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On twisted representations of vertex algebras ☆

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Abstract

In this paper, we develop a formalism for working with representations of vertex and conformal algebras by generalized fields—formal power series involving non-integer powers of the variable. The main application of our technique is the construction of a large family of representations for the vertex superalgebra $\mathfrak{V}_\Lambda$ corresponding to an integer lattice Λ. For an automorphism $\hat{\sigma}:\mathfrak{V}_\Lambda\to\mathfrak{V}_\Lambda$ coming from a finite-order automorphism $\sigma:\Lambda\to\Lambda$ we find the conditions for existence of twisted modules of $\mathfrak{V}_\Lambda$. We show that the category of twisted representations of $\mathfrak{V}_\Lambda$ is semisimple with finitely many isomorphism classes of simple objects.

Keywords: Vertex algebras; Conformal algebras; Lattice vertex algebras; Twisted modules; Semisimple categories

0. Introduction

One of the most fascinating features of vertex algebras is their "sensitivity" to finite-order automorphisms. This is best illustrated by the construction of Moonshine representation of the Monster simple group [2,12].

In general, vertex algebras are represented by formal power series, also called vertex operators, of the form

$$\alpha(z)=\sum_{n\in\mathbb{Z}}\alpha(n)z^{-n-1},\quad \alpha(n)\in \mathrm{gl}(V),$$

with coefficients in the algebra of linear operators on some space V. In this case V is a module over the vertex algebra. Having a finite-order automorphism on a vertex

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algebra $\mathfrak{A}$ means that there is a grading $\mathfrak{A} = \bigoplus_{\lambda \in \Gamma} \mathfrak{A}^{\lambda}$ on $\mathfrak{A}$ by a finite cyclic group Γ. Assume that $\Gamma \subset \mathbb{C}/\mathbb{Z}$, then it is tempting to represent the vertex algebra $\mathfrak{A}$ by the so-called *twisted* or *generalized* vertex operators, involving non-integer powers of z, so that an element $\alpha \in \mathfrak{A}^{\lambda}$ is represented by series of the form

$$\alpha(z) = \sum_{n \equiv \lambda \bmod \mathbb{Z}} \alpha(n)\, z^{-n-1}, \quad \alpha(n) \in \mathrm{gl}(V).$$

This indeed could be done, and then V becomes a *twisted module* over $\mathfrak{A}$. In fact this construction works for any group $\Gamma \subset \mathbb{C}/\mathbb{Z}$.

The idea of twisted realizations of vertex algebras goes back to the work of Lepowsky and Wilson [20], who introduced the so-called twisted vertex operators. These operators were systematically studied in [12,17,19]. Twisted modules of vertex algebras were defined in [5,10], see also [7,17,22]. Closely related is the theory of orbifolds—the invariant subalgebras of vertex algebras under an action of a finite group of automorphisms. Twisted representations of the vertex algebra yield ordinary (non-twisted) representations of its orbifold, see e.g. [1,3].

In this paper, we develop a formalism for working with generalized vertex operators and use it to construct generalized representations of conformal and vertex algebras in a very general setting. Then we consider two applications of this techniques.

First, we study realizations of conformal algebras by twisted formal series. Conformal algebras were introduced by Kac [15], see also [16,27,28]. They proved to be a valuable tool in studying vertex algebras, the relation between the former and the latter is somewhat like the relation between Lie and associative algebras.

The second application is the construction of generalized representations of the vertex (super)algebra $\mathfrak{V}_\Lambda$ corresponding to an integer lattice Λ. Our result is similar to that of Dong and Lepowsky [7], but we use different techniques and get a slightly more general construction. Lattice vertex algebras were extensively studied in [4,6,12,15,24]. They play a very important role in different areas of mathematics and physics, in particular the Moonshine vertex algebra $V^\natural$, mentioned above, is closely related to the lattice vertex algebra of certain even unimodular lattice of rank 24, called the Leech lattice. In fact, one needs to consider a twisted representation of the vertex algebra of the Leech lattice in order to construct the Moonshine vertex algebra.

An automorphism $\sigma : \Lambda \to \Lambda$ of the lattice can be extended (in a non-unique way) to an automorphism $\hat{\sigma} : \mathfrak{V}_\Lambda \to \mathfrak{V}_\Lambda$ of the lattice vertex algebra. Our construction of generalized $\mathfrak{V}_\Lambda$-modules yields all reasonable twisted modules of $\mathfrak{V}_\Lambda$ corresponding to the grading by the root spaces of $\hat{\sigma}$. It turns out that sometimes there is no twisted representation for either continuation of σ. In this case the representations of the orbifold vertex algebra $\mathfrak{V}_\Lambda^{\hat{\sigma}}$ do not come from the twisted representation of $\mathfrak{V}_\Lambda$.

For an automorphism $\hat{\sigma} : \mathfrak{V}_\Lambda \to \mathfrak{V}_\Lambda$ as above we define a category $\mathcal{O}_{\hat{\sigma}}$ of twisted representation, analogous to the category $\mathcal{O}$ of representations of Kac–Moody Lie algebras. All reasonable twisted $\mathfrak{V}_\Lambda$-modules, including those that satisfy a traditional definition of a module over a vertex operator algebra as in [10,12], belong to $\mathcal{O}_{\hat{\sigma}}$. We prove that the category $\mathcal{O}_{\hat{\sigma}}$ is semisimple with finitely many isomorphism classes of simple objects, that is, every module $V \in \mathcal{O}_{\hat{\sigma}}$ is decomposed into a direct sum of irreducible submodules, and there are only finitely many irreducible modules, up to an isomorphism. This result has been also obtained recently by Bakalov et al. [1]. Some special cases were known before, for example the case when $\sigma = -1$ was studied by Dong [5] and Dong and Nagatomo [9], the case when Λ is a simply laced root lattice and σ is an element of the Weyl group of corresponding affine Kac–Moody algebra was studied by Kac and Peterson [17].

