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The summation of power series and Fourier series

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Abstract: The well-known correspondence of a power series with a certain Stieltjes integral is exploited for summation of the series by numerical integration. The emphasis in this paper is thus on actual summation of series, rather than mere acceleration of convergence. It is assumed that the coefficients of the series are given analytically, and then the numerator of the integrand is determined by the aid of the inverse of the two-sided Laplace transform, while the denominator is standard (and known) for all power series.

Since Fourier series can be expressed in terms of power series, the method is applicable also to them.

The treatment is extended to divergent series, and a fair number of numerical examples are given, in order to illustrate various techniques for the numerical evaluation of the resulting integrals.

Keywords: Summation of series.

1. Introduction

We start by considering a power series, which it is convenient to write in the form

$$s(x)=\mu_1-\mu_2x+\mu_3x^2\ldots, \tag{1}$$

for reasons which will become apparent presently. Here there is no intention to limit ourselves to alternating series since, even if the μ_k are all of one sign, x may take negative and even complex values. It will be assumed in this paper that the μ_k are real. The reason for writing (1) with alternating signs is that in many applications x will be real, and negative values of x in (1) (for which (1) will usually be a monotonic series) are associated with a special difficulty, as will become clear in the sequel.

We put aside, for the moment, the question of convergence of (1), and whether convergent or divergent, the series will be considered as a formal expansion in powers of x of the Stieltjes integral

$$I(x)=\int_0^\infty \frac{f(u)}{l+xu}\,\mathrm{d}u. \tag{2}$$

Performing this expansion by means of the formal relation

$$(1+xu)^{-1}=1-xu+x^2u^2+x^3u^3+\ldots \tag{3}$$

followed by integration, the μ_k's are revealed as moments

$$\mu_k = \int_0^\infty u^{k-1} f(u)\, \mathrm{d}u \tag{4}$$

of the function $f(u)$.

It is worth noting at this point that in many applications $f(u)$ will vanish identically for all $u > a$, where a is a positive number, so that the integrals (2), (4) will, in many cases, be effectively over a finite range from $u = 0$ to $u = a$.

The method employed in this paper for the summation of the series (1) consists in the evaluation of $I(x)$ by numerical quadrature. There is already evident at this stage the special difficulty mentioned above, that may be associated with negative real values of x, namely that due to the possible existence of a singularity at the point $u = -1/x$ in the range of integration, in the case that $a \geqslant -1/x$.

The basic problem now is to determine the function $f(u)$ from the moments μ_k, and this essentially involves the inversion of the Mellin transform (4). Some consideration of the problem has been given by the author in an earlier paper [7], but there only the effectively finite Mellin transform was mentioned. In order to make use of the elaborate machinery of the two-sided Laplace transform [10], we now write the moments μ_k in the form

$$\mu_p = \bar{h}(p), \quad p = k = 1, 2, 3, \ldots, \tag{5}$$

to emphasise the fact that we now regard them as the values, for positive integral, p, of the two-sided Laplace transform

$$\bar{h}(p) = \int_{-\infty}^{\infty} \mathrm{e}^{-pt} h(t)\, \mathrm{d}t, \quad \alpha < \operatorname{Re} p < \beta, \tag{6}$$

of a function $h(t)$. It should be noted that Van der Pol and Bremmer in their book [10] define the two-sided Laplace transform with a factor p before the integral, so that appropriate corrections should be made before applying their results to the method outlined in this paper. Here the inequalities following the integral in (6) represent the strip of convergence on the integral, and for our purpose we require $\beta = \infty$. On occasion also the rather restricted tables of the Mellin transform and its inverse [3,8] may also be useful.

In order now to find $f(u)$, our first step is to invert the Laplace transform (6), thus obtaining $h(t)$ from $\bar{h}(p)$. Next in (6) we make the transformation

$$\mathrm{e}^{-t} = u, \qquad t = -\ln u, \qquad \mathrm{d}t = -\mathrm{d}u/u \tag{7}$$

giving (4) for $p = k = 1, 2, 3, \ldots,$ where we have now written

$$f(u) = h(-\ln u). \tag{8}$$

We have in this convenient way inverted the Mellin transform to find $f(u)$ from the moments μ_k.

Once $f(u)$ has been obtained we 'sum' the series (1) by numerical evaluation of the corresponding integral (2).

In Section 3 we give some numerical examples to illustrate the procedure. Of course simple (non-power) series may be treated by taking $x = 1$ or $x = -1$.

