariant mass, the decay spectrum of $K_L \to \mu^+\mu^-\gamma$ deviates from the Dalitz pair spectrum, as a result of interference between conversion ($K_L \to \gamma^*\gamma \to \mu^+\mu^-\gamma$) and bremsstrahlung amplitudes. The latter is proportional to the $K_L \to \mu^+\mu^-$ matrix element, whose $2\gamma$-absorptive part appears to dominate the observed $K_L \to \mu^+\mu^-$ decay rate. We examine the extent to which a scrutiny of the $K_L \to \mu^+\mu^-\gamma$ spectrum in the end-point region could provide evidence on the real part of the $K_L \to \mu^+\mu^-$ amplitude. As a by-product, we obtain the absorptive part of the $K_L \to \gamma^*\gamma$ form factor, using data on the $K_L \to \pi^+\pi^-\gamma$ spectrum.

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It is customary to interpret the decays $K_L \to e^+e^-\gamma$ and $K_L \to \mu^+\mu^-\gamma$ in terms of a Dalitz pair process $K_L \to \gamma^*\gamma \to l^+l^-\gamma$. The branching ratio and the lepton mass spectrum are fitted to an $s$-dependent $K_L\gamma^*\gamma$ vertex $f_{K\gamma\gamma}(s) = f_{K\gamma\gamma}(0) f(s)$, where $f_{K\gamma\gamma}(0)$ is a dimensionless parameter related to the decay width of $K_L \to \gamma\gamma$ by

$$\left| f_{K\gamma\gamma}(0) \right|^2 = \frac{64\pi}{m_K} \Gamma(K_L \to \gamma\gamma) \tag{1}$$
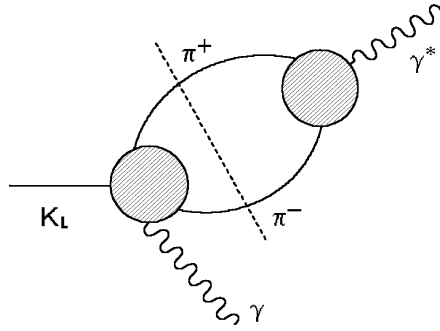
and $f(s)$ is a form factor. The spectrum in the invariant mass of the lepton pair ($s = (p_+ + p_-)^2$) is given by [1]

$$\frac{d\Gamma(K_L \to l^+l^-\gamma)/ds}{\Gamma(K_L \to \gamma\gamma)} = \frac{2\alpha}{3\pi} \left(1 - \frac{s}{m_K^2}\right)^3 \left(1 + \frac{2m_l^2}{s}\right) \left(1 - \frac{4m_l^2}{s}\right)^{1/2} \frac{1}{s} |f(s)|^2. \tag{2}$$

A popular choice of the form factor $f(s)$ is the BMS parametrization [2]

$$f_{\mathrm{BMS}}(s) = \frac{1}{1 - \frac{s}{m_\rho^2}} + \frac{2.3\alpha_{K^*}}{1 - \frac{s}{m_{K^*}^2}} \left[\frac{4}{3} - \frac{1}{1 - \frac{s}{m_\rho^2}} - \frac{1}{9} \frac{1}{1 - \frac{s}{m_\omega^2}} - \frac{2}{9} \frac{1}{1 - \frac{s}{m_\phi^2}}\right], \tag{3}$$

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Fig. 1. Model for imaginary part of $K_L \gamma^* \gamma$ vertex.

where the first term is the $\rho$-dominance approximation, and the remainder is a correction term depending on an unknown parameter $\alpha_{K^*}$. Measurements on the decays $K_L \rightarrow e^+ e^- \gamma$ [3] and $K_L \rightarrow \mu^+ \mu^- \gamma$ [4,5] can be understood in a fairly consistent way with a value $\alpha_{K^*} \approx -0.19$. Other one-parameter forms for $f(s)$ have also been discussed in the literature [6].