Organization of the manuscript: We start with giving formal definitions of conformal and vertex algebras. For more details the reader can consult the books [6,12,15]. Then, in Sections 1.2–1.7 we discuss some properties of these algebras in the context of generalized formal series. In Section 1.3 we derive a nice formula for the products of generalized series, which is probably new. In Section 1.5 we prove that conformal and vertex algebras are exactly the algebraic structures formed by generalized series with coefficients in a Lie algebra and generalized vertex operators, respectively. For vertex algebras this was proved by Li [22] in a slightly less general setting and using different methods. For the non-twisted case this result is well known [15,23].

In Section 2 we show how the approach developed in Section 1 works for conformal algebras. As in the non-twisted case, there is a universal realization of a conformal algebra $\mathfrak{L}$ with coefficients in a certain Lie algebra $\mathrm{Coeff}_\Gamma \mathfrak{L}$. This Lie algebra can be constructed explicitly from $\mathfrak{L}$. In Section 2.2 we illustrate this by the example of an affine conformal algebra. Similar ideas appeared also in [16].

In Section 3 we study generalized representations of lattice vertex superalgebra $\mathfrak{V}_\Lambda$. After some preliminary information on representations of Heisenberg algebras (Section 3.1) and Fock spaces (Section 3.2) we define in Section 3.3 the twisted vertex operators, first introduced by Lepowsky [19]. In Section 3.4 we show that these operators generate a representation of the lattice vertex algebra.

In Section 3.6 we show that we have in fact constructed all reasonable generalized representations of lattice vertex algebras. The argument uses an idea of Lepowsky and Wilson [21], which was also used in [4,5,12,24].

Finally in Sections 3.7–3.8, we study the twisted representations of $\mathfrak{V}_\Lambda$, i.e. generalized representations, which are homogeneous with respect to the grading induced by an automorphism $\hat{\sigma}$ of $\mathfrak{V}_\Lambda$. In Section 3.7 we find the conditions on generalized $\mathfrak{V}_\Lambda$-module, constructed in Section 3.4, to be $\hat{\sigma}$-twisted, while in Section 3.8 we define the category $\mathcal{O}_{\hat{\sigma}}$ of twisted $\mathfrak{V}_\Lambda$-modules and prove that this category is semisimple with finitely many isomorphism classes of simple objects.

The key idea is to show that the category $\mathcal{O}_{\hat{\sigma}}$ is equivalent to the category of graded representations of certain graded associative algebra A, which turns out to be graded semisimple. This idea is similar to the idea of Zhu algebra, introduced by Zhu [29] and then generalized for the twisted case by Dong et al. [8]. However, our algebra A is quite different from the Zhu algebra of $\mathfrak{V}_\Lambda$.

1. Vertex algebras

In this section, we give an abstract definition of the main objects of this paper—conformal and vertex algebras, and then show how they can be represented by generalized formal power series. Conformal algebras were defined by Kac [15,16], see also [27,28]. The first axiomatic definition of vertex algebras is due to Borcherds [2], see also [11,12,15]. We recall here some basic properties of these algebras and develop some techniques that will be used in Sections 2 and 3.

All algebras and spaces are over a ground field $\mathbf{k}$ of characteristic 0.

1.1. Definitions of conformal and vertex algebras

A *Conformal algebra* is a vector space $\mathfrak{L}$ equipped with a sequence of bilinear products $\boxed{n}: \mathfrak{L} \otimes \mathfrak{L} \to \mathfrak{L}$, $n \in \mathbb{Z}_+$, and a linear operator $D: \mathfrak{L} \to \mathfrak{L}$, such that the following axioms hold for all $a, b, c \in \mathfrak{L}$ and $n \in \mathbb{Z}_+$:

(C1) (Locality) $a \boxed{n} b = 0$ for $n \geqslant 0$.

(C2) $D(a \boxed{n} b) = (Da) \boxed{n} b + a \boxed{n} (Db) = -na \boxed{n-1} b + a \boxed{n} (Db)$.

(C3) (Quasisymmetry)

$$a \boxed{n} b = -\sum_{i \geqslant 0} (-1)^{n+i} \frac{1}{i!} D^i (b \boxed{n+i} a).$$

(C4) (Conformal Jacoby identity)

$$(a \boxed{n} b) \boxed{m} c = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (a \boxed{n-i} (b \boxed{m+i} c) - b \boxed{m+i} (a \boxed{n-i} c)).$$

Example. The Virasoro conformal algebra $\mathfrak{Vir}$ is generated over $\mathbf{k}[D]$ by elements υ and c, such that $D\mathsf{c} = 0$ (and therefore all products with c are 0 due to (C2)) and the products of υ with itself are

$$\upsilon \boxed{0} \upsilon = D\upsilon, \quad \upsilon \boxed{1} \upsilon = 2\upsilon, \quad \upsilon \boxed{3} \upsilon = \mathsf{c}. \tag{1}$$

The rest of the products are 0.

A *vertex algebra* can be defined axiomatically as follows. Let $\mathfrak{A}$ be a linear space endowed with a sequence of bilinear operations $\boxed{n}: \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$, $n \in \mathbb{Z}$, and a distinguished element $\mathbb{1} \in \mathfrak{A}$. Let $D: \mathfrak{A} \to \mathfrak{A}$ be a linear map given by $Da = a \boxed{-2} \mathbb{1}$. Then $\mathfrak{A}$ is a vertex algebra if it satisfies the following conditions for any $a, b, c \in \mathfrak{A}$ and $m, n \in \mathbb{Z}$:

(V1) (Locality) $a \boxed{n} b = 0$ for $n \geqslant 0$.

(V2) (Identity) $\mathbb{1}\boxed{n}a = \delta_{n,-1}a,\ \ a\boxed{n}\mathbb{1} = \begin{cases} 0 & \text{if } n \geqslant 0, \\ \frac{1}{(-n-1)!}D^{-n-1}a & \text{if } n < 0. \end{cases}$

(V3) (Associativity)

$$(a\boxed{n}b)\boxed{m}c = \sum_{i\geqslant 0}(-1)^i\binom{n}{i}a\boxed{n-i}(b\boxed{m+i}c) - \sum_{i\leqslant n}(-1)^i\binom{n}{n-i}b\boxed{m+i}(a\boxed{n-i}c).$$

(V4) (Commutation)

$$a\boxed{m}(b\boxed{n}c) - b\boxed{n}(a\boxed{m}c) = \sum_{i\geqslant 0}\binom{m}{i}(a\boxed{i}b)\boxed{m+n-i}c.$$

This is not the shortest possible list of axioms. See the references cited above for other equivalent definitions.