2. Application to Fourier series

Let us start with a Fourier cosine series

$$g_1(\theta) = \sum_{n=0}^{\infty} a_n \cos n\,\theta. \tag{9}$$

Defining

$$\mu_k = a_{k-1}, \quad k = 1, 2, 3, \ldots, \tag{10}$$

and setting

$$x = \mathrm{e}^{\mathrm{i}\theta}, \tag{11}$$

we have

$$g_1(\theta) = \mathrm{Re}\left(\mu_1 + \mu_2 x + \mu_3 x^2 + \ldots\right). \tag{12}$$

Writing then $\mu_k = \bar{h}_1(k)$, we seek the inverse $h_1(t)$ as before, and then define $f_1(u) = h_1(-\ln u)$. Then in accordance with the above theory, the sum of our series (9) is

$$g_1(\theta) = \mathrm{Re}\int_0^{\infty} \frac{f_1(u)}{1 - xu}\,\mathrm{d}u = \int_0^{\infty} \frac{(1 - u\cos\theta) f_1(u)}{1 - 2u\cos\theta + u^2}\,\mathrm{d}u, \tag{13}$$

and the method then is to evaluate the last integral in (13) by numerical quadrature.

Turning now to the Fourier sine series

$$g_2(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta, \tag{14}$$

this is conveniently written in the form

$$g_2(\theta) = I\left[x\left(\nu_1 + \nu_2 x + \nu_3 x^2 + \ldots\right)\right], \tag{15}$$

where we have defined

$$\nu_k = b_k, \quad k = 1, 2, 3, \ldots, \tag{16}$$

and $x = \mathrm{e}^{\mathrm{i}\theta}$ as before. In accordance with our procedure we write $\nu_k = \bar{h}_2(k)$, seek the inverse $h_2(t)$, and then define $f_2(u) = h_2(-\ln u)$. Then the sum of our series (14) is

$$g_2(\theta) = I\left[x\int_0^{\infty} \frac{f_2(u)}{1 - xu}\,\mathrm{d}u\right] = \sin\theta \int_0^{\infty} \frac{f_2(u)}{1 - 2u\cos\theta + u^2}\,\mathrm{d}u. \tag{17}$$

An application of (13), (17) to a specific example is given at the end of the next section.

3. Numerical examples

Let us start with the series

$$s(x) = 1 - \frac{x}{2^2} + \frac{x^2}{3^2} - \frac{x^3}{4^2} + \ldots \tag{18}$$

which for $x = 1$ has the sum $\frac{1}{12}\pi^2$, and for $x = -1$ the sum $\frac{1}{6}\pi^2$. Here we have

$$\bar{h}(p) = \mu_p = p^{-2}, \quad p = k = 1, 2, 3, \ldots \tag{19}$$

which gives us immediately the inverse

$$h(t) = t\, U(t), \quad \text{Re } p > 0, \tag{20}$$

where $U(t)$ is the unit function defined by

$$U(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{21}$$

Going over to the variable u, we now have

$$f(u) = h(-\ln u) = -\ln u\; U(-\ln u), \tag{22}$$

so that the 'sum' of our series (18) is

$$I(x) = \int_0^\infty \frac{(-\ln u)}{1+xu} U(-\ln u)\, \mathrm{d}u = \int_0^1 \frac{(-\ln u)}{1+xu} \mathrm{d}u, \tag{23}$$

and we see that in this case the upper limit of integration is effectively $u = 1$.

This integral converges for any positive (or zero) x, and thus gives a 'sum' for the series (18), even when it is divergent, if x is positive. $I(x)$, $x \geqslant 0$, is readily evaluated by numerical quadrature, and for this purpose it is desirable to eliminate the singularity in the integrand at $u = 0$. Using the simple integral

$$\int_0^1 (-\ln u)\mathrm{d}u = [u - u \ln u]_0^1 = 1, \tag{24}$$

we find

$$I(x) = 1 + x\int_0^1 \frac{u \ln u}{1+xu} \mathrm{d}u, \tag{25}$$

where the integrand is now finite throughout the range of integration, and the integral is now readily evaluated by a simple quadrature rule. For example, Simpson's rule with 32 ordinates gives for $x = 1$ the result

$$I(1) = 0.82254,$$

which may be compared with the exact value

$$\tfrac{1}{12}\pi^2 = 0.82247\ldots.$$

Greater accuracy is, of course, readily achieved by increasing the number of ordinates or using a Gaussian rule.