In the end-point region $s \approx m_K^2$, one expects the decay $K_L \rightarrow l^+ l^- \gamma$ to show a deviation from the Dalitz pair spectrum Eq. (2), due to internal bremsstrahlung from the underlying transition $K_L \rightarrow l^+ l^-$. Because of the chiral suppression of the latter, the bremsstrahlung effect is of relevance mainly for the channel $K_L \rightarrow \mu^+ \mu^- \gamma$. Considering that the branching ratio $Br(K_L \rightarrow \mu^+ \mu^-)$ is $\approx 7 \times 10^{-9}$, one expects the bremsstrahlung contribution to $Br(K_L \rightarrow \mu^+ \mu^- \gamma)$ to be of order $10^{-11}$, and confined to photons of very low energy. There is, however, the possibility of an interference effect between the bremsstrahlung and Dalitz pair amplitudes that could conceivably probe the real part of the $K_L \rightarrow \mu^+ \mu^-$ amplitude. It is this possibility that we wish to explore in this Letter.

It may be recalled that if the $K_L \rightarrow \mu^+ \mu^-$ amplitude is parametrized as $f_{Kll} \bar{u} \gamma_5 v$, the decay rate is

$$\Gamma\left(K_L \rightarrow l^+ l^-\right) = |f_{Kll}|^2 \frac{m_K v_0}{8\pi}, \quad v_0 = \left(1 - \frac{4m_l^2}{m_K^2}\right)^{1/2}. \tag{4}$$

The imaginary part of the $K_L \rightarrow l^+ l^-$ amplitude can be reliably calculated in terms of $K_L \rightarrow 2\gamma$, by considering the absorptive contribution of the two-photon intermediate state ($K_L \rightarrow \gamma\gamma \rightarrow l^+ l^-$), with the result [7]

$$\left|\operatorname{Im} f_{Kll}\right| = \frac{\alpha}{4v_0} \frac{m_l}{m_K} \ln\left(\frac{1 + v_0}{1 - v_0}\right) \left|f_{K\gamma\gamma}(0)\right|. \tag{5}$$

The measured branching ratio of $K_L \rightarrow \mu^+ \mu^-$ is almost saturated by the two-photon absorptive part, the real part being limited by [8]

$$\left|\operatorname{Re} f_{Kll} / \operatorname{Im} f_{Kll}\right| \leqslant 0.23 \,(90\% \text{ C.L.}). \tag{6}$$

In considering the possible interference of bremsstrahlung and conversion amplitudes, it is necessary to take account of a possible imaginary part in the $K_L \gamma^* \gamma$ form factor $f(s)$. Such an imaginary part is expected, quite generally, in the region $s > 4m_\pi^2$, and is not included in the BMS parametrization Eq. (3). As a simple model for $\operatorname{Im} f(s)$, we consider a $\pi^+ \pi^-$ intermediate state in the virtual photon channel. One can then use unitarity to obtain $\operatorname{Im} f(s)$ in terms of the form factor characterising the $K_L \rightarrow \pi^+ \pi^- \gamma$ vertex, and the electromagnetic form factor of the pion (Fig. 1). We define the invariant amplitude for the direct emission ($M1$) transition $K_L \rightarrow \pi^+ \pi^- \gamma$ in the conventional way [9]

