Appl. Math. Lett. Vol. 3, No. 3, pp. 39–41, 1990
Printed in Great Britain.

0893-9659/90 $3.00 + 0.00

Inverse Inequalities for Chebyshev Approximations in L^∞ Norms

DANIELE FUNARO

Dipartimento di Matematica - Università di Pavia

(Received April 1990)

Abstract. Inverse inequalities in the space of polynomials, relating the maximum norm in [-1,1] and weighted Sobolev norms, are shown.

Statement of the problem. We first introduce the following norms in the space of continuous functions:

$$||\phi||_0 = \left(\int_{-1}^{1} \phi^2 \omega dx\right)^{1/2},$$

$$||\phi||_\infty = max_{x\in[-1,1]}|\phi(x)|,$$

$$||\phi||_1 = \left(||\phi||_0^2 + ||\phi_x||_0^2\right)^{1/2}, \ \phi \in C^o([-1,1]),$$

where $\omega(x) = \frac{1}{\sqrt{1-x^2}}$ is the Chebyshev weight. It is well-known that it is possible to find two constants $C_1, C_2 > 0$ such that:

$$C_1||\phi||_0 \leq ||\phi||_\infty \leq C_2||\phi||_1, \ \forall\phi \in C^o([-1,1]). \tag{1}$$

On the other hand, if we denote by $\mathbf{P}_N$ the space of polynomials whose degree is less or equal to N, the following *inverse inequalities* hold (see [1] and [4]):

$$||\phi||_1 \leq C_1 N^{3/2}||\phi||_\infty \leq C_2 N^2||\phi||_0, \ \forall\phi \in \mathbf{P}_N, \tag{2}$$

where $C_1, C_2 > 0$ do not depend on N.
Besides, denoting by $x_j^{(N)} = \cos\frac{j\pi}{N}$, $j = 0, ..., N$ the Chebyshev Gauss-Lobatto nodes in [-1,1], we can consider the norms in $\mathbf{P}_N$:

$$||\phi||_{N,0} = \left(\frac{\pi}{N}\sum_{j=0}^{N}{}'' \phi_j^2\right)^{1/2},$$

$$||\phi||_{N,\infty} = max_{\ 0\leq j\leq N}|\phi_j| ,$$

$$||\phi||_{N,1} = (||\phi||_{N,0} + ||\phi_x||_{N,0})^{1/2}, \ \phi \in \mathbf{P}_N,$$

where $\phi_j = \phi(x_j^{(N)})$ and the symbol $\sum''$ indicates that the first and the last terms in the summation are halved. Let us remark that these norms are those actually used in computations.

In [2] it is shown that discrete and continuous Sobolev norms are uniformly equivalent; i.e., we can determine $C_1, C_2 > 0$ such that:

$$C_1\|\phi\|_{N,i} \leq \|\phi\|_i \leq C_2\|\phi\|_{N,i} \ , \ \forall\phi \in \mathbf{P}_N, \ i = 0,1. \tag{3}$$

This does not apply anymore for the maximum norm. Actually we have:

$$\|\phi\|_{N,\infty} \leq \|\phi\|_\infty \leq \sigma(N) \ \|\phi\|_{N,\infty} \ , \ \forall\phi \in \mathbf{P}_N, \tag{4}$$

where σ is an increasing function of N which grows at least like $\log(N)$ (see [5], p.13).

Here we shall show the exact equivalent of (2) for the discrete norms; more exactly we can prove the existence of two constants $C_1, C_2 > 0$ such that:

$$\|\phi\|_{N,1} \leq C_1 N^{3/2}\|\phi\|_{N,\infty} \leq C_2 N^2\|\phi\|_{N,0} \ , \ \forall\phi \in \mathbf{P}_N. \tag{5}$$

We remark that (5) cannot be trivially obtained by (2), (3) and (4).

Proof. The right hand side inequality in (5) is easily obtained by noting that:

$$\|\phi\|_{N,\infty} \leq \left(\sum_{j=0}^{N}{}'' \phi_j^2\right)^{1/2} = \sqrt{\frac{N}{\pi}} \ \|\phi\|_{N,0} \ , \ \forall\phi \in \mathbf{P}_N.$$

To prove the left hand side inequality, we first note that:

$$\|\phi\|_{N,0} \leq \|\phi\|_{N,\infty} \left(\frac{\pi}{N} \sum_{j=0}^{N}{}'' 1\right)^{1/2} = \sqrt{\pi} \ \|\phi\|_{N,\infty} \ , \ \forall\phi \in \mathbf{P}_N. \tag{6}$$

Afterwards, we define $\xi_i^{(N)} = \cos \frac{2i-1}{2N}\pi \ , i = 1, ..., N$; therefore we have the quadrature formula (see [5]):

$$\int_{-1}^{1} \psi\omega dx = \frac{\pi}{N} \sum_{j=1}^{N} \psi(\xi_j^{(N)}) \ , \ \forall\psi \in \mathbf{P}_{2N-1}.$$

In [4] it is shown that for any $\phi \in \mathbf{P}_N$ we have:

$$|\phi_x(\xi_j^{(N)})| \leq \frac{N}{\sqrt{1-(\xi_j^{(N)})^2}} \ \|\phi\|_{N,\infty} \ , \ j = 1, ..., N.$$

Thus, one gets by (3):

$$\|\phi_x\|_{N,0}^2 \leq \frac{1}{C_1}\|\phi_x\|_0^2 = \frac{1}{C_1}\frac{\pi}{N}\sum_{j=1}^{N}\phi_x^2(\xi_j^{(N)}) \leq \frac{\pi N}{C_1}\|\phi\|_{N,\infty}^2 \sum_{j=1}^{N} \frac{1}{1-(\xi_j^{(N)})^2} \ , \ \forall\phi \in \mathbf{P}_N.$$

Now, recalling that $\sin z \geq \frac{2z}{\pi}$, $z \in [0, \frac{\pi}{2}]$ we obtain the estimates:

$$\begin{aligned}\sum_{j=1}^{N} \frac{1}{1-(\xi_j^{(N)})^2} &= \sum_{j=1}^{N} \left(\sin \frac{2j-1}{2N}\pi\right)^{-2} \leq 1 + 2\sum_{j=1}^{[N/2]} \left(\sin \frac{2j-1}{2N}\pi\right)^{-2} \\ &\leq 1 + 2\sum_{j=1}^{[N/2]} \left(\frac{N}{2j-1}\right)^2 \leq 1 + 2N^2 \sum_{j=1}^{\infty} \frac{1}{j^2} \leq C_3 N^2.\end{aligned} \tag{8}$$

Combining (6), (7) and (8), we can easily conclude.

References

1. C.Canuto and A.Quarteroni, Approximation Results for Orthogonal Polynomials in Sobolev Spaces, *Math. Comput.* **38**, 67–86 (1982).
2. C.Canuto and A.Quarteroni, Error Estimates for Spectral and Pseudo-Spectral Approximation of Hyperbolic Equations, *SIAM J. Numer. Anal.* **19**, 629–642 (1982).
3. R.J.Duffin and A.C.Shaeffer, A Refinement of an Inequality of the Brothers Markoff, *Trans. Amer. Math. Soc.* **50**, 517–528 (1941).
4. A.Quarteroni, Some Results of Bernstein and Jackson Type for Polynomial Approximation in L^p *– spaces*, *Jpn. J. Appl. Math.* **1**, 173–181.
5. T.J.Rivlin, *The Chebyshev polynomials*, John Wiley & Sons, New York, (1974).

Dipartimento di Matematica - Università di Pavia, Strada Nuova 65, 27100 Pavia, Italy.