For negative values of $x < -1$ the integral (25) diverges but still has a finite Cauchy principal value. The general question of the relation of such Cauchy principal values with the series (1) will be the subject of a separate investigation. For a convenient method of numerical evaluation of Cauchy principal values of integrals see Longman [6]. For the case $x = -1$, the integral (25) is still convergent since there is then no pole at $u = -1/x = 1$ because of the vanishing of the numerator $u \ln u$ of the integrand at this point. We have now, from (25),

$$I(-1) = 1 - \int_0^1 \frac{u \ln u}{1-u} \mathrm{d}u, \tag{26}$$

and the integrand is finite over the range of integration, since

$$\lim_{u \to 1} \frac{u \ln u}{1-u} = -1. \tag{27}$$

Application of Simpson's rule with 32 ordinates yields the result

$$I(-1) = 1.64486$$

which may be compared with the exact

$$\tfrac{1}{6}\pi^2 = 1.64493\ldots.$$

As a second example we consider the series

$$s(x) = 1 - 1!x + 2!x^2 - 3!x^3 + \ldots \tag{28}$$

which is famous in divergent series theory [12]. For general theory of divergent series the reader is referred to Hardy [4]. Here we have

$$\bar{h}(p) = \mu_p = (p-1)!, \quad p = k = 1, 2, 3, \ldots. \tag{29}$$

Inverting the Laplace transform we immediately have [10]

$$h(t) = \mathrm{e}^{-\mathrm{e}^{-t}}, \quad -\infty < \operatorname{Re} p < \infty, \tag{30}$$

so that

$$f(u) = \mathrm{e}^{-u} \tag{31}$$

and

$$I(x) = \int_0^\infty \frac{\mathrm{e}^{-u}}{1 + xu}\,\mathrm{d}u. \tag{32}$$

In particular for the series

$$s(1) = 1 - 1! + 2! - 3! + \ldots, \tag{33}$$

we expect to have the 'sum'

$$I(1) = \int_0^\infty \frac{\mathrm{e}^{-u}}{1 + u}\,\mathrm{d}u. \tag{34}$$

In order to evaluate this by a simple quadrature, we *could* make a transformation

$$u = -\ln v, \qquad \mathrm{d}u = -\mathrm{d}v/v, \tag{35}$$

to obtain

$$I(1) = \int_0^1 \frac{\mathrm{d}v}{1 - \ln v}, \tag{36}$$

but this integral is not so suited to simple numerical quadrature owing to the extremely steep nature of the integrand near $v = 0$. A better transformation is $u = -2 \ln v$, which yields the result

$$I(1) = 2\int_0^1 \frac{v\,\mathrm{d}v}{1 - 2\ln v}. \tag{37}$$

Simpson's rule with 32 ordinates yields $I(1)$ correct to five places of decimals, the exact value being

$$I(1) = eE_1(1) = 0.59635\ldots, \tag{38}$$

where $E_1(x)$ is the exponential integral [1].

As a further example, let us consider the evaluation of Euler's constant C defined by

$$C = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right). \tag{39}$$

Evidently we may write C as the sum for $x = -1$ of the power series

$$s(x) = 1 - (\tfrac{1}{2} - \ln\tfrac{2}{1})x + (\tfrac{1}{3} - \ln\tfrac{3}{2})x^2 - (\tfrac{1}{4} - \ln\tfrac{4}{3})x^3 + \ldots, \tag{40}$$

for which

$$\mu_k = \frac{1}{k} - \ln\left(\frac{k}{k-1}\right), \quad k > 1 \tag{41}$$

but

$$\mu_1 = 1. \tag{42}$$

Since μ_1 is not given by the general formula (41) for μ_k, it is desirable to take out the first term and consider $C - 1$ as the sum of the series

$$s(x) = (\tfrac{1}{2} - \ln\tfrac{2}{1}) - (\tfrac{1}{3} - \ln\tfrac{3}{2})x + (\tfrac{1}{4} + \ln\tfrac{4}{3})x^2 \ldots \tag{43}$$

for $x = -1$. Here we have

$$\bar{h}(p) = \mu_p = \frac{1}{p+1} - \ln\left(1 + \frac{1}{p}\right), \quad p = k = 1, 2, 3\ldots, \tag{44}$$

and $\bar{h}(p)$ has the known inverse

$$h(t) = \left[e^{-t} - \frac{1 - e^{-t}}{t}\right]U(t), \quad \operatorname{Re} p > 0, \tag{45}$$

so that

$$f(u) = \left[u + \frac{1-u}{\ln u}\right]U(-\ln u). \tag{46}$$

Taking the case $x = -1$ we therefore expect

$$C = 1 + \int_0^1 \left(\frac{u}{1-u} + \frac{1}{\ln u}\right)\mathrm{d}u. \tag{47}$$