$$\mathcal{M}\left(K_L \rightarrow \pi^+(p_+)\pi^-(p_-)\gamma(\epsilon, k)\right) = \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\mu} k^\nu p_+^\rho p_-^\sigma C g_{M1}(s), \tag{7}$$
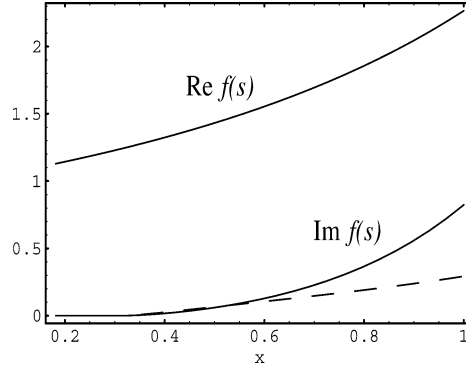
Fig. 2. Comparison of Im $f(s)$ calculated in this Letter, with Re $f(s)$ as given by BMS parametrization with $\alpha_{K^*} = -0.19$. Dotted line indicates Im $f(s)$ as obtained from a complex $\rho$-pole $\left(1 - \frac{s}{m_\rho^2} - i\frac{\Gamma_\rho}{m_\rho}v^3\right)^{-1}$.

where $s = (p_+ + p_-)^2$ and $C$ is a normalization factor given by

$$C = e\frac{|f_S|}{m_K^4}, \quad |f_S| = \left[\frac{16\pi m_K}{\beta_0}\Gamma\left(K_S \to \pi^+\pi^-\right)\right]^{1/2}, \quad \beta_0 = \left(1 - \frac{4m_\pi^2}{m_K^2}\right)^{1/2}. \tag{8}$$

Experiments have determined the form factor $g_{M1}(s)$ to be [10]

$$g_{M1}(s) = \frac{a_1}{m_\rho^2 - s} + a_2 \tag{9}$$

with $a_2 = -1.35$ and $a_1/a_2 = -0.737$ GeV$^2$. The pion charge form factor is adequately represented by

$$f_\pi^{em}(s) = \frac{1}{1 - \frac{s}{m_\rho^2}}. \tag{10}$$

In terms of $g_{M1}(s)$ and $f_\pi^{em}(s)$, we have calculated the imaginary part of the $K_L\gamma^*\gamma$ form factor to be

$$\mathrm{Im}\, f(s) = \left[\frac{\alpha}{6}\frac{|f_S|}{m_K}\frac{s}{m_K^2}\frac{1}{f_{K\gamma\gamma}(0)}\right]f_\pi^{em}(s)g_{M1}(s)v^3\Theta\left(s - 4m_\pi^2\right) \tag{11}$$

with $v = \sqrt{1 - \frac{s}{m_K^2}}$, $s$ being the mass of the virtual photon. This is plotted in Fig. 2, where we also show the real part of $f(s)$, taken to be the BMS form factor. For comparison, we also indicate in Fig. 2 the result for Im $f(s)$ that one would obtain from postulating a complex $\rho$-pole with a momentum-dependent decay width,

$$f(s) \approx \left(1 - \frac{s}{m_\rho^2} - i\frac{\Gamma_\rho}{m_\rho}v^3\right)^{-1}.$$

We are now in a position to calculate the decay spectrum of $K_L \to \mu^+\mu^-\gamma$, taking into account the interference of conversion and bremsstrahlung. Defining $x_\gamma = \frac{2E_\gamma}{m_K} = (1 - \frac{s}{m_K^2})$ as the scaled photon energy ($0 < x_\gamma < 1 - 4r$, $r = \frac{m_l^2}{m_K^2}$), the distribution in $x_\gamma$ is given by

