Physics Letters B 761 (2016) 119–124

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Global adiabaticity and non-Gaussianity consistency condition

Antonio Enea Romano [a,c,*], Sander Mooij [b], Misao Sasaki [c]

[a] *Instituto de Fisica, Universidad de Antioquia, A.A.1226, Medellin, Colombia*
[b] *Grupo de Cosmología y Astrofísica Teórica, Departamento de Física, FCFM, Universidad de Chile, Blanco Encalada 2008, Santiago, Chile*
[c] *Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

ARTICLE INFO

Article history:
Received 24 June 2016
Received in revised form 11 August 2016
Accepted 11 August 2016
Available online 17 August 2016
Editor: M. Trodden

ABSTRACT

In the context of single-field inflation, the conservation of the curvature perturbation on comoving slices, $\mathcal{R}_c$, on super-horizon scales is one of the assumptions necessary to derive the consistency condition between the squeezed limit of the bispectrum and the spectrum of the primordial curvature perturbation. However, the conservation of $\mathcal{R}_c$ holds only after the perturbation has reached the adiabatic limit where the constant mode of $\mathcal{R}_c$ dominates over the other (usually decaying) mode. In this case, the non-adiabatic pressure perturbation defined in the thermodynamic sense, $\delta P_{nad} \equiv \delta P - c_w^2 \delta\rho$ where $c_w^2 = \dot{P}/\dot{\rho}$, usually becomes also negligible on superhorizon scales. Therefore one might think that the adiabatic limit is the same as thermodynamic adiabaticity. This is in fact not true. In other words, thermodynamic adiabaticity is not a sufficient condition for the conservation of $\mathcal{R}_c$ on super-horizon scales. In this paper, we consider models that satisfy $\delta P_{nad} = 0$ on all scales, which we call global adiabaticity (GA), which is guaranteed if $c_w^2 = c_s^2$, where c_s is the phase velocity of the propagation of the perturbation. A known example is the case of ultra-slow-roll (USR) inflation in which $c_w^2 = c_s^2 = 1$. In order to generalize USR we develop a method to find the Lagrangian of GA K-inflation models from the behavior of background quantities as functions of the scale factor. Applying this method we show that there indeed exists a wide class of GA models with $c_w^2 = c_s^2$, which allows $\mathcal{R}_c$ to grow on superhorizon scales, and hence violates the non-Gaussianity consistency condition.

1. Introduction

A period of accelerated expansion during the early stages of the evolution of the Universe, called inflation [1–3], is able to account for several otherwise difficult to explain features of the observed Universe such as the high level of isotropy of the CMB [4] radiation and the small value of the curvature. Some of the simplest inflationary models are based on a single slowly-rolling scalar field, and they are in good agreement with observations. It is commonly assumed in slow-roll models that adiabaticity in the thermodynamic sense, $\delta P_{nad} \equiv \delta P - c_w^2 \delta\rho = 0$ where $c_w^2 = \dot{P}/\dot{\rho}$, implies the conservation of the curvature perturbation on uniform density slices ζ, and hence the conservation of the curvature perturbation on comoving slices $\mathcal{R}_c$, on super-horizon scales.

In [5] it was shown that there can be important exceptions, i.e. in some cases thermodynamic adiabaticity does not necessarily imply the super-horizon conservation of $\mathcal{R}_c$ and ζ, and that they can differ from each other. This can happen even for models in which $c_w^2 = c_s^2$. Here c_s is the speed of propagation of the curvature perturbation. It turns out that it may be defined as $c_s^2 \equiv (\delta P/\delta\rho)_c$, where the suffix "c" means a quantity evaluated on comoving slices defined by $\delta T^i_0 = 0$ (or equivalently slices on which the scalar field is homogeneous). An example is ultra-slow-roll (USR) inflation [6,7], in which the flat potential $V(\phi) = V_0$ yields exact adiabaticity $\delta P_{nad} = 0$ on all scales. USR inflation could in principle last for 60 e-folds, but then it would be difficult to make it consistent with observation. Alternatively, one can study models in which a USR phase is followed by a conventional slow-roll phase [8], at which stage $\mathcal{R}_c$ becomes conserved. In USR inflation, both $\mathcal{R}_c$ and ζ exhibit super-horizon growth but their behavior is very different from each other. As it has been stressed in [8], the non-freezing of $\mathcal{R}_c$ has important phenomenological consequences. Since the freezing of $\mathcal{R}_c$ on superhorizon scales is a necessary ingredient [9] for Maldacena's consistency relation [10] to hold, models that do not conserve $\mathcal{R}_c$ can actually violate that consistency condition. We note as well that another consequence

* Corresponding author.
E-mail address: aer@phys.ntu.edu.tw (A.E. Romano).

http://dx.doi.org/10.1016/j.physletb.2016.08.025

of superhorizon growth of $\mathcal{R}_c$ is that it should be evaluated at the end of inflation, instead of just after horizon crossing [7].