It is easily seen that the integrand is finite throughout the range of integration, tending to 0 as $u \to 0$, and to $-\frac{1}{2}$ as $u \to 1$. As in the previous example, however, the logarithm causes extremely steep behaviour of the integrand near $x = 0$, and so we obtain an integral more amenable to numerical quadrature by making a substitution $u = v^2$. Then we find

$$C = 1 + \int_0^1 \left(\frac{2v^3}{1-v^2} + \frac{v}{\ln v}\right)\mathrm{d}v \tag{48}$$

Evaluation by Simpson's rule with 32 ordinates yields the result

$$C = 0.577213,$$

which may be compared with the exact value

$$C = 0.577216\ldots.$$

We now consider the very slowly convergent series

$$1 - 2^{-1/2} + 3^{-1/2} - 4^{-1/2} + \ldots \tag{49}$$

which is known [5,2] to have the sum

$$(1-2^{1/2})\zeta(\tfrac{1}{2}) = 0.604898643\ldots, \tag{50}$$

where ζ denotes the Riemann zeta function. We are thus led to consider the summation of the series

$$s(x) = 1 - 2^{-1/2}x + 3^{-1/2}x^2 - 4^{-1/2}x^3 + \ldots \tag{51}$$

for which

$$\bar{h}(p) = \mu p = p^{-1/2}, \quad p = k = 1, 2, 3, \ldots, \tag{52}$$

Inverting, we have immediately

$$h(t) = (\pi t)^{-1/2}U(t), \quad \mathrm{Re}\, p > 0, \tag{53}$$

so that

$$f(u) = (-\pi \ln u)^{-1/2}U(-\ln u). \tag{54}$$

This gives us

$$I(x) = \pi^{-1/2}\int_0^1 \frac{(-\ln u)^{-1/2}}{1+xu}\,\mathrm{d}u, \tag{55}$$

and the sum of our series (49) can be evaluated by numerical quadrature, after putting $x = 1$. Before doing this, however, it is desirable to get rid of the singularity of the integrand at $u = 1$ by making the substitution

$$(-\ln u)^{1/2} = -\ln v, \tag{56}$$

i.e.

$$u = \mathrm{e}^{-(\ln v)^2}, \qquad \mathrm{d}u = \frac{-2\ln v}{v}e^{-(\ln v)^2}\mathrm{d}v. \tag{57}$$

Then we find

$$I(1) = 2\pi^{-1/2}\int_0^1 \frac{\mathrm{e}^{-(\ln v)^2}}{v\left[1+\mathrm{e}^{-(\ln v)^2}\right]}\,\mathrm{d}v, \tag{58}$$

which has now no singularity of the integrand in the range of integration, the integrand tending to zero as $v \to 0$.

Numerical integration by Simpson's rule, using 16 ordinates, yielded the result

$$I(1) \doteqdot -0.60485.$$

If we transform back from (55) to the variable t, we find

$$I(1) = \pi^{-1/2}\int_0^\infty \frac{t^{-1/2}}{1+\mathrm{e}^t}\,\mathrm{d}t, \tag{59}$$

which is a known expression [5] for the sum of the series (49).

The following example is included in order to show how the values of the first few coefficients in the power series may be used to facilitate numerical evaluation of the integral $I(x)$. Starting from

$$\bar{h}(p)=\mu_p=\frac{p}{p^2+1}, \quad p=k=1,2,3,\ldots, \tag{60}$$

we immediately have

$$h(t)=\cos t\, U(t), \quad \operatorname{Re} p>0, \tag{61}$$

so that the sum of our series (1) with coefficients as in (60) is

$$I(x)=\int_0^1 \frac{\cos(\ln u)}{1+xu}\,\mathrm{d}u. \tag{62}$$

Now this integral, as it stands, is not convenient for numerical quadrature, owing to the infinite number of oscillations (of finite amplitude) of the integrand near $u=0$. We can get over this difficulty in the following way. From the Laplace transform of $U(t)\cos t$ we know that

$$\int_0^\infty \mathrm{e}^{-kt}\cos t\,\mathrm{d}t=\frac{k}{k^2+1}, \quad k=1,2,3,\ldots, \tag{63}$$

and going over to the u variable we thus have of course

$$\mu_k=\int_0^1 u^{k-1}\cos(\ln u)\,\mathrm{d}u=\frac{k}{k^2+1}, \quad k=1,2,3\ldots. \tag{64}$$