For $n, m \geqslant 0$, associativity (V3) is exactly the conformal Jacoby identity (C4). Among other properties of vertex algebras are formulas (C2) as well as the quasisymmetry identity (C3), which holds for all integer n. So vertex algebras are a special case of conformal algebras.

There is an additional axiom which is often imposed on vertex algebras, see [6,11,12,23]. A vertex algebra $\mathfrak{A}$ is called a *vertex operator algebra* if

(V5) $\mathfrak{A} = \oplus_{n\in\mathbb{Z}}\mathfrak{A}_n$ is graded so that $\mathbb{1} \in \mathfrak{A}_0$ and $\mathfrak{A}_i\boxed{n}\mathfrak{A}_j \subset \mathfrak{A}_{i+j-n-1}$.

(V6) There exists an element $\upsilon \in \mathfrak{A}_2$ generating the Virasoro conformal algebra $\mathfrak{Vir} \subset \mathfrak{A}$, so that relations (1) hold, where $\mathsf{c} = c\mathbb{1}$ for some $c \in \mathbf{k}$. Also, $\upsilon\boxed{0}a = Da$, $\upsilon\boxed{1}a = (\deg a)a$ for all homogeneous $a \in \mathfrak{A}$. The number $2c$ is called the *conformal charge* of $\mathfrak{A}$.

In Section 1.5 we show that conformal and vertex algebras are precisely the algebraic structures formed by certain formal infinite series.

1.2. Formal series and locality

Let L be a Lie algebra. Denote by $L\{z\}$ the $\mathbf{k}$-linear span of all series of the form

$$\sum_{n\in\lambda+\mathbb{Z}}\alpha(n)z^{-n-1}, \quad \alpha(n) \in L, \ \ \lambda \in \mathbf{k}.$$

For a linear space V, let $\mathrm{vo}\{V\} \subset (\mathrm{gl}\ V)\{z\}$ be the space of all such series with coefficients in the Lie algebra $\mathrm{gl}\ V$ with the property that $\alpha(n)v = 0$ for $n \gg 0$ for any fixed $v \in V$. We call $\mathrm{vo}\{V\}$ the space of *generalized vertex operators*. It contains the space $\mathrm{vo}(V)$ of ordinary vertex operators, that involve only integer powers of z. Denote by $\mathbb{1} \in \mathrm{vo}(V)$ the vertex operator with the only non-zero coefficient being $\mathbb{1}(-1) = \mathrm{id}$.

The space $L\{z\} = \bigoplus_{[\lambda] \in \mathbf{k}/\mathbb{Z}} L\{z\}^{[\lambda]}$ is graded by the group $\mathbf{k}/\mathbb{Z}$ so that $L\{z\}^{[\lambda]}$ is the space of all series of the form $z^{-\lambda}\alpha_\lambda(z)$, $\alpha_\lambda(z) \in L[[z^{\pm 1}]] = L\{z\}^{[0]}$. The space of vertex operators $F\{V\}$ is a homogeneous subspace of $(\mathrm{gl}\,V)\{z\}$ and $\mathrm{vo}(V) = \mathrm{vo}\{V\}^{[0]}$.

A pair of series $\alpha, \beta \in L\{z\}$ are said to be *local of order* $N \in \mathbb{Z}_+$ [6,15] if $(z - w)^N[\alpha(w), \beta(z)] = 0$, or, equivalently,

$$\sum_{s=0}^{N} (-1)^s \binom{N}{s} [\alpha(n-s), \beta(m+s)] = 0 \quad \forall m, n \in \mathbf{k}.$$

The same applies for $\alpha, \beta \in \mathrm{vo}\{V\}$. It is easy to see that if series $\alpha = \sum_\lambda z^{-\lambda}\alpha_\lambda$, $\beta = \sum_\lambda z^{-\lambda}\beta_\lambda \in L\{z\}$ are local of order N, then any two homogeneous components $\alpha_\lambda, \beta_\mu \in L[[z^{\pm 1}]]$ are local of the same order.

Remark. The space $\mathrm{vo}(V)$ of vertex operators over V is a linear space over the field $F = \mathbf{k}((z))$ of formal power series in z. Let $E \supset F$ be the field extension of F generated by z^λ for all $\lambda \in \mathbf{k}$. Both F and E are differential fields, and E is a differential Galois extension of F. Then $\mathrm{vo}\{V\}$ is a linear space over E, and in fact $\mathrm{vo}\{V\} = \mathrm{vo}(V) \otimes_F E$. We observe that the multiplication by the elements of E does not affect the locality of vertex operators.

It is possible to consider a more general extension E of $\mathbf{k}((z))$, for example involving $\log z$. Most of the results in this paper can be generalized to this more general setting.

1.3. The products of formal series

Recall that for two ordinary vertex operators $\alpha, \beta \in \mathrm{vo}(V)$, one can define products $\alpha \boxed{n} \beta \in \mathrm{vo}(V)$, $n \in \mathbb{Z}$, in the following way. Let $\iota_{w,z} : \mathbf{k}(w,z) \to \mathbf{k}((w^{-1}, z))$ and $\iota_{z,w} : \mathbf{k}(w,z) \to \mathbf{k}((w, z^{-1}))$ be the expansions of a rational function into Laurent series at $(w,z) = (\infty, 0)$ and $(w,z) = (0, \infty)$, respectively, so that

$$\iota_{w,z}(w-z)^n = \sum_{i \geqslant 0} (-1)^{n+i} \binom{n}{i} w^{n-i} z^i,$$

$$\iota_{z,w}(w-z)^n = \sum_{i \geqslant 0} (-1)^i \binom{n}{i} w^i z^{n-i}. \tag{2}$$

Of course, if $n \geqslant 0$ then $\imath_{w,z}(w-z)^n = \imath_{z,w}(w-z)^n$. We define

$$(\alpha \boxed{n} \beta)(z) = \mathrm{Res}_w(\alpha(w)\beta(z)\imath_{w,z}(w-z)^n - \beta(z)\alpha(w)\imath_{z,w}(w-z)^n). \tag{3}$$