In this paper focusing on K-inflation, i.e., Einstein-scalar models with a general kinetic term, we explore in a general way other single field models which have $c_w^2 = c_s^2$, hence satisfy $\delta P_{nad} = 0$ on all scales which we call globally adiabatic (GA), but which may not conserve $\mathcal{R}_c$. We find a generalization of the USR model. A different generalization without imposing the condition $c_w^2 = c_s^2$ was discussed in [11,12].

The method we adopt is based on establishing a general condition for the non-conservation of $\mathcal{R}_c$ in terms of the dependence of the background quantities, in particular the slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$ and the sound velocity c_s, on the scale factor a.

We first derive the necessary condition for the comoving curvature perturbation $\mathcal{R}_c$ to grow on superhorizon scales. Next we determine $\rho(a)$ and $P(a)$ by solving the continuity equation. Then using the equivalence between barotropic fluids and K-inflationary models which satisfy the condition $c_w^2 = c_s^2$ [13,14], we determine the corresponding Lagrangian for the equivalent scalar field model. Using this method we obtain a new class of GA scalar field models which do not conserve $\mathcal{R}_c$.

Throughout the paper we denote the proper-time derivative by a dot ($^{\cdot} = d/dt$), the conformal-time derivative by a prime ($' = d/d\eta = a\, d/dt$) and the Hubble expansion rates in proper and conformal times by $H = \dot{a}/a$ and $\mathcal{H} = a'/a$, respectively. We also use the terminology "adiabaticity" for thermodynamic adiabaticity $\delta P_{nad} = 0$ throughout the paper.

2. Conservation of $\mathcal{R}_c$ and global adiabaticity

We set the perturbed metric as

$$ds^2 = a^2\Big[-(1+2A)d\eta^2 + 2\partial_j B dx^j d\eta + \Big\{\delta_{ij}(1+2\mathcal{R}) + 2\partial_i\partial_j E\Big\}dx^i dx^j\Big\}\Big]. \tag{1}$$

In [5] it was shown that independently of the gravity theory and for generic matter the energy–momentum conservation equations imply

$$\delta P_{nad} = \left[\left(\frac{c_w}{c_s}\right)^2 - 1\right](\rho + P)A_c\,. \tag{2}$$

In the case of general relativity, the additional relation $A_c = \dot{\mathcal{R}}_c/H$ gives an important relation for the time derivative of $\mathcal{R}_c$

$$\delta P_{nad} = \left[\left(\frac{c_w}{c_s}\right)^2 - 1\right](\rho + P)\frac{\dot{\mathcal{R}}_c}{H}. \tag{3}$$

The non-adiabatic pressure perturbation is given according to its thermodynamics definition

$$\delta P_{nad} \equiv \delta P - c_w^2 \delta\rho. \tag{4}$$

This definition of δP_{nad} is important because it is gauge invariant and $\delta P_{nad} = \delta P_{ud}$, where δP_{ud} is the pressure perturbation on uniform density ($\delta\rho = 0$) slices. It appears in the equation for the curvature perturbation on uniform density slices $\zeta \equiv \mathcal{R}_{ud}$ obtained from the energy conservation law [15],

$$\zeta' = -\frac{\mathcal{H}\delta P_{nad}}{(\rho + P)} + \frac{1}{3}\overset{(3)}{\Delta}\left(v - E'\right)_{ud} \tag{5}$$

where v is the 3-velocity potential ($v = \delta\phi/\phi'$ for a scalar field). In general, the curvature perturbations on uniform density and comoving slices are related as

$$\zeta = \mathcal{R}_c + \frac{\delta P_{nad}}{3(\rho + P)(c_s^2 - c_w^2)}\,. \tag{6}$$

A common interpretation of these equations (see for example [16,17]) is that when $\delta P_{nad} \approx 0$ with $c_w^2 \neq c_s^2$, ζ and $\mathcal{R}_c$ are approximately equal because of eq. (6), and they are both approximately conserved on super-horizon scales because of eq. (3). This is in agreement with the well-known coincidence of ζ and $\mathcal{R}_c$ on super-horizon scales for slow roll-models in general relativity, since in this case $c_s \neq c_w$ and $\delta P_{nad} \approx 0$ on superhorizon scales.