$$\frac{d\Gamma(K_L \to \mu^+\mu^-\gamma)}{dx_\gamma} = \left(\frac{1}{12}\frac{\alpha\pi}{(2\pi)^3}m_K^3\right)[Dal + Int + Brem] \tag{12}$$
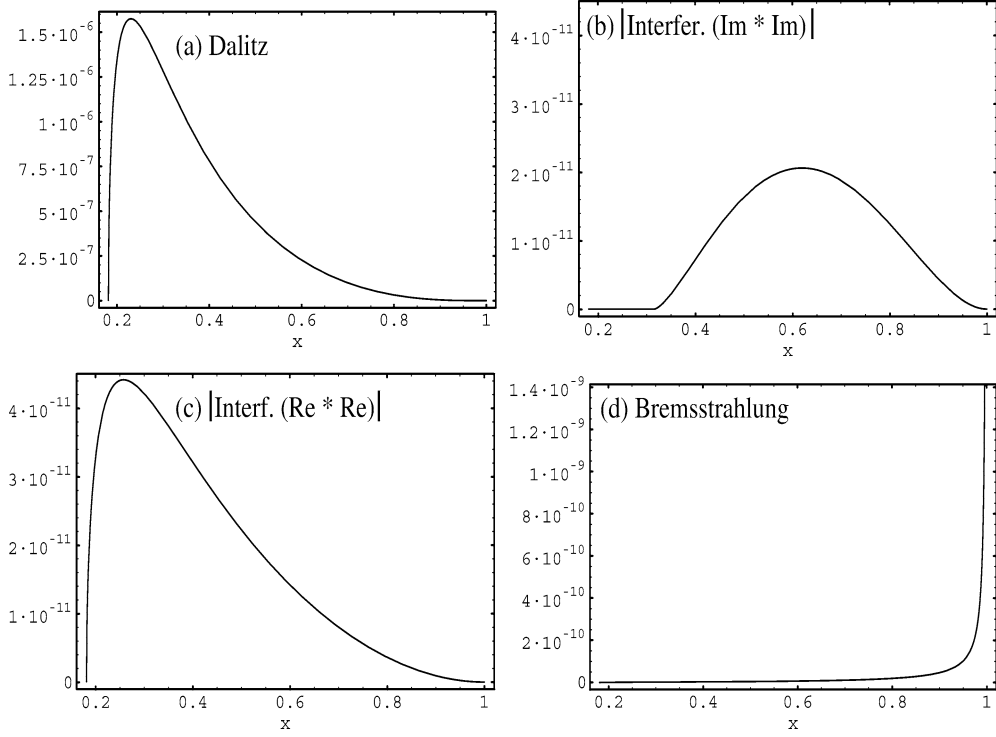
Fig. 3. Contributions to $dBr(K_L \to \mu^+ \mu^- \gamma)/dx$ from (a) Dalitz-pair spectrum, (b) interference of bremsstrahlung with Dalitz amplitude (imaginary parts), (c) interference between real parts, (d) pure bremsstrahlung. (Interference terms shown in modulus only.)

where the three terms, denoting Dalitz, interference and bremsstrahlung contributions, are given by

$$Dal = \left| f_{K\gamma\gamma}(0) \right|^2 \left| f(s) \right|^2 \frac{m_K^2}{s^2} x_\gamma^2 v(1 - x_\gamma + 2r),$$

$$Int = f_{K\gamma\gamma}(0) \, \text{Re}\left( f_{Kll}^* f(s) \right) \frac{1}{s} \frac{m_\mu}{m_K} x_\gamma^2 \ln \frac{1+v}{1-v},$$

$$Brem = -|f_{Kll}|^2 \frac{1}{m_K^2} \left[ 2v \frac{1 - x_\gamma}{x_\gamma} + \left( 2 + 4\frac{r}{x_\gamma} - \frac{2}{x_\gamma} - x_\gamma \right) \ln \frac{1+v}{1-v} \right]. \tag{13}$$

The different contributions to the decay rate $\frac{d\Gamma(K_L \to \mu^+ \mu^- \gamma)}{dx}$ ($x \equiv 1 - x_\gamma = \frac{s}{m_K^2}$) are plotted in Fig. 3. The Dalitz pair spectrum shown in Fig. 3(a) dominates up to $x \approx 0.95$, and essentially accounts for the measured branching ratio $Br(K_L \to \mu^+ \mu^- \gamma) \sim 3.7 \times 10^{-7}$. The interference terms proportional to Re $f_{Kll}$ Re $f(s)$ and Im $f_{Kll}$ Im $f(s)$ are shown separately in Fig. 3(b) and (c), where we have chosen $|\text{Re} \, f_{Kll}|$ equal to the maximum value allowed by the experimental rate of $K_L \to \mu^+ \mu^-$ (Eq. (6)). The interference terms are competitive with the Dalitz pair spectrum only in the region $x \geqslant 0.97$, where the branching ratio is of order $10^{-11}$ per unit of $x$. In the same region the pure bremsstrahlung contribution shown in Fig. 3(d) begins to dominate the spectrum.

