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The Role of Modular Functions in a Class-Number Problem

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The author has previously shown that there are exactly nine complex quadratic fields of class-number one. Here we show that the proof rests on some extremely interesting identities in modular functions which also provide a connection with the work of Heegner.

1. *Introduction.* In [3], a complete determination of all complex quadratic fields of class-number one was made. Nevertheless, many obvious questions arise upon reading [3]; here we will discuss some of these questions and their answers.

2. *The method of* [3] *and the Questions that Arise.* Let $d = -\Delta < 0$ denote the discriminant of a complex quadratic field and $h(d)$ the class-number of the field. Further let k denote the discriminant of a real quadratic field such that $(k, d) = 1$, let $\chi(n) = \chi_k(n) = (k/n)$ be the real primitive character associated with this field and let ε_0 denote the fundamental unit of this field. Let $Q = (a, b, c)$ denote the positive definite quadratic form $ax^2 + bxy + cy^2$ of discriminant d, $Q' = (a, -b, c)$, and r denote a complex number with positive imaginary part. Set

$$L(s, \chi, Q, r) = \tfrac{1}{2}(r-\bar{r})^s \sum_{x, y \neq 0, 0} \frac{\chi(Q(x, y))}{[(x+ry)(x+\bar{r}y)]^s} \tag{1}$$

where the principal value of $z^s = e^{s \log z}$ is used. We have

$$\frac{12L(1, \chi, Q, r)}{\pi i k \prod_{p|k} (1-p^{-2})} = f_Q(r) + f_{Q'}(-\bar{r}) - \frac{2\pi i \chi(a)}{k} \tag{2}$$

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where

$$f_Q(r) = f_{Q,k}(r) = -\chi(a)\frac{2\pi i(r-\frac{1}{2})}{k} - \frac{12\chi(a)}{k^2\prod_{p|k}(1-p^{-2})}\cdot\begin{cases}\log p \text{ if } k \text{ is a power of } p\\ 0 \text{ if } k \text{ is not a power of a prime}\end{cases}$$

$$+\frac{24}{k^2\prod_{p|k}(1-p^{-2})}\sum_{n=1}^{\infty} e^{\frac{2\pi i n r}{k}} \sum_{y|n} y^{-1} \sum_{j=1}^{k} \chi(Q(j,y))e^{\frac{2\pi i n j}{ky}}, \quad (3)$$

and p denotes primes only. Generalizations for all s of (2) are derived in [4] for $r = (b+\sqrt{d})/2a$ but there is no need for such a restriction on r; indeed we may even derive (2) with $-\bar{r}$ replaced by r' such that $\operatorname{Im} r' > 0$ and $\operatorname{Re}(r+r') = 0$. When $b = a$ and $\operatorname{Re} r = \frac{1}{2}$,

$$f_Q(r) = f_{Q'}(-\bar{r}) - \frac{2\pi i\chi(a)}{k}.$$

Thus, thanks to the connection between $L(s,\chi,Q,(b+\sqrt{d})/2a)$ and ordinary Dirichlet L-functions when $h(d) = 1$, we have for $h(d) = 1$ and d odd,

$$\frac{12\omega h(kd)h(k)\log\varepsilon_0}{k^2\prod_{p|k}(1-p^{-2})} = f_{\left(1,1,\frac{\Delta+1}{4}\right)}\left(\frac{1+\sqrt{d}}{2}\right)$$

$$\left(\text{where } \omega = \begin{cases}3 \text{ if } d=-3\\ 1 \text{ if } d<-4\end{cases}\right). \quad (4)$$