The equation (3) is the key relation to understand how $\mathcal{R}_c$ depends on the non-adiabatic pressure δP_{nad}. First of all let us note that this equation is valid on any scale. The advantage of it with respect to eq. (5) is that it does not involve gradient terms, so it allows us to directly relate $\delta P_{\rm nad}$ to $\dot{\mathcal{R}}_c$ if $c_w^2 \neq c_s^2$, while in eq. (5) $\dot{\zeta}$ depends on spatial gradients, which in the case of USR are not negligible on super-horizon scales [5]. This explains why in USR in which $c_w^2 = c_s^2 = 1$, both $\mathcal{R}_c$ and ζ are not conserved despite $\delta P_{nad} = 0$.

It should be noted here that for slow-roll attractor models $c_w^2 \neq c_s^2$ in general, and $\mathcal{R}_c$ is time-varying on sub-horizon scales. This implies that the non-adiabatic pressure perturbation δP_{nad} on sub-horizon scales is not zero. In other words, the attractor models are adiabatic only on super-horizon scales, and we call these models super-horizon adiabatic (SHA).

From eq. (3) we can immediately deduce that in general relativity there are two possible scenarios for the non-conservation of $\mathcal{R}_c$,

$$\begin{aligned}&(1)\quad c_s^2 = c_w^2\,, \quad \delta P_{nad} = 0\,,\\ &(2)\quad c_s^2 \neq c_w^2\,, \quad \delta P_{nad} \neq 0\,.\end{aligned} \tag{7}$$

The second case was studied in [11,12]. Here we focus on the first case. It is trivial to see that because of the gauge invariance of δP_{nad} the condition $c_w^2 = c_s^2$ automatically implies $\delta P_{nad} = 0$. The models satisfying the condition $c_s^2 - c_w^2 = \delta P_{nad} = 0$ are adiabatic on any scale, and because of this we call them globally adiabatic (GA). In GA models an explicit calculation can reveal the super-horizon behavior of $\mathcal{R}_c$, and ζ, as was shown in [5] in the case of USR. Below, we develop an inversion method to find a new class of models that violate the conservation of $\mathcal{R}_c$ without solving the perturbations equations.

3. Globally adiabatic K-essence models

The condition $c_w^2 = c_s^2$ has been studied in the context of K-inflation [13] described by the action ($X \equiv -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$)

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left[M_{Pl}^2 R + 2P(X,\phi)\right], \tag{8}$$

and it was shown that it is satisfied by scalar field models with the Lagrangian of the form,

$$P(X,\phi) = u(Xg(\phi)) \equiv u(Y) \tag{9}$$

where u and g are arbitrary functions. These models are equivalent to a barotropic perfect fluid, i.e. a fluid with equation of state $P(\rho)$. See also [18–21]. We note again that these models *are adiabatic on any scale* (GA), contrary to the slow-roll attractor models, which are adiabatic *only on super-horizon scales* (SHA). The fact that they are mutually exclusive can be readily seen by considering the hypothetical case of $\delta P_{nad} = 0$ *and* $c_w^2 \neq c_s^2$. In this case eq. (3) which is valid on any scale would mean $\mathcal{R}_c$ should be *frozen on all scales*. In contrast, the condition $c_w^2 = c_s^2$ allows for the curvature perturbation to *evolve* both on *sub-horizon* and *super-horizon* scales.

In [13] it was shown that it is possible to associate any barotropic perfect fluid with an equivalent K-inflation model according to

$$2\int^{P}\frac{du}{F(u)}=\log(Y)\,, \tag{10}$$

where $F(P)=\rho(P)+P$ and $Y=g(\phi)X$. These models are the ones which could violate the conservation of $\mathcal{R}_c$ for adiabatic perturbations, since they satisfy $c_w^2=c_s^2$. It is noted of course that the global adiabaticity is not the sufficient condition for the non-conservation of $\mathcal{R}_c$. Not all GA models violate the conservation of $\mathcal{R}_c$ on super-horizon scales.

4. General conditions for super-horizon growth of $\mathcal{R}_c$

From the equation for the curvature perturbation on comoving slices,

$$\frac{\partial}{\partial t}\left(\frac{a^3\epsilon}{c_s^2}\frac{\partial}{\partial t}\mathcal{R}_c\right)-a\epsilon\Delta\mathcal{R}_c=0\,, \tag{11}$$

we can deduce, after re-expressing the time derivative in terms of the derivative respect to the scale factor a, that on superhorizon scales there is (apart from a constant solution) a solution of the form,

$$\mathcal{R}_c\propto\int^{a}\frac{da}{a}f(a)\,;\quad f(a)\equiv\frac{c_s^2(a)}{Ha^3\epsilon(a)}\,, \tag{12}$$

where we have introduced the function $f(a)$ for later convenience. In conventional slow-roll inflation c_s^2 and ϵ are both slowly varying, hence the integral rapidly approaches a constant, rendering $\mathcal{R}_c$ conserved. The time dependent part of the above solution corresponds to the decaying mode.