The mth coefficient of $\alpha \boxed{n} \beta$ is given by

$$\begin{aligned}(\alpha \boxed{n} \beta)(m) = & \sum_{s \geqslant 0} (-1)^s \binom{n}{s} \alpha(n-s)\beta(m+s) \\ & - \sum_{s \leqslant n} (-1)^s \binom{n}{n-s} \beta(m+s)\alpha(n-s).\end{aligned} \tag{4}$$

If $n \geqslant 0$ then the products $\alpha \boxed{n} \beta$ make sense for formal series $\alpha, \beta \in L[[z^{\pm 1}]]$ as well. In this case (4) simplifies to

$$(\alpha \boxed{n} \beta)(m) = \sum_s (-1)^s \binom{n}{s} [\alpha(n-s), \beta(m+s)].$$

One can solve these equations with respect to the commutators and thus recover the bracket in L from the products in $L[[z^{\pm 1}]]$:

$$[\alpha(m), \beta(n)] = \sum_s \binom{m}{s} (\alpha \boxed{s} \beta)(m+n-s) \tag{5}$$

for every $m \in \mathbb{Z}_+,\ n \in \mathbb{Z}$. If α and β are local, then (5) holds for all $m, n \in \mathbb{Z}$.

On the other hand, the -1st product is given by $\alpha \boxed{-1} \beta = \alpha_- \beta + \beta \alpha_+$, where $\alpha_\pm(z) = \sum_{n \geqslant 0} \alpha(n)\, z^{-n-1}$. It is also called *the normally ordered product* and is sometimes denoted by $:\alpha\beta:$. For any $n<0$, we have $\alpha \boxed{n} \beta = \frac{1}{(-n-1)!} {:}(D^{-n-1}\alpha)\beta{:}$, where $D = d/dz\colon\ \mathrm{vo}\{V\} \to \mathrm{vo}\{V\}$ is the operator of differentiation. It follows that vertex operators satisfy relations (V2).

Now we show how to expand these products to $\mathrm{vo}\{V\}$ and $L\{z\}$. For homogeneous $\alpha = z^\lambda \alpha_0$ and $\beta = z^\mu \beta_0$, where $\lambda, \mu \in \mathbf{k}$ and $\alpha_0,\ \beta_0$ involve only integer powers of z, we set

$$\alpha \boxed{n} \beta = \sum_{j \geqslant 0} \binom{\lambda}{j} (\alpha_0 \boxed{n+j} \beta_0) z^{\mu+\lambda-j}. \tag{6}$$

Here

$$\binom{\lambda}{j} = \frac{\lambda(\lambda-1)\cdots(\lambda-j+1)}{j!}.$$

Note that the summation in (6) is finite due to the locality of α and β, and hence of α_0 and β_0. One can easily check that products (3) satisfy identity (6) if we substitute

$\alpha = z^{\lambda}\alpha_0,\ \beta = z^{\mu}\beta_0$ for $\lambda, \mu \in \mathbb{Z}$. We extend the definition of the products $\boxed{n}$ by linearity to arbitrary $\alpha, \beta \in \mathrm{vo}\{V\}$ and $\alpha, \beta \in L\{z\}$ when $n \geqslant 0$. Note that $\alpha \boxed{n} \beta \in \mathrm{vo}\{V\}^{[\lambda+\mu]}$ if $\alpha \in \mathrm{vo}\{V\}^{[\lambda]}$ and $\beta \in \mathrm{vo}\{V\}^{[\mu]}$.

It is clear that if α and β are local of order N then $\alpha \boxed{n} \beta = 0$ for $n \geqslant N$. It could also be shown that (5) remains valid for generalized series as well if α and β are homogeneous and $m \equiv \deg \alpha \bmod \mathbb{Z},\ n \equiv \deg \beta \bmod \mathbb{Z}$.

We now write explicitly the formula for the products of twisted vertex operators $\alpha, \beta \in \mathrm{vo}\{V\}$, analogous to (3). Let N be the order of locality of α and β. Denote

$$\Delta(w,z) = \sum_{\lambda} \sum_{j=0}^{N-n-1} \binom{-\lambda}{j} w^{\lambda} z^{-\lambda-j} (w-z)^j,$$

where λ runs over the set of degrees that appear in α.

Then

$$\begin{aligned}(\alpha \boxed{n} \beta)(z) &= \sum_{\lambda} (z^{-\lambda}\alpha_{\lambda}) \boxed{n} \beta \\ &= \sum_{\lambda} \sum_{j \geqslant 0} \binom{-\lambda}{j} (\alpha \boxed{n+j} \beta) z^{-\lambda-j} \\ &= \mathrm{Res}_w((\alpha(w)\beta(z)\, \iota_{w,z}(w-z)^n - \beta(z)\alpha(w)\iota_{z,w}(w-z)^n)\Delta(w,z)). \quad (7)\end{aligned}$$

It is easy to see that identity (V2) holds for every $a \in \mathrm{vo}\{V\}$ and $n \in \mathbb{Z}$.

1.4. The case of finite grading

Of a particular interest is the case when all the degrees of α are rational numbers with common denominator p, i.e.

$$\alpha(z) = \sum_{q=0}^{p-1} z^{-q/p}\, \alpha_q(z), \quad \alpha_q \in \mathrm{vo}(V).$$

In this case we can rewrite (7) in a different way. Some calculations show that

$$\Delta(w,z) = \sum_{q=0}^{p-1} \sum_{j=0}^{m} \binom{-q/p}{j} w^{q/p} z^{-q/p-j} (w-z)^j = \left(\frac{w-z}{w^{1/p} - z^{1/p}} \right)^{m+1} F_p(m),$$

where

$$\begin{aligned}F_p(m) = \sum_{l=1-p}^{1-p+m} &\left(\sum_{q=0}^{p-1} \sum_{k \geqslant 0} (-1)^{l+q+kp} \binom{-q/p+k}{m} \binom{m+1}{l+q-kp} \right) \\ &\times w^{m-l+1/p-1} z^{l/p-m}.\end{aligned}$$