We can now write, for example.

$$I(x)=\int_0^1\left(\frac{1}{1+xu}-1+xu-x^2u^2\right)\cos(\ln u)\,\mathrm{d}u+\mu_1-\mu_2x+\mu_3x^2,$$

so that

$$I(x)=\mu_1-\mu_2x+\mu_3x^2-x^3\int_0^1\frac{u^3\cos(\ln u)}{1+xu}\,\mathrm{d}u. \tag{65}$$

In (65) the oscillations in the integrand near $u=0$ are virtually annulled by the factor u^3. Thus e.g. for $x=1$ we have the formula

$$I(1)=\frac{2}{5}-\int_0^1\frac{u^3\cos(\ln u)}{1+u}\,\mathrm{d}u \tag{66}$$

for the sum of the series

$$s(1)=\frac{1}{1^2+1}-\frac{2}{2^2+1}+\frac{3}{3^2+1}-\frac{4}{4^2+1}+\ldots. \tag{67}$$

Using Simpson's rule with 32 ordinates we readily find the result

$$s(1)\doteqdot 0.26961,$$

and this is correct to all places of decimals given.

We conclude this section with an example for the summation of a Fourier series which arises in the solution of a problem in steady-state heat flow. This example, though simple, is chosen to illustrate certain features in the summation of a typical Fourier series.

Suppose we have a semi-infinite strip of conducting material whose edges in the (x, y)-plane are the lines $x=0,\ y\geqslant 0$; $x=\pi,\ y\geqslant 0$; $y=0,\ 0\leqslant x\leqslant\pi$. Further suppose that the temperature

$v(x, y)$ has the value $v=0$ on the first two edges (the 'sides'), while $v=1$ on the third edge (the 'end' of the strip).

Then the temperature distribution $v(x\,y)$ on the strip is the solution of Laplace's equation

$$\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2 = 0, \tag{68}$$

together with the boundary conditions

$$\left.\begin{aligned} &v(0, y) = 0,\ y > 0,\\ &v(\pi, y) = 0, y > 0,\\ &v(x, 0) = 1, 0 < x < \pi. \end{aligned}\right\} \tag{69}$$

Of course there are discontinuities at the corners (0, 0) and (π, 0), but this fact does not interfere with our solving this idealised problem.

Solving (68), (69), by the usual method of separation of variables, we are led to the result

$$v(x, y) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{e^{-(2n+1)y}}{2n+1} \sin(2n+1)x, \tag{70}$$

in which, of course, the solution $v(x, y)$ is expressed as a Fourier series in x.

In order to have convenient application for our theory, we write (70) in the form

$$v(x, y) = \frac{4}{\pi} \cos x \sum_{n=1}^{\infty} \frac{e^{-(2n+1)y}}{2n+1} \sin n(2x) + \frac{4}{\pi} \sin x \sum_{n=0}^{\infty} \frac{e^{-(2n+1)y}}{2n+1} \cos n(2x). \tag{71}$$

We first sum the series

$$g_1(\theta) = \sum_{n=0}^{\infty} \frac{e^{-(2n+1)y}}{2n+1} \cos n\,\theta. \tag{72}$$

Referring to our theory we have here

$$\mu_{n+1} = \frac{e^{-(2n+1)y}}{2n+1}, \quad n = 0, 1, 2, \ldots,$$

so that

$$\bar{h}_1(p) = \mu_p = \frac{e^{-(2p-1)y}}{2p-1}, \quad p = k = 1, 2, 3, \ldots, \tag{73}$$

and inverting,

$$h_1(t) = \tfrac{1}{2}\, e^{t/2} U(t - 2y), \quad \operatorname{Re} p > \tfrac{1}{2}. \tag{74}$$

This leads to

$$f_1(u) = \tfrac{1}{2} u^{-1/2} U(-\ln u - 2y), \tag{75}$$

so that

$$g_1(\theta) = \int_0^{e^{-2y}} \frac{u^{-1/2}}{2} \frac{1 - u\cos\theta}{1 - 2u\cos\theta + u^2}\, du. \tag{76}$$