The necessary and sufficient condition for super-horizon freezing is that there exists some $\delta>0$ for which

$$\lim_{a\to\infty}a^{\delta}f(a)=0. \tag{13}$$

By definition of inflation, H must be sufficiently slowly varying; $\epsilon=-\dot{H}/H^2\ll 1$. So we may neglect the time dependence of H in eq. (12) at leading order, while ϵ and c_s^2 may vary rapidly in time. For models for which $\epsilon\approx a^{-n}$ and $c_s^2\approx a^q$ we get

$$f\propto a^{q+n-3}\,, \tag{14}$$

hence the condition for freezing is

$$q+n-3<0\,. \tag{15}$$

If this condition is violated, i.e. $q+n-3\geq 0$, then the solution (12) will grow on super-horizon scales. This happens for example in USR, which corresponds to $c_s^2=1$ and $\epsilon\propto a^{-6}$, i.e. $q=0$, and $n=6$. (The super-horizon growth of $\mathcal{R}_c$ in USR can also be understood as a direct consequence of the non-attractor nature of USR [22].) In general, we expect that q would not become very large. This implies ϵ should decrease sufficiently rapidly. Conversely, if ϵ decreases sufficiently rapidly, then the growth of $\mathcal{R}_c$ on superhorizon scales will follow.

5. Barotropic model

We have shown that GA models could violate the super-horizon conservation of $\mathcal{R}_c$, so now we will look for GA K-essence models which do indeed violate it, based on the freezing condition in eq. (13). Inspired by the equivalence between barotropic fluids and GA K-essence models [13] we will first look for barotropic fluids that can give the growing curvature perturbation on superhorizon scales. From the very beginning we will set $c_w^2=c_s^2$.

Using the Friedmann equation we can write the slow-roll parameter ϵ as

$$\epsilon=-\frac{\dot{H}}{H^2}=\frac{3}{2}\frac{\rho+P}{\rho}\,. \tag{16}$$

In terms of the scale factor and ϵ the energy conservation equation reads

$$\frac{d\rho}{da}+\frac{3}{a}(\rho+p)=\frac{d\rho}{da}+\frac{2\epsilon\rho}{a}=0. \tag{17}$$

We may now define the quantity $b(a)=2\epsilon\rho$. It appears naturally in the continuity equation and plays a crucial role in regards to the super-horizon behavior of curvature perturbations because the function $f(a)$ can be re-written in terms of it as

$$f(a)\propto\frac{Hc_s^2}{a^3b(a)}\,. \tag{18}$$

Integrating the energy conservation equation we get

$$\rho(a)=\rho_0\exp\left[-2\int_{a_0}^{a}\frac{\epsilon}{a}da\right]=\int-\frac{b(a)}{a}da\,. \tag{19}$$

Using eq. (16), we then obtain

$$P(a)=\left(\frac{2}{3}\epsilon-1\right)\rho\,. \tag{20}$$

The sound velocity is given by

$$\begin{aligned}c_w^2=c_s^2=\frac{dP}{d\rho}&=-1+\frac{1}{3}\frac{db(a)}{d\rho}\\&=-1+\frac{1}{3}\frac{db(a)}{da}\Big/\left(\frac{d\rho}{da}\right)\\&=-1-\frac{a}{3b(a)}\frac{db(a)}{da}\,.\end{aligned} \tag{21}$$

We now consider the behavior of $f(a)$ introduced in (12). As mentioned before, we consider the case when ϵ decreases sufficiently rapidly. In this case, $\rho=3H^2M_P^2$ approaches a constant rapidly. Hence the time dependence of ρ may be neglected compared to that of other quantities that vary far more rapidly. With this approximation, assuming $\epsilon\propto a^{-n}$, we find

$$c_s^2\approx\frac{n-3}{3}\,, \tag{22}$$

which means $q\approx 0$, and

$$f(a)=\frac{c_s^2(a)}{Ha^3\epsilon(a)}\propto a^{n-3}\,, \tag{23}$$

which satisfies the condition for the growth if $n>3$, in accordance with the original anticipation. In passing, it is interesting to note that the condition $n>3$ implies $c_s^2>0$, a necessary condition to avoid the gradient instability of the perturbation. Thus virtually all GA models that are free from the gradient instability exhibit superhorizon growth of the comoving curvature perturbation $\mathcal{R}_c$.