This leads us to the idea of exponentiating f. Set

$$F_Q(r) = F_{Q,k}(r) = e^{f_Q(r)}. \quad (5)$$

In [3] we considered the following function for $\Delta \equiv 3 \pmod 8$ and $k = 8$:

$$H_Q(r) = \begin{cases}\frac{1}{2}[F_Q(r)^2 - F_Q(r)^{-2}] - \sqrt{2}[R^{-1/2}F_Q(r) + R^{1/2}F_Q(r)^{-1}], & \frac{\Delta+1}{4} \equiv 1 \pmod 8\\ \frac{1}{2}[F_Q(r)^2 - F_Q(r)^{-2}] + \sqrt{2}[R^{1/2}F_Q(r) + R^{-1/2}F_Q(r)^{-1}], & \frac{\Delta+1}{4} \equiv 3 \pmod 8\\ \frac{1}{2}[F_Q(r)^2 - F_Q(r)^{-2}] + \sqrt{2}[R^{-1/2}F_Q(r) + R^{1/2}F_Q(r)^{-1}], & \frac{\Delta+1}{4} \equiv 5 \pmod 8\\ \frac{1}{2}[F_Q(r)^2 - F_Q(r)^{-2}] - \sqrt{2}[R^{1/2}F_Q(r) + R^{-1/2}F_Q(r)^{-1}], & \frac{\Delta+1}{4} \equiv 7 \pmod 8\end{cases} \quad (6)$$

where

$$Q = \left(1, 1, \frac{\Delta+1}{4}\right) \quad \text{and} \quad R = 1+\sqrt{2}.$$

In the notation of [3],

$$H_Q\left(\frac{1+\sqrt{d}}{2}\right) = \begin{cases} z_{2N+1} - 4y_N & \frac{\Delta+1}{4} \equiv 1 \pmod 8 \\ z_{2N+1} + 4y_{N+1} & \frac{\Delta+1}{4} \equiv 3 \pmod 8 \\ z_{2N+1} + 4y_N & \frac{\Delta+1}{4} \equiv 5 \pmod 8 \\ z_{2N+1} - 4y_{N+1} & \frac{\Delta+1}{4} \equiv 7 \pmod 8 \end{cases}$$

We also considered for $k = 12$ and $\Delta \equiv 19 \pmod{24}$, the function

$$a_Q(r) = \begin{cases} \frac{1}{\sqrt{2}}[F_Q(r) - F_Q(r)^{-1}] + 1, & \frac{\Delta+1}{4} \equiv 1 \pmod 4 \\ \frac{1}{\sqrt{2}}[F_Q(r) - F_Q(r)^{-1}] - 1, & \frac{\Delta+1}{4} \equiv 3 \pmod 4 \end{cases} \tag{7}$$

where $Q = (1, 1, (\Delta+1)/4)$. Thanks to certain congruence conditions on $h(kd)$, we found in [3] that for $\Delta \equiv 19 \pmod{24}$ and $h(d) = 1$, $H_Q((1+\sqrt{d})/2)$ and $a_Q((1+\sqrt{d})/2)$ are rational integers and that further if $\Delta \geq 200$ then

$$H_Q(r) = \left\{\begin{matrix} a_Q(r)^3 + 3, & \frac{\Delta+1}{4} \equiv 1 \pmod 4 \\ a_Q(r)^3 - 3, & \frac{\Delta+1}{4} \equiv 3 \pmod 4 \end{matrix}\right\} \left(r = \frac{1+\sqrt{d}}{2}\right). \tag{8}$$

Equation (8) is a Diophantine equation which was solved in [3]; there are seven solutions, four of which correspond to $d = -19, -43, -67, -163$, and the other three are extraneous solutions which can't correspond to any $d \leq -200$. The use of $\Delta \geq 200$ was governed by the necessity of showing that the two sides of (8) differ by less than one, and hence, being rational integers, are equal.

Several questions present themselves of which perhaps the most obvious is

Question 1. Is (8) an identity, valid for all r? If so this would eliminate the necessity of dealing with $\Delta \geq 200$ in [3], this being the messiest and least satisfactory part of [3]. Furthermore, if (8) isn't an identity, valid for all r, then for sufficiently large Δ, there could not possibly be equality in (8) at $r=(1+\sqrt{d})/2$. In this case, we needn't have bothered to solve a Diophantine equation in [3].

Question 2. Are the three extraneous solutions to (8) really extraneous or do they somehow correspond to other fields?