Using this we get

$$(\alpha \boxed{n}\, \beta)(z) = \mathrm{Res}_w \Bigg((\alpha(w)\beta(z)\iota_{w,z}(w-z)^n - \beta(z)\alpha(w)\iota_{z,w}(w-z)^n) \times \left(\frac{w-z}{w^{1/p}-z^{1/p}}\right)^{N-n} F_p(N-n-1)\Bigg). \tag{8}$$

We also remark that the polynomial $F_p(m)$ has the following property:

$$F_p(m)|_{w^{1/p}=z^{1/p}} = p^{-m}\, z^{(m+1)(1-p)/p}. \tag{9}$$

1.5. *Algebras of formal series*

It is well known [15,23] that a subspace $\mathfrak{A} \subset \mathrm{vo}(V)$ of pairwise local vertex operators such that $\mathbb{1} \in \mathfrak{A}$ and $\mathfrak{A}\, \boxed{n}\, \mathfrak{A} \subset \mathfrak{A}$ for all $n \in \mathbb{Z}$ is a vertex algebra. Similarly, a subspace $\mathfrak{L} \subset L[[z^{\pm 1}]]$ of local formal series such that $\mathfrak{L}\, \boxed{n}\, \mathfrak{L} \subset \mathfrak{L}$ and $DL \subseteq L$ is a conformal algebra. Moreover, all vertex and conformal algebras are obtained in that way. Now we generalize this result to the case of generalized vertex operators and series.

Theorem 1. (a) *Let $\mathfrak{A} \subset \mathrm{vo}\{V\}$ be a subspace of pairwise local generalized vertex operators such that $\mathbb{1} \in \mathfrak{A}$ and $\mathfrak{A}\, \boxed{n}\, \mathfrak{A} \subseteq \mathfrak{A}$ for all $n \in \mathbb{Z}$. Then $\mathfrak{A}$ is a vertex algebra.*

(b) *Let $\mathfrak{L} \subset L\{z\}$ be a subspace of pairwise local generalized series with coefficients in a Lie algebra L, such that $D\mathfrak{L} \subseteq \mathfrak{L}$ and $\mathfrak{L}\, \boxed{n}\, \mathfrak{L} \subseteq \mathfrak{L}$ for all $n \in \mathbb{Z}$. Then $\mathfrak{L}$ is a conformal algebra.*

Proof. We will prove (a), statement (b) is proved in the same way. We have to show that $\mathfrak{A}$ satisfies identities (V3) and (V4), since (V1) and (V2) hold by assumption. These identities are linear combinations of "vertex monomials" of the form $a_1 \boxed{n_1} \dots \boxed{n_{l-1}}\, a_l$ (with some order of parentheses) where a_i's are either equal to $\mathbb{1}$ or are formal variables. We have to show that for any specification $a_i \in \mathfrak{A}$ the identity vanishes. Note that these identities are multilinear.

Let $\mathfrak{A}^{\mathrm{gr}} \subset \mathrm{vo}\{V\}$ be the graded closure of $\mathfrak{A}$, i.e. the minimal graded subspace of $\mathrm{vo}\{V\}$ containing $\mathfrak{A}$. Since all homogeneous components of the vertex operators from $\mathfrak{A}$ are pairwise local, the space $\mathfrak{A}^{\mathrm{gr}}$ satisfies all the assumptions of the theorem, so we can assume that $\mathfrak{A} = \mathfrak{A}^{\mathrm{gr}}$ is graded.

Consider all the vertex operators $\beta \in \mathrm{vo}(V)$ such that $z^\lambda \beta \in \mathfrak{A}$ for some $\lambda \in \mathbf{k}$. Every two such vertex operators are local, so since the theorem is known to be true in the case of ordinary vertex operators, these β's generate a vertex algebra $\mathfrak{B} \subset \mathrm{vo}(V)$.

Let $R(a_1, \dots, a_l) = 0$ be an identity which we have to check. Since R is multilinear, it is enough to check it for homogeneous $a_i = z^{-\lambda_i} b_i$, where $b_i \in \mathfrak{B}$, $\lambda_i \in \mathbf{k}$, $1 \leqslant i \leqslant l$. Sometimes we must set $b_i = \mathbb{1}$ and $\lambda_i = 0$. When we substitute these expressions for a_i into $R(a_1, \dots, a_k) = 0$ and apply (6), we get a linear combination of vertex monomials of the form

$$P(\lambda_1, \dots, \lambda_l) z^{-\lambda_1 - \dots - \lambda_l} z^m (b_{\kappa(1)} \boxed{n_1} \dots \boxed{n_{l-1}} b_{\kappa(l)}),$$

where $P(\lambda_1, \dots, \lambda_l)$ is a polynomial in $\lambda_1, \dots, \lambda_l$, $m \in \mathbb{Z}$, κ is a permutation of $\{1, 2, \dots, l\}$ and the products $\boxed{n_i}$'s are applied according to some order of parentheses. We can cancel the common factor $z^{-\lambda_1 - \dots - \lambda_l}$. Now we observe that the only remaining factor in these monomials that depends on λ_i's is P, therefore, for fixed $b_1, \dots, b_l$ the map $(\lambda_1, \dots, \lambda_l) \mapsto R(z^{-\lambda_1} b_1, \dots, z^{-\lambda_l} b_l)$ is a polynomial map from $\mathbf{k}^l$ to $\mathrm{vo}\{V\}$. But when all $\lambda_i \in \mathbb{Z}$ this map is equal to 0, because then $a_i \in \mathrm{vo}(V)$ generate a vertex algebra which satisfies the identity R, therefore, since $|\mathbf{k}| = \infty$, R must be identically 0. $\square$

Remark. There is a more conceptual argument illustrating Theorem 1. Recall that $\mathrm{vo}(V)$ is a linear space over the field $F = \mathbf{k}((z))$, and $\mathrm{vo}\{V\}$ is a linear space over the field E generated over F by z^λ's, see Remark at the end of Section 1.2. Let $\mathfrak{B} \subset \mathrm{vo}(V)$ be the vertex algebra generated over F by all vertex operators β such that βz^λ belongs to the graded closure of $\mathfrak{A}$ for some $\lambda \in \mathbf{k}$. Recall that a commutative algebra A with derivation d can be thought of as a vertex algebra if we set $a \boxed{n} b = 0$ for $n \geqslant 0$ and $a \boxed{n} b = \frac{1}{(-n-1)!} d^{-n-1}(a) b$ for $n < 0$, so we can treat both E and F as vertex algebras. Recall also that there is a notion of tensor product of vertex algebras [2,15], which can easily be generalized for the case of vertex algebras over some ground vertex algebra, so that one can consider the vertex algebra $\mathfrak{B} \otimes_F E$. The idea is that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B} \otimes_F E$.