6. Scalar field model

Let us now find a scalar field model that corresponds to the barotropic model discussed in the previous section. As a warm-up, let us consider the USR case, whose fluid interpretation has already been studied in [23]. In this case, we exactly have $c_s^2=1$. From eq. (21), this implies $b/2 = \epsilon\rho \; \left(= 3(\rho + P)/2\right) \propto a^{-6}$. Also $c_s^2 = 1$ implies $\rho = P + const$. Inserting this into eq. (10) gives

$$\frac{2dP}{2P + const.} = \frac{dY}{Y}\,. \tag{24}$$

Thus up to a constant term P and Y are the same,

$$P = Y + const. \tag{25}$$

Absorbing $g(\phi)$ in Y into the definition of the scalar field by $g^{1/2}d\phi \to d\phi$, this is indeed the Lagrangian for a minimally coupled massless scalar with a cosmological constant:

$$L = P(\phi, X) = X - V_0\,. \tag{26}$$

This is consistent with $\rho + P = 2X \propto \epsilon\rho \propto a^{-6}$.

Let us generalize the USR case. As in the previous section, we consider models that have the behavior of $\epsilon\rho$ as

$$2\epsilon\rho = b(a)\,, \tag{27}$$

where $b(a)$ should decrease faster than a^{-3} asymptotically at $a \to \infty$ but otherwise is an arbitrary function. Then we have

$$F(P) \equiv \rho + P = 2H^2\epsilon = \frac{2\epsilon\rho}{3} = \frac{b(a)}{3}\,, \tag{28}$$

which gives

$$\frac{dY}{Y} = 2\frac{dP}{F(P)} = 6\frac{dP}{2\epsilon\rho} = 6\frac{dP}{b(a)}\,. \tag{29}$$

For dP, using the energy conservation law, we may rewrite it as

$$\begin{aligned} dP &= d\left(-\rho + F(P)\right) = -d\rho + \frac{db(a)}{3} \\ &= 3\frac{da}{a}(\rho + P) + \frac{db(a)}{3} = b(a)\frac{da}{a} + \frac{db(a)}{3}\,. \end{aligned} \tag{30}$$

Therefore we have

$$\frac{dY}{Y} = 6\frac{dP}{b(a)} = 6\frac{da}{a} + 2\frac{db}{b}\,. \tag{31}$$

Hence

$$Y \propto a^6 b^2\,. \tag{32}$$

This is consistent with the USR case in which $b(a) \propto a^{-6}$ and $Y = X \propto a^{-6}$.

This relation is quite useful since it allows to rewrite the freezing function $f(a)$ as

$$f(a) \propto \frac{Hc_s^2}{\sqrt{Y}}\,, \tag{33}$$

from which we can deduce that $Y(a)$ determines the super-horizon behavior of $\mathcal{R}_c$. In particular, for the models we are considering in which c_s is constant, we infer that super-horizon growth can happen in the limit $Y \to 0$.

For a given choice of $b(a)$, eq. (32) can be inverted to give the scale factor as a function of Y, $a = a(Y)$. Also eq. (30) can be integrated to give $P = P(a)$. Combining these two, one can obtain the Lagrangian for the scalar field, $L = P = P(Y)$.

Note that in GA models there is a one-to-one correspondence between the scale factor and state variables such as $P(a)$ and $\rho(a)$, which is the reason why we can also write a barotropic equation of state $P(\rho) = P(a(\rho))$. Once any of the functions $P(a), \rho(a), b(a), \epsilon(a), Y(a)$ is specified, all the others are specified too, as well as the equation of state $P(\rho)$ or its scalar field equivalent Lagrangian $P(Y)$, which is in fact the basis of the inversion method that we are developing in this paper.

7. Examples

Here we give a couple of specific K-inflation models that are globally adiabatic and violate the conservation of $\mathcal{R}_c$. Given the parametric behavior of $b \equiv 2\epsilon\rho$, our inversion method allows us to deduce the Lagrangian.

7.1. Ex 1: generalized USR

Let us consider a specific case where $b(a)$ is a power-law function,

$$2\epsilon\rho = b(a) = ca^{-n}\,, \tag{34}$$

where c is a constant. We assume $n > 3$ in order to have the growth on superhorizon scales.

From eq. (32) we have

$$a \propto Y^{1/(6-2n)}\,. \tag{35}$$

Now eq. (30) gives

$$\begin{aligned} P &= \int^a \left(b(a)\frac{da}{a} + \frac{db(a)}{3}\right) \\ &= -\frac{c}{n}a^{-n} + \frac{c}{3}a^{-n} + const. \\ &= \frac{n-3}{3n}b(a) + const. \end{aligned} \tag{36}$$