Question 3. In [3] we expanded $H_Q(r)$ and $a_Q(r)$ in powers of $q^{1/8}$ and $q^{1/12}$ respectively where $q=\exp[2\pi i(r-\frac{1}{2})]$. R. S. Lehman has noted in a private letter that these expansions (as far as they were given) are considerably simpler if we use $q=64\exp[2\pi i(r-\frac{1}{2})]$. Is this true for the entire expansions of $H_Q(r)$ and $a_Q(r)$ and if so why?

Question 4. After (8) has been reduced, the equations that must ultimately be solved in [3] are the same equations to which Heegner's equation ([1], [5]) reduces. Is this a coincidence or are we dealing with the same functions?

Question 5. Suppose $\Delta\equiv 3 \pmod 8$. Thanks to the connection with ordinary Dirichlet L-functions, we are able to say that

$$F_{(1,1,(\Delta+1)/4),8}((1+\sqrt{d})/2)$$

is a unit in $Q(\sqrt{2})$ when $h(d)=1$. What can we say when $h(d)>1$?

3. *The method of Siegel.* Siegel has noted [2] that (2) is a form of Kronecker's limit formula. The simplest case is with $k=5$, $(d,k)=1$, when we get

$$F_Q(r)=F_{Q,5}(r)=-5^{-\chi(a)/2}\eta(5r)^{-\chi(a)}\prod_{j=0}^{4}\left[\eta\left(\frac{r+24j}{5}\right)\right]^{-\chi(Q(24j,1))} \tag{9}$$

where

$$\eta(r)=e^{\frac{\pi ir}{12}}\prod_{n=1}^{\infty}(1-e^{2\pi inr}).$$

In dealing with $h(d)=1$, $\chi_5(d)=-1$ for $d<-19$ (and this is also true for $d=-3,-7,-8$) and thus it is natural to require $\chi_5(d)=-1$. In this case we may restrict our attention to $F_{(1,1,1)}(r)$ since the other $F_Q(r)$ may be obtained from $F_{(1,1,1)}(r)$ by a unimodular transformation.

One finds easily that for $F=F_{(1,1,1)}(r)$ and $\varepsilon=(1+\sqrt{5})/2$,

$$-\frac{j(r)}{25\sqrt{5}}=\left[\left(\frac{F}{\varepsilon^3}\right)^{1/2}-\left(\frac{F}{\varepsilon^3}\right)^{-1/2}\right]\cdot$$
$$\cdot\left[(\varepsilon F^3)^{1/2}-(\varepsilon F^3)^{-1/2}-(\varepsilon F)^{1/2}-(\varepsilon F)^{-1/2}\right]^3 \tag{10}$$

where

$$j(r) = \frac{\left\{1+240 \sum_{n=1}^{\infty} \sigma_3(n)\, e^{2\pi inr}\right\}^3}{e^{2\pi ir} \prod_{n=1}^{\infty} (1-e^{2\pi inr})^{24}}.$$

Now we make use of (4) and the fact that $\omega h(5d) = 4N+2$ to evaluate (10) as

$$-j\left(\frac{1+\sqrt{d}}{2}\right) = \begin{cases} 125F_{N-1}(L_{3N+2}-L_{N+1})^3, & N \text{ odd} \\ 625L_{N-1}(F_{3N+2}-F_{N+1})^3, & N \text{ even} \end{cases} \quad \left(\frac{\Delta+1}{4} \equiv 1 \pmod 5\right) \tag{11}$$

where F_n and L_n are solutions of

$$x_{n+2} = x_{n+1}+x_n; \quad F_0 = 0,\ F_1 = 1; \quad L_0 = 2,\ L_1 = 1.$$

We can transform (10) to cover the case of $(\Delta+1)/4 \equiv 2 \pmod 5$. We get

$$-j\left(\frac{1+\sqrt{d}}{2}\right) = \begin{cases} 125F_{N+2}(L_{3N+1}+L_N)^3, & N \text{ even} \\ 625L_{N+2}(F_{3N+1}+F_N)^3, & N \text{ odd} \end{cases} \quad \left(\frac{\Delta+1}{4} \equiv 2 \pmod 5\right)\cdot \tag{12}$$