Finally we put $\theta = 2x$ to obtain the sum

$$\sum_{n=0}^{\infty} \frac{e^{-(2n+1)y}}{2n+1} \cos 2nx = \int_0^{e^{-2y}} \frac{u^{-1/2}}{2} \frac{1 - u\cos 2x}{1 - 2u\cos 2x + u^2}\, du. \tag{77}$$

Turning now to the series

$$g_2(\theta)=\sum_{n=1}^{\infty}\frac{e^{-(2n+1)y}}{2n+1}\sin n\theta, \tag{78}$$

we have now

$$\bar{h}_2(p)=v_p=\frac{e^{-(2p+1)y}}{2p+1},\quad p=k=1,2,3,\ldots, \tag{79}$$

and thus

$$h_2(t)=\tfrac{1}{2}\,e^{-t/2}U(t-2y),\quad \operatorname{Re} p>-\tfrac{1}{2}, \tag{80}$$

yielding

$$f_2(u)=\tfrac{1}{2}u^{1/2}U(-\ln u-2y). \tag{81}$$

Thus we find

$$g_2(\theta)=\int_0^{e^{-2y}}\tfrac{1}{2}u^{1/2}\frac{\sin\theta}{1-2u\cos\theta}\,du, \tag{82}$$

so that

$$\sum_{n=1}^{\infty}\frac{e^{-(2n+1)y}}{2n+1}\sin 2nx=\int_0^{e^{-2y}}\tfrac{1}{2}u^{1/2}\frac{\sin 2x}{1-2u\cos 2x+u^2}\,du. \tag{83}$$

Combining (71) with (77), (83) we find after some simplification

$$v(x,y)=\frac{4}{\pi}\sin x\int_0^{e^{-2y}}\frac{(u^{1/2}+u^{-1/2})}{2(1-2u\cos 2x+u^2)}\,du. \tag{84}$$

For numerical quadrature it is desirable to eliminate the singularity in the integrand at $u=0$, and this is achieved by making the substitution

$$u=w^2,\quad du=2w\,dw$$

to yield

$$v(x,y)=\frac{4}{\pi}\sin x\int_0^{e^{-y}}\frac{1+w^2}{1-2w^2\cos 2x+w^4}\,dw. \tag{85}$$

As a verification of this result we may note:

(i) For $x=\frac{1}{2}\pi$ and $y=0$ we have from (85)

$$v(\tfrac{1}{2}\pi,0)=\frac{4}{\pi}\int_0^1\frac{dw}{1+w^2}=1$$

which is correct, in view of the boundary conditions.

(ii) For general $y>0$, but $x=\frac{1}{2}\pi$ we find

$$v(\tfrac{1}{2}\pi,y)=\frac{4}{\pi}\int_0^{e^{-y}}\frac{dw}{1+w^2}=\frac{4}{\pi}\tan^{-1}(e^{-y})=\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sin hy}\right).$$

This result is also correct, since it is not difficult to show that the exact solution to our temperature destribution problem is

$$v(x,y)=\frac{2}{\pi}\tan^{-1}\left(\frac{\sin x}{\sin h\,y}\right). \tag{86}$$

(iii) As a final verification of (85), let us consider the case $x = y = 1$. Then (85) gives

$$v(1, 1) = \frac{4}{\pi} \sin 1 \int_0^{e^{-1}} \frac{(1 + w^2)\, dw}{1 - 2w^2 \cos 2 + w^4}.$$

Numerical quadrature by Simpson's rule using 16 ordinates then yields

$$v(1, 1) = 0.39559479,$$

and comparison with (86) shows the above result to be correct to all places of decimals shown.

4. Convergence

In the not infrequent cases when $a = 1$, it is clear that for $|x| < 1$ the expansion of (2) in the form (1) is justified, since the series (3) is then absolutely and uniformly convergent with respect to u in $0 \leqslant u \leqslant 1$, and so the subsequent term-by-term multiplication by $f(u)$ and integration with respect to u in [0,1]. Thus the series (1) then converges to the intergral (2), when the moments μ_k are defined as in (4). In other cases the series (1) 'corresponds' [9] to the integral (2), and if the series diverges, the integral can be regarded as the 'anti-limit' [12] of the sequence of partial sums.

In the case where x is real and negative, and $a > -1/x$, there may be a pole inside the effective interval of integration. In this case the integral (2) can be evaluated in the sense of the Cauchy principal value [6]. It is intended that the relation of this value to the (possibly divergent) series be the subject of a further publication.

For the whole question of divergent series, the reader is referred to the literature. See for example Hardy [4], Shanks [11,12]. It is not the purpose of this paper to go further into this subject.

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