Note that this technique works for a more general field extension E of $\mathbf{k}((z))$.

Remark. The products of the generalized vertex operators were first introduced by Li [22]. However, he deals only with generating functions of these products, formulas (7) and (8) seem to be new. Li also proves Theorem 1(a) using more straightforward techniques. Realizations of conformal algebras by generalized series were mentioned by Kac [16].

1.6. Differentiation of vertex operators

Assume that we are given a pair (V, D) consisting of a linear space V and a linear map $D : V \to V$. Let $\Delta : \mathrm{vo}\{V\} \to \mathrm{vo}\{V\}$ be the linear operator on the space of generalized vertex operators defined by $\Delta\phi = \frac{d}{dz}\phi - [D, \phi]$. A vertex operator $\phi \in \mathrm{vo}\{V\}$ is said to be of weight $\lambda \in \mathbf{k}$ if $\Delta\phi = \lambda z^{-1}\phi$. Denote by $\mathrm{vo}\{V\}_\lambda$ the space of all vertex operators of weight λ. It is easy to see that if $\phi \in \mathrm{vo}\{V\}_\lambda$ then $z^\mu \phi \in \mathrm{vo}\{V\}_{\lambda+\mu}$.

In general, it is not true that any vertex operator can be represented as a sum of homogeneous vertex operators. However the following is true.

Proposition 1. *Let $S \subset \mathrm{vo}\{V\}$ be a set of pairwise local vertex operators, and let $\mathfrak{A} \subset \mathrm{vo}\{V\}$ be the vertex algebra generated by S. If $S \subset \mathrm{vo}\{V\}_0$ then also $\mathfrak{A} \subset \mathrm{vo}\{V\}_0$ and if $S \subset \bigoplus_{\lambda \in \mathbf{k}} \mathrm{vo}\{V\}_\lambda$ then also $\mathfrak{A} \subset \bigoplus_{\lambda \in \mathbf{k}} \mathrm{vo}\{V\}_\lambda$.*

Proof. First of all we note that Δ is a derivation of all products:

$$\Delta(\alpha \boxed{n} \beta) = (\Delta\alpha) \boxed{n} \beta + \alpha \boxed{n} (\Delta\beta),$$

because so are both $\mathrm{ad}\, D$ and $\frac{d}{dz}$. Therefore if $\Delta\alpha = \Delta\beta = 0$ then $\Delta(\alpha \boxed{n} \beta) = 0$ and the first statement follows. For the second statement, it is enough to assume that all the generators from S are homogeneous. A pair $\alpha, \beta \in S$ can be written as $\alpha = z^\lambda \alpha_0,\ \beta = z^\mu \beta_0$ for some $\lambda, \mu \in \mathbf{k}$ and $\alpha_0, \beta_0 \in \mathrm{vo}\{V\}_0$. Now the statement follows from formula (6). $\square$

1.7. Modules over vertex algebras

Now we give several definitions of modules for vertex algebras. We will call a vector space V a *module* over a vertex algebra $\mathfrak{A}$ if there is a vertex algebra homomorphism $\pi : \mathfrak{A} \to \mathrm{vo}(V)$. In other words, for any $a, b \in \mathfrak{A}$, the vertex operators $\pi(a)$ and $\pi(b)$ are local and $\pi(a \boxed{n} b) = \pi(a) \boxed{n} \pi(b),\ \pi(\mathbb{1}) = \mathbb{1}$. We remark that sometimes what we call a module is called a weak module.

If instead of a homomorphism $\pi : \mathfrak{A} \to \mathrm{vo}(V)$ we have a homomorphism $\pi : \mathfrak{A} \to \mathrm{vo}\{V\}$ of $\mathfrak{A}$ into the space of generalized vertex operators over V, then V is called a *generalized module*.

Assume now that a vertex algebra $\mathfrak{A} = \bigoplus_{[\lambda] \in \Gamma} \mathfrak{A}^{[\lambda]}$ is graded by a group $\Gamma \subset \mathbf{k}/\mathbb{Z}$ so that $\mathfrak{A}^{[\lambda]} \boxed{n} \mathfrak{A}^{[\mu]} \subset \mathfrak{A}^{[\lambda+\mu]}$. Then a generalized module V is called *twisted* if the representation homomorphism $\pi : \mathfrak{A} \to \mathrm{vo}\{V\}$ is homogeneous, that is, $\pi\mathfrak{A}^{[\lambda]} \subset \mathrm{vo}\{V\}^{[\lambda]}$. This definition is due to Li [22]. Equivalently, twisted modules can be defined using the so-called twisted Jacoby identity, see e.g. [5,8,10].

Remark. We can generalize the definition of twisted representation to the case when $\mathrm{vo}\{V\} = F(V) \otimes_F E$ for an arbitrary differential Galois extension E of the field $F = \mathbf{k}((z))$, see Remark at the end of Section 1.2. Note that the Galois group $\mathrm{Gal}(E/F)$ acts on $\mathrm{vo}\{V\}$ in a natural way. Let $\Gamma \subset \mathrm{Aut}\, \mathfrak{A}$ be a group of automorphisms of the vertex algebra $\mathfrak{A}$ and fix a group homomorphism $\rho : \Gamma \to \mathrm{Gal}(E/F)$. A representation $\pi : A \to \mathrm{vo}\{V\}$ is called twisted if it is equivariant with the action of Γ: $\pi(\gamma a) = \rho(\gamma)\pi(a)$ for any $\gamma \in \Gamma$ and $a \in \mathfrak{A}$. Note that for different homomorphisms $\rho : \Gamma \to \mathrm{Gal}(E/F)$ we will get different categories of twisted representations.