Plugging eq. (35) into this, we finally obtain

$$L = P(Y) = Y^{n/(2n-6)} - V_0\,. \tag{37}$$

Since this may be regarded as a natural generalization of the USR case, which corresponds to the case $n = 6$, we call it the generalized USR (GUSR) model. Lagrangians involving Y^α terms have already been studied in [11,24,25], but those models are either not exactly globally adiabatic because of the presence of a non-constant potential or they satisfy the relation $\epsilon \propto a^{-n}$ only approximately and during a limited time range, while for GUSR $\epsilon \propto a^{-n}$ is an exact relation and is valid at any time. As the Lagrangian is of the type described in eqs. (9) and (26) (remember that after a field transformation Y can be made equal to X), we understand that this scalar field model is indeed equivalent to a barotropic fluid. Hence we have $c_w^2 = c_s^2$, and therefore $\delta P_{nad} = 0$. Indeed the second condition for super-horizon growth of $\mathcal{R}_c$ given in eq. (7) is satisfied. More precisely, we note that for the GUSR model, the sound velocity is exactly constant,

$$c_w^2 = c_s^2 = \frac{n-3}{3}\,. \tag{38}$$

The power spectrum of the comoving curvature perturbation can be explicitly computed for this model. One finds [26] that the spectral index is a function of n: $n_s - 1 = 6 - n$, in agreement with the scale invariant spectrum of the original ultra slow-roll inflation in which one has $n = 6$. Hence, the model can be constrained by the observational value. Note as well, from eq. (38), that to have a slightly red-tilted spectrum, we need a slightly superluminal speed of sound.

7.2. Ex 2: Lambert inflation

As another example, let us consider the case when ϵ is a power-law function,

$$\epsilon(a) = \epsilon_0 a^{-n}\,. \tag{39}$$

As before, we assume $n > 3$. In this case, since $d\log\rho/d\log a = -2\epsilon \propto a^{-n}$, we find

$$\rho(a) = \rho_0 \exp\left[\frac{2\epsilon}{n}\right]. \tag{40}$$

It is clear that ρ approaches a constant ρ_0 asymptotically at $a \to \infty$.

Inserting eq. (39) and eq. (40) into eq. (21), the sound velocity is given by

$$c_w^2 = c_s^2 = -1 - \frac{1}{3}\left(\frac{d\log\epsilon}{d\log a} + \frac{d\log\rho}{d\log a}\right) = \frac{n-3+2\epsilon}{3}. \tag{41}$$

Thus c_s^2 is time dependent, but it rapidly approaches a constant as ϵ decays out. Also from eq. (39) and eq. (40), we find

$$b(a) = 2\epsilon\rho = 2\epsilon\rho_0 \exp\left[\frac{2\epsilon}{n}\right]. \tag{42}$$

Thus we have

$$Y \propto a^6 b^2 \propto a^{6-2n} \exp\left[\frac{4\epsilon}{n}\right] \propto \epsilon^{(2n-6)/n} \exp\left[\frac{4\epsilon}{n}\right], \tag{43}$$

which implies

$$Y^{n/(2n-6)} \propto \frac{4\epsilon}{2n-6} \exp\left[\frac{4\epsilon}{2n-6}\right]. \tag{44}$$

To find the Lagrangian, we manipulate eq. (30) as

$$\begin{aligned} dP &= b\frac{da}{a} + \frac{db}{3} = -\frac{b}{n}\frac{d\epsilon}{\epsilon} + \frac{db}{3} \\ &= -\frac{2}{n}\rho_0 e^{2\epsilon/n} d\epsilon + \frac{db}{3}. \end{aligned} \tag{45}$$

Therefore, integrating this we obtain

$$P = \rho_0 e^{2\epsilon/n}\left(-1 + \frac{2\epsilon}{3}\right) + const. \tag{46}$$

One can invert eq. (44) to find ϵ as a function of Y, and then insert it into the above to obtain the Lagrangian.

Specifically, we introduce the Lambert function $W(x)$ defined by the inverse function of $X(z) = ze^z$,

$$z = X^{-1}(ze^z) \equiv W(ze^z). \tag{47}$$

Setting

$$Y^{n/(2n-6)} = ze^z; \quad z = \frac{4\epsilon}{2n-6}, \tag{48}$$

we have

$$\frac{4\epsilon}{2n-6} = W(y); \quad y \equiv Y^{n/(2n-6)}. \tag{49}$$

Inserting this into eq. (46), we finally obtain

$$\begin{aligned} L &= P(Y) \\ &= \rho_0 \left(\frac{n-3}{3} W(y) - 1\right) \exp\left[\frac{n-3}{n} W(y)\right] - V_0, \end{aligned} \tag{50}$$

where $y = y(Y)$ is given in eq. (49).