The field with $d = -8$ may even be included by sending $F_{(1,1,1)}(r)$ to $-e^{-\pi i/5} F_{(1,0,2)}(r)$ and evaluating at $r = \sqrt{-2}$:

$$j(\sqrt{-2}) = 125F_{N-1}(L_{3N+2}+L_{N+1})^3 = 20^3 \quad (\Delta = 8,\ N = 0). \tag{13}$$

In equations (11) and (12), $j((1+\sqrt{d})/2)$ is a perfect cube and since no L_n is divisible by 5, we are reduced to finding all F_n which are cubes with n even, $n \geq 0$ (in fact $n \geq 2$ in (12)). These are $F_0 = 0$, $F_2 = 1$, $F_6 = 8$. These values correspond to $d = -3, -43, -163$ in (11) and the last two correspond to $d = -7$, $d = -67$ in (12).

The hallmarks of Siegel's method are that only one value of k is necessary and the Diophantine equations arise from identities in modular functions.

4. *The cases* $k = 8$ *and* 12. We likewise find that $F_{Q,8}(r)$ and $F_{Q,12}(r)$ may be expressed in terms of η functions although the expressions are more complicated. In fact we are dealing with $\eta(r)^{1/2}$ when $k = 8$ and $\eta(r)^{1/4}$ when $k = 12$. For $k = 8$,

$$F_{Q,8}(r) = \zeta_Q \cdot 2^{-\frac{\chi(a)}{4}} \cdot \left[\frac{\eta(8r)}{\eta\left(\frac{4r+3}{2}\right)}\right]^{-\frac{\chi(a)}{2}} \cdot \prod_{j=0}^{3} \left[\eta\left(\frac{2r+3j}{4}\right)\right]^{-\frac{\chi(Q(3j,2))}{2}} \cdot \prod_{j=0}^{7} \left[\eta\left(\frac{r+3j}{8}\right)\right]^{-\frac{\chi(Q(3j,1))}{2}}$$

where ζ_Q is a 16th root of unity and as an example, $\zeta_{(1,1,1)} = e^{2\pi i/8}$. However, we are able to show that for $\Delta \equiv 3 \pmod 8$, $F_{Q,8}(r)$ is invariant under transformations of level 8 and for $\Delta \equiv 19 \pmod{24}$, $F_{Q,12}(r)$ is invariant under transformations of level 12. These facts are corollaries of more general results on how $F_{Q,k}(r)$ transforms under unimodular transformations for $k = 8$ and 12. In fact if we set

$$G_{Q,8}(r) = \chi_8(a)\chi_8\left(\frac{a^2-b^2}{4}+ac\right)\exp\frac{2\pi i}{8}\left[\left(\frac{b-a}{2}\right)a\chi_8(a)\right]F_{Q,8}(r),$$
$$G_{Q,12}(r) = \chi_{12}(a)\chi_{-4}\left(\frac{a^2-b^2}{4}+ac\right)\exp\frac{2\pi i}{12}\left[\left(\frac{b-a}{2}\right)a\chi_{12}(a)\right]F_{Q,12}(r), \tag{14}$$

then for $k = 8$ and $\Delta \equiv 3 \pmod 8$ or $k = 12$ and $\Delta \equiv 19 \pmod{24}$,

$$G_{Q,k}\left(\frac{\alpha r+\beta}{\gamma r+\delta}\right) = G_{Q_1,k}(r) \tag{15}$$

where α, β, γ, δ are integers, $\alpha\delta-\beta\gamma = 1$ and

$$Q(\alpha x-\beta y, -\gamma x+\delta y) \equiv Q_1(x, y) \pmod k. \tag{16}$$

We can now answer Question 1.

THEOREM 1. *Equation* (8) *is an identity in r.*

In particular this means that $H_Q(r)$ is invariant under transformations of level 4. We also find that

$$H_{1,1,5}(r) = H_{1,1,1}(r), \quad H_{1,1,7}(r) = H_{1,1,3}(r),$$
$$H_{1,1,3}(r) = -H_{1,1,1}(r+2). \tag{17}$$

These results are proved in the standard way of showing that the differences have no poles and are therefore constant.