Assume that an $\mathfrak{A}$-module V (generalized or not) has a linear map $D : V \to V$. Then the module V is called *strong* if $\pi(\mathfrak{A}) \subset \mathrm{vo}\{V\}_0$, see Section 1.6. We note that if $\mathfrak{A}$ contains an element υ such that $Y(\upsilon)(0) = D$ (as it is the case when $\mathfrak{A}$ is a vertex operator algebra), and V is a module over $\mathfrak{A}$ such that $\pi(\upsilon) \in \mathrm{vo}(V) = \mathrm{vo}\{V\}^{[0]}$, then V is a strong module. Indeed, by (5) we have $[\upsilon(0), a(n)] = (\upsilon \boxed{0}\, a)(n) = (Da)(n)$ for any $a \in \mathfrak{A}^{[n]}$.

Let again V be a module (generalized or not) over $\mathfrak{A}$ and let $\pi : \mathfrak{A} \to \mathrm{vo}\{V\}$ be the representation map. Assume $\mathfrak{A}$ is a vertex operator algebra and let $\upsilon \in \mathfrak{A}$ be the Virasoro element. We say that V is a module over the vertex *operator* algebra if $V = \bigoplus_{n \in \mathbf{k}} V_n$ is graded, $\pi(\upsilon) \in \mathrm{vo}(V)$ and $\pi(\upsilon)(1)|_{V_n} = n$. As it was mentioned above, in this case V is necessarily a strong module over $\mathfrak{A}$.

Remark. Let $\mathfrak{A}$ be a vertex algebra and let V be a strong twisted module over $\mathfrak{A}$. Then the semidirect product $\mathfrak{A} \ltimes V$ has a structure of *generalized vertex algebra*, introduced by Dong and Lepowsky [6]. The products in a generalized vertex algebra are indexed not necessarily by integers. If $a \in \mathfrak{A}$ and $v \in V$ then $a \boxed{n}\, v = a(n)v$ and the products $v \boxed{n}\, a$ are defined using the quasisymmetry identity (C3), see Section 1.1.

2. Twisted realizations of conformal algebras

In this section, we consider realizations of conformal algebras by generalized formal series. We construct the twisted coefficient algebra of a conformal algebra, which gives the universal realization of this type. As an example we consider affine conformal algebras in Section 2.2.

2.1. The coefficient algebra

Let $\mathfrak{L}$ be a conformal algebra. Assume that $\mathfrak{L} = \bigoplus_{[\lambda] \in \Gamma} \mathfrak{L}^{[\lambda]}$ is graded by a group $\Gamma \subset \mathbf{k}/\mathbb{Z}$ so that $\mathfrak{L}^{[\lambda]} \boxed{n}\, \mathfrak{L}^{[\mu]} \subseteq \mathfrak{L}^{[\lambda+\mu]}$, $D\mathfrak{L}^{[\lambda]} \subseteq \mathfrak{L}^{[\lambda]}$.

Define a Lie algebra $\mathrm{Coeff}_\Gamma\, \mathfrak{L}$ in the following way. The underlying linear space of $\mathrm{Coeff}_\Gamma\, \mathfrak{L}$ is spanned by the symbols $a(n)$ for all homogeneous $a \in \mathfrak{L}$ and $\mathbf{k} \ni n \equiv \deg a \bmod \mathbb{Z}$ subject to the linear relations $(Da)(n) = -na(n-1)$. The brackets in $\mathrm{Coeff}_\Gamma\, \mathfrak{L}$ are defined by the formula (5) for $a = \alpha \in \mathfrak{L}^{[m]}$ and $b = \beta \in \mathfrak{L}^{[n]}$. For a non-homogeneous $a \in \mathfrak{L}$ denote $a(n) = a^{[n]}(n)$, where $a^{[n]}$ is the projection of a onto the space $\mathfrak{L}^{[n]}$.

This construction generalizes the construction of usual coefficient algebra $\mathrm{Coeff}\, \mathfrak{L} = \mathrm{Coeff}_{[0]}\, \mathfrak{L}$ done in [15,16,28]. If $\mathfrak{L}$ is a vertex algebra, then $\mathrm{Coeff}_\Gamma\, \mathfrak{L}$ was considered in [8], at least when Γ is a finite cyclic group.

Let $\phi : \mathfrak{L} \to (\mathrm{Coeff}_\Gamma\, \mathfrak{L})\{z\}$ be a map given by $\phi(a) = \sum_{n \in \Gamma + \mathbb{Z}} a(n) z^{-n-1}$. It is easy to see that ϕ is a homomorphism of $\mathfrak{L}$ into a conformal subalgebra of $(\mathrm{Coeff}_\Gamma\, \mathfrak{L})\{z\}$ such that $\phi(\mathfrak{L}^{[\lambda]}) \subset (\mathrm{Coeff}_\Gamma\, \mathfrak{L})\{z\}^{[\lambda]}$. Moreover, this map is universal in the following sense: if $\rho : \mathfrak{L} \to K\{z\}$ is another homogeneous homomorphism of $\mathfrak{L}$ into a conformal subalgebra of series with coefficients in some Lie algebra K, then there is a unique Lie algebra homomorphism $\pi : \mathrm{Coeff}_\Gamma\, \mathfrak{L} \to K$ making the following diagram commutative:

$$\begin{array}{ccc} (\mathrm{Coeff}_\Gamma\,\mathfrak{L})\{z\} & \xrightarrow{\pi} & K\{z\} \\ & \nwarrow^{\phi} \quad {}^{\rho}\nearrow & \\ & \mathfrak{L} & \end{array} \tag{10}$$

The proof of the fact that $\mathrm{Coeff}_\Gamma\,\mathfrak{L}$ is indeed a Lie algebra and of the above universality property is done in the same way as in the case when $\Gamma = 0$.