The principal conclusion of this Letter is embodied in Fig. 4, which shows the full spectrum $\frac{dBr(K_L \to \mu^+ \mu^- \gamma)}{dx}$ in the interesting end-point interval $0.92 < x < 1.0$. In this region, the spectrum deviates in a systematic way from the monotonically decreasing Dalitz pair spectrum, going through a minimum around $x \approx 0.97$, and then rising sharply as $x$ approaches unity, in the manner characteristic of bremsstrahlung. The interference terms affect the detailed shape of the spectrum in the neighbourhood of the minimum, but only at a level $\approx 10^{-11}$ in the differential
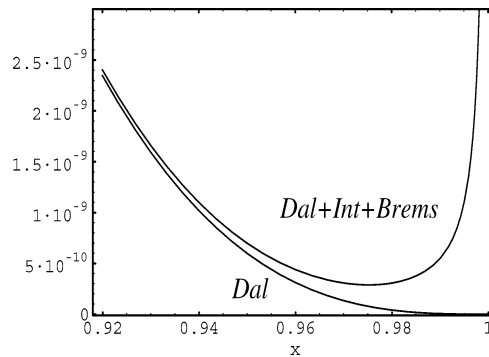
Fig. 4. Shape of $\mu^+\mu^-$ invariant mass spectrum in the end-point region (full spectrum compared to Dalitz-pair contribution).

branching ratio. In particular, the difference between choosing $\mathrm{Re}\, f_{Kll}/\mathrm{Im}\, f_{Kll}$ to be $+0.23$ or $-0.23$ is almost unobservable with the resolution chosen in Fig. 3.

We conclude that while the spectrum of the decay $K_L \to \mu^+\mu^-\gamma$ should show an interesting departure from the Dalitz pair spectrum for large invariant masses, there is little realistic prospect of being able to extract the real part of the $K_L \to \mu^+\mu^-$ amplitude from such a measurement. As a by-product of our analysis, we have determined the imaginary part of the $K_L \to \gamma^*\gamma$ form factor, relying entirely on an empirical measurement of the spectrum of $K_L \to \pi^+\pi^-\gamma$. This imaginary part provides a well-defined correction to existing models of this form factor.

## Acknowledgements

## References

[1] L.M. Sehgal, Phys. Rev. D 7 (1973) 3303, and references therein.
[2] L. Bergström, E. Massó, P. Singer, Phys. Lett. B 249 (1990) 141.
[3] V. Fanti, et al., NA48 Collaboration, Phys. Lett. B 458 (1999) 553;
    J. LaDue, KTeV Collaboration, talk at DPF 2002, May 27, 2002.
[4] A. Alavi-Harati, et al., KTeV Collaboration, Phys. Rev. Lett. 87 (2001) 071801;
    V. Fanti, et al., NA48 Collaboration, Z. Phys. C 76 (1997) 653.
[5] G. Breese Quinn, Ph.D. dissertation, University of Chicago, June 2000.
[6] G. D'Ambrosio, G. Isidori, J. Portelés, Phys. Lett. B 423 (1998) 385.
[7] L.M. Sehgal, Phys. Rev. 183 (1969) 1511.
[8] A. Alavi-Harati, et al., KTeV Collaboration, Phys. Rev. Lett. 87 (2001) 071801.
[9] L.M. Sehgal, J. van Leusen, Phys. Rev. Lett. 83 (1999) 4933.
[10] A. Alavi-Harati, et al., KTeV Collaboration, Phys. Rev. Lett. 86 (2001) 761.