The cracked-beam problem solved by the boundary approximation method

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Abstract \\

The cracked-beam problem, as a variant of Motz’s problem, is discussed, and its very accurate solution in double precision is explicitly provided by the boundary approximation method (BAM) (i.e., the Trefftz method). Half of its expansion coefficients are zero, which is supported by an a posteriori analysis. Finding a good model of singularity problems is important for studying numerical methods. As a singularity model, the cracked-beam problem given in this paper seems to be superior to Motz’s problem in Li et al. [Z.C. Li, R. Mathon, P. Serman, Boundary methods for solving elliptic problem with singularities and interfaces, SIAM J. Numer. Anal. 24 (1987) 487–498]. 

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Fig. 1. Two models of Laplace’s equation with singularities, (a) the Motz’s problem, (b) the cracked-beam problem.

1. Motz’s problem

The singularity problems of elliptic equations have drawn much attention in the last several decades. Variant numerical methods have been studied, and reported in many papers. It is important to find a typical singularity problem such that different methods may be compared with each other in numerical performance, to expose their merits and drawbacks [3,4,10]. Motz’s problem [9] is a benchmark of singularity problems, which solves the Laplace equation on the rectangle \( S = \{(x, y), -1 < x < 1, 0 < y < 1\} \). Let the four corners of \( S \) be \( A(1,0), B(1,1), C(-1,1) \) and \( D(-1,0) \). The mixed type of Dirichlet and Neumann conditions is enforced on its boundary (see Fig. 1(a)),

\[
\begin{align*}
    u_{AB} &= 500, & u_{DD} &= 0, & u_{OA} &= 0, & u_{AB \cup CD} &= 0, \\
    \nu
\end{align*}
\]

where \( O \) is the origin, and \( u_\nu \) is the solution derivative along the outward normal. Many methods have been developed to compute its approximate solutions. In Li et al. [6], the boundary approximation method (BAM) (i.e., the Trefftz method) is proposed to provide the very accurate solution under double precision, which is expressed as

\[
    v_N = \sum_{\ell=0}^{N} \tilde{D}_{\ell} r^{\ell+\frac{\ell}{2}} \cos \left( \ell + \frac{1}{2} \right) \theta,
\]

where \((r, \theta)\) are the polar coordinates at the origin, \( N = 34 \), and the 35 coefficients \( \tilde{D}_\ell \) are explicitly listed in [6], while an error of \( D_{31} \) was pointed out by Lucas and Oh in [8]. The approximation (2) converges to the true solution exponentially; the notorious condition number of the associated matrix also grows exponentially. To reduce the condition number, we may choose piecewise particular solutions, and apply the BAM along the interior and exterior boundary. As a consequence, the condition number decreases, but the errors increase. A strict analysis is given in [6]. Besides, the conformal transformation method (CTM) in [11] provides the very accurate 20 leading coefficients under double precision; the first 100 coefficients by CTM using Mathematica are published in [5], and the first 500 coefficients are collected in http://www.math.nsysu.edu.tw/u/scicomp/ttlu/computing.html.

2. The cracked-beam problem

When the boundary conditions on \( AB \) and on \( BC \) are exchanged as in [2,10,12], see Fig. 1(b),

\[
\begin{align*}
    u_{BC} &= 500, & u_{DO} &= 0, & u_{OA} &= 0, & u_{AB \cup CD} &= 0,
    \quad \nu
\end{align*}
\]
Table 1
The errors and condition numbers for the cracked-beam problem by the BAM with \( v_N \) and \( w = 1/(N + 1) \)

| \( N + 1 \) | \( |u - v|_B \) | \( |u - v|_{\infty, BC} \) | Cond. |
|-------------|-------------|-----------------|-------|
| 12          | 0.157(-1)   | 0.182(-1)       | 112   |
| 20          | 0.947(-4)   | 0.136(-3)       | 0.227(4) |
| 28          | 0.734(-6)   | 0.123(-5)       | 0.427(5) |
| 36          | 0.643(-8)   | 0.121(-7)       | 0.772(6) |
| 44          | 0.606(-10)  | 0.128(-9)       | 0.135(8) |

this gives the cracked-beam problem. The solution can also be expressed in (2). Since \( v_N \) satisfies the Laplace equation in \( S \) and the boundary conditions on \( OD \cup OA \) already, the coefficients \( \tilde{D}_i \) should be chosen to satisfy the rest of the boundary conditions as best as possible. Define the boundary norm on \( AB \cup BC \cup CD \) as

\[
\|\epsilon\|_B = \|u - v\|_B = \left\{ \int_{BC} (v - 500)^2 + w^2 \int_{AB \cup CD} v_\nu^2 \right\}^{\frac{1}{2}}, \quad w = \frac{1}{N + 1}.
\]