We may also find the relation between H and j:

THEOREM 2. *Set* $H = H_{(1,1,1)}(r)$. *Then*

$$(H-3)(H+1)^3 = -\tfrac{1}{64}j(r). \tag{18}$$

This result provides the answer to Questions 2 and 3 as well as incidentally eliminating the need to deal with $k = 12$ in [3]. We see directly from (18) that for $\Delta \equiv 3 \pmod 8$, $h(d) = 1$, $H_{1,1,1}((1+\sqrt{d})/2)-3$ is a cube if $(\Delta+1)/4 \equiv 1 \pmod 4$ and $H_{1,1,3}((1+\sqrt{d})/2)+3$ is a cube if $(\Delta+1)/4 \equiv 3 \pmod 4$. Now we see that the remaining three solutions to (8) in [3] are not extraneous at all: two of them correspond to $d = -3$ and the other corresponds to $d = -11$; the use of $k = 12$ kept us from including these fields in the numerical examples of [3]. The answer to Question 3 also comes from (16): $H_Q(r)$ is a Laurant series in $q = 2\sqrt{2}\exp[(2\pi i/4)(r-\frac{1}{2})]$ with rational coefficients. It then follows from (8) that $a_Q(r)$ is a Laurant series in $q^{1/3}$ with rational coefficients.

5. *The Connections with Heegner's Work.* Heegner [1] reduced the equation

$$\sigma^{24} - e^{-2\pi i/3}\gamma_2(r)\sigma^8 - 16 = 0, \tag{19}$$

where $\gamma_2(r)^3 = j(r)$ and γ_2 is real on the imaginary axis, to an equation

$$\sigma^{12} + 2\zeta\sigma^8 + 2\zeta^2\sigma^4 - 4 = 0, \tag{20}$$

and this in turn to the equation

$$\sigma^6 + 2\alpha\sigma^4 + 2\beta\sigma^2 - 2 = 0. \tag{21}$$

Further, α and β satisfy the equation,

$$(\beta - 2\alpha^2)^2 = 2\alpha(\alpha^3 + 1). \tag{22}$$

If $r = (1+\sqrt{d})/2$ where $\Delta \equiv 3 \pmod 8$ and $3 \nmid \Delta$, then this reduction may be made in such a way [5] (and in fact uniquely so) that α and β are in $Q(j((1+\sqrt{d}/2))$. It should be noted that we have used $e^{-2\pi i/3}\gamma_2(r)$ instead of $\gamma_2(r)$ because we are dealing with $r = (1+\sqrt{d})/2$ instead of the more customary $(-3+\sqrt{d})/2$.

The reduction from Equation (19) to (20) may be made in four different ways, giving rise to four modular functions for ζ. Again, we may go from (20) to (21) in four ways so that there are sixteen possibilities for α and β. Surprisingly enough, it is not always the same function which gives the value in $Q(j((1+\sqrt{d})/2))$. In fact, we have the following results:

THEOREM 3. *If* $\Delta \equiv 3 \pmod 8$ *and* $3 \nmid d$, *then the function* ζ *which is in* $Q(j(r))$ *at* $r = (1+\sqrt{d})/2$ *is given by*

$$\zeta(r)^3 = (-1)^{\frac{\Delta-3}{8}} H_{\left(1,1,\frac{\Delta+1}{4}\right)}(r) - 3, \tag{23}$$

the cube root being chosen which is real on $\mathrm{Re}\, r = \frac{1}{2}$.

COROLLARY. *If* $\Delta \equiv 19 \pmod{24}$, *then the proper choice for* ζ *is*

$$\zeta(r) = (-1)^{\frac{\Delta-3}{8}} a_{\left(1,1,\frac{\Delta+1}{4}\right)}(r). \tag{24}$$

THEOREM 4. *If* $\Delta \equiv 3 \pmod 8$ *and* $3 \nmid d$ *then the function* α *which is in* $Q(j(r))$ at $r = (1+\sqrt{d})/2$ *is given by*

$$\alpha(r)=\begin{cases}\dfrac{\zeta(r)^2}{2+\chi_8\left(\dfrac{\Delta+1}{4}\right)[R^{1/2}F(r)+R^{-1/2}F(r)^{-1}]} & \text{if } \dfrac{\Delta+1}{4}\equiv 1 \pmod 4\\[2ex] \dfrac{\zeta(r)^2}{2-\chi_8\left(\dfrac{\Delta+1}{4}\right)[R^{-1/2}F(r)+R^{1/2}F(r)^{-1}]} & \text{if } \dfrac{\Delta+1}{4}\equiv 3 \pmod 4\end{cases} \tag{25}$$

where $F(r)=F_{(1,1,(\Delta+1)/4),8}(r)$ *and* $\zeta(r)$ *is given by Theorem* 3.