Note that this model has been derived without making any approximation, and it gives exactly $\epsilon \propto a^{-n}$. However, as we mentioned before, in the late time limit, there is no difference between $\epsilon \propto a^{-n}$ and $\rho\epsilon \propto a^{-n}$. Thus the two models discussed above are essentially the same at late times. This can be easily checked by expanding $W(y)$ around $y = 0$,

$$W(y) = y - y^2 + \cdots. \tag{51}$$

At leading order in $y = Y^{n/(2n-6)}$, this gives

$$P(Y) = \frac{n-3}{3}\rho_0 Y^{n/(2n-6)} - \rho_0 - V_0. \tag{52}$$

By absorbing the constant coefficient into $g(\phi)$ in the definition of Y, $Y = g(\phi)X$, and absorbing ρ_0 into the constant V_0, eq. (50) reduces to

$$P = Y^{n/(2n-6)} - V_0, \tag{53}$$

which indeed coincides with the GUSR model, see eq. (37).

Higher order terms in the expansion give an infinite class of models of the type

$$u(Y) = \sum_i \beta_i Y^{n_i}, \tag{54}$$

where β_i are appropriate coefficients.

Finally, note that in USR and as well in the two examples considered here, the shift symmetry in the potential ($V(\phi) = V_0$) is a direct consequence of the demand $c_w^2 = c_s^2$, which in turn follows from the global adiabaticity of the model. That is in line with the general statement [27,28] that for a k-essence theory to describe a fluid, one needs a shift symmetry (i.e., there is no physical clock, the model is of the non-attractor type).

8. Conclusions

In conventional slow-roll models, one has $c_s^2 \neq c_w^2$ and the superhorizon freezing of $\mathcal{R}_c$ can be understood as a result of $\delta P_{\rm nad} \approx 0$ on superhorizon scales. When $c_s^2 = c_w^2$, one has $\delta P_{\rm nad} = 0$ on all scales, but following eq. (3) this does not constrain the superhorizon behavior of $\mathcal{R}_c$ anymore. This behavior now follows from $\mathcal{R}_c$'s equation of motion given in eq. (11), and the condition for superhorizon freezing is given in eq. (13). Violation of this condition leads to superhorizon growth of $\mathcal{R}_c$.

We have developed a method to construct the Lagrangian of a K-essence globally adiabatic (GA) model by specifying the behavior of background quantities such as $\epsilon\rho$ where ϵ is the slow-roll parameter, using the equivalence between barotropic fluids and GA K-essence models. We have applied the method to find the equations of state of the fluids and derive the Lagrangian of the equivalent single scalar field models. Interestingly, we have found that the requirement to avoid the gradient instability, i.e., $c_s^2 > 0$ is almost identical to the condition for the non-conservation on superhorizon scales.

The advantage of our approach is that we did not have to solve any perturbation equation explicitly. We have begun from requiring some behavior for ϵ, or for $b \equiv 2\epsilon\rho$, and have then used our inversion method to find the Lagrangian that produces that behavior.

We have shown that the main difference between attractor models and GA models is that the latter are adiabatic on all scales, while attractor models are approximately adiabatic in the sense of $\delta P_{nad} = 0$ only on super-horizon scales and $c_w^2 \neq c_s^2$.

The detailed study of the new models found in this paper will be done in a separate upcoming work [26] but we can already predict that they can be compatible with observational constraints on the spectral index thanks to the extra parameter n which is not present in USR. Furthermore they can violate the Maldacena's consistency condition and consequently produce large local shape non-Gaussianity.

In the future it will be interesting to apply the inversion method we have developed to other problems related to primordial curvature perturbations, or to develop a similar method for the adiabatic sound speed as function of the scale factor.

Acknowledgements

The work of MS was supported by MEXT KAKENHI No. 15H05888. SM is funded by the FONDECYT 2015 Postdoctoral Grant 3150126. This work was supported by the Dedicacion exclusiva and Sostenibilidad programs at UdeA, the UDEA CODI project IN10219CE and 2015-4044, and COLCIENCIAS mobility project COSOMOLOGY AFTER BICEP.