(4)

The BAM solution \( u_N \) can be obtained by \( \|u - u_N\|_B = \inf_{v \in \{v_N\}} \|u - v\|_B \). Choose the uniform distributed points \( P_i \) on \( AB \cup BC \cup CD \). We may require \( v = 500 \) at \( P_i \in BC \) and \( u_{\nu} = 0 \) at \( P_i \in AB \cup CD \). Let the number \( M \) of \( P_i \) be much larger than \( N + 1 \); we obtain an over-determined system \( Fx = b \), where \( F \) is an \( M \times (N + 1) \) matrix, and \( x \) is the unknown vector consisting of \( \tilde{D}_i \). We employ the least squares method to solve the system, where the condition number is defined by \( \text{Cond.} = \left( \frac{\lambda_{\text{max}}(A)}{\lambda_{\text{min}}(A)} \right)^{\frac{1}{2}} \), where \( A = F^TF \), and \( \lambda_{\text{max}}(A) \) and \( \lambda_{\text{min}}(A) \) are the maximal and minimal eigenvalues of \( A \) respectively. The errors, condition numbers and the leading coefficients at \( N = 43 \) are given in Tables 1 and 2. Since Table 2 shows \( D_{4\ell+2} \approx D_{4\ell+3} \approx 0 \), we may simply seek the following solution expression:

\[
v_N^\nu = \sum_{\ell=0}^{L-1} \sum_{k=0}^{1} \tilde{D}_{4\ell+k} r^{4\ell+k+\frac{1}{2}} \cos \left( \frac{4\ell + k + \frac{1}{2}}{2} \right),
\]

(5)

where \( N + 1 = 4 \times L \). The BAM solution \( \epsilon \{v_N^\nu\} \) is sought by \( \|u - u_N\|_B = \inf_{v \in \{v_N^\nu\}} \|u - v\|_B \), and the results are given in Tables 3 and 4. All the results in Tables 1–4 are obtained by computation in double precision. From Tables 1 and 3 we have observed the empirical asymptotes:

\[
\begin{align*}
\|u - u_N\|_B & = O(0.558^N), \quad \|u - u_N\|_{\infty, BC} = O(0.566^N), \quad \text{Cond.} = O(1.42^N), \quad (6) \\
\|u - u_N\|_B & = O(0.558^N), \quad \|u - u_N\|_{\infty, BC} = O(0.556^N), \quad \text{Cond.} = O(1.39^N). \quad (7)
\end{align*}
\]

Note that the convergent rates in (7) are close to those in (6) and (6), but only half of the coefficients of \( v_N \) are needed. Moreover, the condition number in (7) is smaller than that in (6) and (6). Hence the solutions (5) with Tables 3 and 4 may better serve for testing models of singularity problems. Compared with the more accurate coefficients in [7] using Mathematica with unlimited significant digits, the leading coefficients \( \tilde{D}_0 \) and \( \tilde{D}_1 \) in Tables 2 and 4 have 15 decimal significant digits! In contrast, the leading coefficient \( \tilde{D}_0 = 0.4011624537450(3) \) in [6] only has 12 decimal significant digits¹.

¹ Such a \( \tilde{D}_0 \) is obtained by using the central rule; its accuracy may be improved by using the Gaussian rule.
with the boundary conditions

\[ \begin{align*}
12. \quad & \text{The solutions of the cracked-beam problem can be obtained through a transformation by (cf. [7])} \\
& \text{Let the error } \epsilon_N = u - u_N, \quad N + 1 = 4 \times L \quad \text{and } \| (\epsilon_N)_v \|_{0, BC} \leq K_N \| \epsilon_N \|_{1, S}, \text{ where } K_N (\geq 1) \text{ may be unbounded as } N \to \infty. \text{ Suppose for } w = \frac{1}{N+1}, \\
& \left( K_N + \frac{1}{w} \right) \| \epsilon_N \|_B \to 0, \text{ as } N \to \infty. \tag{10}
\end{align*} \]

Then the solution of the cracked-beam problem can be expressed by

\[ u = \sum_{\ell=0}^{\infty} \sum_{k=0}^{1} D_{4\ell+k} r^{4\ell+k+\frac{1}{2}} \cos \left( 4\ell + k + \frac{1}{2} \right) \theta. \tag{11} \]
Table 3
The errors and condition numbers for the cracked-beam problem by the BAM with \( v_N^* \) and \( w = 1/(N + 1) \)

| \( N + 1 \) | \( |u - v|_B \) | \( |u - v|_{\infty,BC} \) | Cond. |
|------------|----------------|----------------|--------|
| 12         | 0.163(-1)      | 0.158(-1)      | 13.8   |
| 20         | 0.983(-4)      | 0.997(-4)      | 165    |
| 28         | 0.764(-6)      | 0.764(-6)      | 0.219(4) |
| 36         | 0.671(-8)      | 0.655(-8)      | 0.307(5) |
| 44         | 0.633(-10)     | 0.602(-10)     | 0.442(6) |