Along these lines, if we reduce the equation

$$(\sigma^{24}-16)^3=\sigma^{24}j(r)$$

to a cubic equation in σ^{12} (the constant term being -2^6), then the four possible coefficients of σ^{24} are $-4H_{1,1,1}(r+g)$, $g=0,1,2,3$. There are also less obvious relations available in the next reduction of this equation to a cubic in σ^6. These facts help simplify the proofs of Theorems 3 and 4 considerably.

We can now answer Question 4. Indeed, the equation following (103) of [3] is nothing more than

$$\alpha^3+1=\pm\square \quad \text{with} \quad \alpha=\pm 2\square, \tag{26}$$

while the equation following (105) of [3] is nothing more than

$$\alpha^3+1=\pm 2\square \quad \text{with} \quad \alpha=\pm\square, \tag{27}$$

with α being given by Theorem 4 in each case.

Equations (26) and (27) are precisely the equations that arise when we attempt to find all rational integral solutions of (22).

6. *Kronecker's Limit Formula for* $k=8$. After Theorem 4, it is not surprising that we can determine the field containing $R^{-1/2}F_{Q,8}(r)$. In fact we find

THEOREM 5. *If* $\Delta\equiv 3 \pmod 8$, *then* $R^{-1/2}F_{(1,1,(\Delta+1)/4),8}((1+\sqrt{d})/2)$ *is a unit in the field* $Q(j((1+\sqrt{d})/2), \sqrt{2})$.

In actual fact, we first prove that $H_{(1,1,(\Delta+1)/4)}((1+\sqrt{d})/2)$ and $F_{(1,1,(\Delta+1)/4),8}((1+\sqrt{d})/2)$ are in the proper fields in a manner similar to the method of Weber [6] in proving that $j((1+\sqrt{d})/2)$ is a cube. It is here that the answer to Question 3 is useful since it enables us to say that our transformation equations have coefficients in the correct fields. We then use these results to prove Theorems 3 and 4. When $h(d)=2$, we can make the description of Theorem 5 much more precise.

THEOREM 6. *If* $h(d) = 2$ *and* $\Delta \equiv 3 \pmod 8$, *then*

$$R^{-1/2} F_{\left(1,1,\frac{\Delta+1}{4}\right),8}\left(\frac{1+\sqrt{d}}{2}\right) = R^n \varepsilon^m$$

where ε *is either the fundamental unit in* $Q(\sqrt{2p})$ *or its square root (which is then in* $Q(\sqrt{p}, \sqrt{2})$*),* p *is the unique prime* $\equiv 1 \pmod 4$ *dividing* Δ *and*

$$n = \frac{h(8d)-4}{8}.$$

There are ten known fields to which Theorem 6 applies. Table 1 presents the results.

TABLE 1

VALUES OF n, ε, AND m IN THEOREM 6

d	$\frac{\Delta+1}{4} \pmod 8$	p	n	ε	m
−35	1	5	0	$3+\sqrt{10}$	1
−51	5	17	0	$3\sqrt{2}+\sqrt{17}$	1
−91	7	13	1	$5+\sqrt{26}$	1
−115	5	5	2	$3+\sqrt{10}$	1
−123	7	41	1	$9+\sqrt{82}$	1
−187	7	17	3	$3\sqrt{2}+\sqrt{17}$	1
−235	3	5	2	$3+\sqrt{10}$	2
−267	3	89	2	$20\sqrt{2}+3\sqrt{89}$	1
−403	5	13	3	$5+\sqrt{26}$	2
−427	3	61	5	$11+\sqrt{122}$	1

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