References

[1] A.D. Linde, A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems, Phys. Lett. B 108 (1982) 389.
[2] A. Albrecht, P.J. Steinhardt, Cosmology for grand unified theories with radiatively induced symmetry breaking, Phys. Rev. Lett. 48 (1982) 1220.
[3] A.A. Starobinsky, Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations, Phys. Lett. B 117 (1982) 175.
[4] P.A.R. Ade, et al., Planck Collaboration, Planck 2015 results. XX. Constraints on inflation, arXiv:1502.02114 [astro-ph.CO].
[5] A.E. Romano, S. Mooij, M. Sasaki, Adiabaticity and gravity theory independent conservation laws for cosmological perturbations, Phys. Lett. B 755 (2016) 464, arXiv:1512.05757 [gr-qc].
[6] N.C. Tsamis, R.P. Woodard, Improved estimates of cosmological perturbations, Phys. Rev. D 69 (2004) 084005, arXiv:astro-ph/0307463.
[7] W.H. Kinney, Horizon crossing and inflation with large eta, Phys. Rev. D 72 (2005) 023515, arXiv:gr-qc/0503017.
[8] M.H. Namjoo, H. Firouzjahi, M. Sasaki, Violation of non-Gaussianity consistency relation in a single field inflationary model, Europhys. Lett. 101 (2013) 39001, arXiv:1210.3692 [astro-ph.CO].
[9] P. Creminelli, M. Zaldarriaga, Single field consistency relation for the 3-point function, J. Cosmol. Astropart. Phys. 0410 (2004) 006, arXiv:astro-ph/0407059.
[10] J.M. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, J. High Energy Phys. 0305 (2003) 013, arXiv:astro-ph/0210603.
[11] X. Chen, H. Firouzjahi, M.H. Namjoo, M. Sasaki, A single field inflation model with large local non-Gaussianity, Europhys. Lett. 102 (2013) 59001, arXiv:1301.5699 [hep-th].
[12] J. Martin, H. Motohashi, T. Suyama, Ultra slow-roll inflation and the non-Gaussianity consistency relation, Phys. Rev. D 87 (2) (2013) 023514, arXiv:1211.0083 [astro-ph.CO].
[13] F. Arroja, M. Sasaki, A note on the equivalence of a barotropic perfect fluid with a K-essence scalar field, Phys. Rev. D 81 (2010) 107301, arXiv:1002.1376 [astro-ph.CO].
[14] S. Unnikrishnan, L. Sriramkumar, A note on perfect scalar fields, Phys. Rev. D 81 (2010) 103511, arXiv:1002.0820 [astro-ph.CO].
[15] D. Wands, K.A. Malik, D.H. Lyth, A.R. Liddle, A new approach to the evolution of cosmological perturbations on large scales, Phys. Rev. D 62 (2000) 043527, arXiv:astro-ph/0003278.
[16] D.H. Lyth, K.A. Malik, M. Sasaki, A general proof of the conservation of the curvature perturbation, J. Cosmol. Astropart. Phys. 0505 (2005) 004, arXiv:astro-ph/0411220.
[17] A.J. Christopherson, K.A. Malik, The non-adiabatic pressure in general scalar field systems, Phys. Lett. B 675 (2009) 159, arXiv:0809.3518 [astro-ph].
[18] C. Quercellini, M. Bruni, A. Balbi, Affine equation of state from quintessence and k-essence fields, Class. Quantum Gravity 24 (2007) 5413, arXiv:0706.3667 [astro-ph].
[19] V. Faraoni, The correspondence between a scalar field and an effective perfect fluid, Phys. Rev. D 85 (2012) 024040, arXiv:1201.1448 [gr-qc].
[20] A. Diez-Tejedor, Note on scalars, perfect fluids, constrained field theories, and all that, Phys. Lett. B 727 (2013) 27, arXiv:1309.4756 [gr-qc].
[21] P. Wongjun, A perfect fluid in Lagrangian formulation due to generalized three-form field, arXiv:1602.00682 [gr-qc].
[22] S. Mooij, G.A. Palma, Consistently violating the non-Gaussian consistency relation, J. Cosmol. Astropart. Phys. 1511 (11) (2015) 025, arXiv:1502.03458 [astro-ph.CO].
[23] X. Chen, H. Firouzjahi, M.H. Namjoo, M. Sasaki, Fluid inflation, J. Cosmol. Astropart. Phys. 1309 (2013) 012, arXiv:1306.2901 [hep-th].
[24] X. Chen, H. Firouzjahi, E. Komatsu, M.H. Namjoo, M. Sasaki, In-in and δN calculations of the bispectrum from non-attractor single-field inflation, J. Cosmol. Astropart. Phys. 1312 (2013) 039, arXiv:1308.5341 [astro-ph.CO].
[25] S. Hirano, T. Kobayashi, S. Yokoyama, Ultra slow-roll G-inflation, arXiv:1604.00141 [astro-ph.CO].
[26] A.E. Romano, S. Mooij, in preparation.
[27] R. Akhoury, C.S. Gauthier, A. Vikman, Stationary configurations imply shift symmetry: no Bondi accretion for quintessence / k-essence, J. High Energy Phys. 0903 (2009) 082, arXiv:0811.1620 [astro-ph].
[28] I. Sawicki, I.D. Saltas, L. Amendola, M. Kunz, Consistent perturbations in an imperfect fluid, J. Cosmol. Astropart. Phys. 1301 (2013) 004, arXiv:1208.4855 [astro-ph.CO].