Table 4
The coefficients for the cracked-beam problem by the BAM with \( v_N^* \) at \( N = 43 \)

<table>
<thead>
<tr>
<th>( \ell )</th>
<th>( \bar{D}_\ell )</th>
<th>( \ell )</th>
<th>( \bar{D}_\ell )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-.540565122713627(3))</td>
<td>24</td>
<td>(-.122003356039312(-7))</td>
</tr>
<tr>
<td>1</td>
<td>(-.167041350909274(3))</td>
<td>25</td>
<td>(-.143271475445432(-6))</td>
</tr>
<tr>
<td>4</td>
<td>(-.221801471698037(1))</td>
<td>28</td>
<td>(-.52001354219428(-9))</td>
</tr>
<tr>
<td>5</td>
<td>(-.168233110389616(1))</td>
<td>29</td>
<td>(-.716837433124135(-8))</td>
</tr>
<tr>
<td>8</td>
<td>(-.72271267633515(-2))</td>
<td>32</td>
<td>(-.228129611768236(-10))</td>
</tr>
<tr>
<td>9</td>
<td>(-.419620077505091(-1))</td>
<td>33</td>
<td>(-.36216184602297(-9))</td>
</tr>
<tr>
<td>12</td>
<td>(-.349003797759028(-3))</td>
<td>36</td>
<td>(-.949261568428855(-12))</td>
</tr>
<tr>
<td>13</td>
<td>(-.154580008074137(-2))</td>
<td>37</td>
<td>(-.16714175107239(-10))</td>
</tr>
<tr>
<td>16</td>
<td>(-.824172479435014(-5))</td>
<td>40</td>
<td>(-.265402936544629(-13))</td>
</tr>
<tr>
<td>17</td>
<td>(-.64951270910120(-4))</td>
<td>41</td>
<td>(-.499048861493208(-12))</td>
</tr>
<tr>
<td>20</td>
<td>(-.317915859974737(-6))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>(-.296970814870737(-5))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** From [6] we have

\[
\|\epsilon_N\|_{1,S} = \|u - u_N\|_{1,S} \leq C \left( K_N + \frac{1}{w} \right) \|\epsilon_N\|_B,
\]  

(12)

where \( C \) is a bounded constant independent of \( N \). From (10) and (12), \( \{\epsilon_N\} \) is a bounded sequence. Based on the Kandrasov or Rellich theorem [1], any bounded sequence in the space \( H^1(S) \) contains a subsequence that converges in \( H^0(S) \). Then there must exist a subsequence \( \{\epsilon_N^+\} \) in \( H^0(S) \), i.e., \( \lim_{N \to \infty} \epsilon_N^+ = \bar{\epsilon} \). Since \( \{\epsilon_N^+\} \) is also bounded in \( H^1(S) \), the convergent limit \( \bar{\epsilon} \in H^1(S) \). This implies that \( \lim_{N \to \infty} u_N^+ \to \tilde{u}(= u - \bar{\epsilon}) \in H^1(S) \). Moreover, since \( K_N \geq 1 \) and \( w = \frac{1}{N+1} \), we conclude from (4), (10) and (12) that \( \|\tilde{u} - 500\|_{0,BC} = 0 \) and \( \|\tilde{u}\|_{0,\partial S \setminus \Gamma_D} = 0 \). Hence \( \tilde{u} \) must be the unique solution of the cracked-beam problem. Obviously, the entire sequence \( u_N \) also converges to \( \tilde{u}(= u) \) based on \( \|u - u_N\|_{1,S} \to 0 \) as \( N \to \infty \) from (10) and (12). This completes the proof of proposition by replacing \( u_N \) by \( u_N^+ \). \( \square \)

Usually the constant \( K_N = O(N^\beta) \), \( \frac{1}{4} \leq \beta \leq 2 \). When \( w = \frac{1}{N+1} \), the exponential convergent rates in (7) guarantee (10). The analysis of proposition is made based on the a posteriori numerical results, and is then called a posteriori analysis. Proposition implies that \( D_{4\ell+2} = D_{4\ell+3} = 0 \), \( \forall \ell \geq 0 \). Also note that condition (10) is stronger than the condition \( \|\epsilon_N\|_B \to 0 \) as \( N \to \infty \).
In summary, the cracked-beam problem, as a benchmark of singularity problems, seems to be superior to Motz’s problem, because half of its expansion coefficients are zero, and because the solutions from the BAM have higher accuracy and better stability.

References