We also give another construction of $\mathrm{Coeff}_\Gamma\,\mathfrak{L}$. Let $L = \mathrm{Coeff}\,\mathfrak{L}$ be the ordinary coefficient Lie algebra of $\mathfrak{L}$. Let $\mathfrak{L}_\Gamma \subset L\{z\}$ be the conformal algebra generated by all the series of the form $z^n a$ for $n \in \mathbf{k}$ and $a \in \mathfrak{L}^{[n]}$. The algebra $\mathfrak{L}_\Gamma$ is closed under the multiplication by $z^{\pm 1}$, in fact we have $z^{-\lambda}\mathfrak{L}_\Gamma^{[\lambda]} = \mathbf{k}[z^{\pm 1}]\mathfrak{L}^{[\lambda]}$.

Consider the coefficient Lie algebra $\mathrm{Coeff}\,\mathfrak{L}_\Gamma$ of $\mathfrak{L}_\Gamma$. Let $L_\Gamma \subset \mathrm{Coeff}\,\mathfrak{L}_\Gamma$ be its subalgebra consisting of all elements of the form $a(0)$ for $a \in \mathfrak{L}_\Gamma$. In other words, $L_\Gamma = \mathrm{Ker}\,D$, where $D\colon \mathrm{Coeff}\,\mathfrak{L}_\Gamma \to \mathrm{Coeff}\,\mathfrak{L}_\Gamma$ is the derivation given by $a(n) \mapsto -na(n-1)$. There is a Lie algebra homomorphism $\mathrm{Coeff}\,\mathfrak{L}_\Gamma \to L_\Gamma$ given by $a(n) \mapsto (z^n a)(0)$ for $a \in \mathfrak{L}^{[n]}$, which induces the conformal algebra homomorphism $\eta : \mathfrak{L}_\Gamma \to L_\Gamma[[w^{\pm 1}]]$ such that $\eta(z^k a) = w^k \eta(a)$ for $k \in \mathbb{Z}$. It could be shown that η is the universal among all realizations of $\mathfrak{L}_\Gamma$ by integral formal series that commute with multiplication by the variable.

We construct a homomorphism $\rho : \mathfrak{L} \to L_\Gamma\{w\}$ in the following way. Let $a \in \mathfrak{L}^{[\lambda]}$, define $\rho(a) = \eta(z^\lambda a)w^{-\lambda} \in L_\Gamma\{w\}$. Clearly, ρ is a linear map and it does not depend on the choice of representative $\lambda \in [\lambda]$. Now we show that ρ is indeed a homomorphism.

Let $a_1, \dots, a_k \in \mathfrak{L}$ be homogeneous elements of $\mathfrak{L}$ satisfying a conformal identity $R(a_1, \dots, a_k) = 0$. Here R is a linear combination of conformal monomials in a_i's with coefficients in $\mathbf{k}[D]$. We have to show that $R(\rho(a_1), \dots, \rho(a_k)) = 0$ in $L_\Gamma\{w\}$. Without loss of generality, we can assume that R is homogeneous with respect to the gradation by Γ. Let $\deg a_i = [\lambda_i]$. Substitute $a_i = z^{-\lambda_i} b_i$ in R, apply formula (6) and then cancel the common factor $z^{-\lambda_1 - \cdots - \lambda_k}$. We get an identity $R_1(b_1, \dots, b_k) = 0$, which holds in $\mathfrak{L}_\Gamma \subset L\{z\}$, where R_1 is a combination of conformal monomials in $b_1, \dots, b_k$ with coefficients in $\mathbf{k}[D, z^{\pm 1}]$. Since $\eta : \mathfrak{L}_\Gamma \to L_\Gamma[[w^{\pm 1}]]$ is a $\mathbf{k}[D, z^{\pm 1}]$-module homomorphism, we have $R_1(\eta(b_1), \dots, \eta(b_k)) = 0$ in $L_\Gamma[[w^{\pm 1}]]$. Substitute now $\eta(b_i) = w^{\lambda_i}\rho(a_i)$ and apply (6) again. After dividing by $w^{\lambda_1 + \cdots + \lambda_k}$, we get $R(\rho(a_1), \dots, \rho(a_k)) = 0$.

By the universality of $\mathrm{Coeff}_\Gamma\,\mathfrak{L}$ we get a map $\pi\colon \mathrm{Coeff}_\Gamma\,\mathfrak{L} \to L_\Gamma$ such that diagram (10) commutes for $K = L_\Gamma$.

Proposition 2. *The map* $\pi\colon \mathrm{Coeff}_\Gamma\,\mathfrak{L} \to L_\Gamma$ *is an isomorphism.*

Proof. We will show that the homomorphism $\rho : \mathfrak{L} \to L_\Gamma\{w\}$ constructed above has the same universality property as $\mathrm{Coeff}_\Gamma\,\mathfrak{L}$. Then the fact that $\pi\colon \mathrm{Coeff}_\Gamma\,\mathfrak{L} \to L_\Gamma$ is an isomorphism follows from uniqueness of $\mathrm{Coeff}_\Gamma\,\mathfrak{L}$.

Let $\psi : \mathfrak{L} \to K\{w\}$ be a homogeneous homomorphism of $\mathfrak{L}$ into a space of generalized series with coefficients in some Lie algebra K. This induces a homomorphism $\theta : \mathfrak{L}_\Gamma \to K[[w^{\pm 1}]]$ defined by $\theta(z^n a) = w^n \psi(a)$ for any $a \in \mathfrak{L}^{[\lambda]}$ and $n \in \lambda + \mathbb{Z}$. It is easy to see that $\theta(z^n a) = w^n \theta(a)$ for every $a \in \mathfrak{L}_\Gamma$ and $n \in \mathbb{Z}$. Hence, there is a homomorphism $L_\Gamma \to K$ such that all the corresponding diagrams commute. $\square$

2.2. Twisted affine algebras

We illustrate the construction of Section 2.1 by the example of affine conformal algebras.

Let $\mathfrak{g}$ be a Lie algebra with an invariant bilinear form $(\cdot\,|\,\cdot)$, i.e. such that $([a,b]\,|\,c) = (a\,|\,[b,c])$. Assume that the algebra $\mathfrak{g} = \bigoplus_{[\lambda]\in\Gamma} \mathfrak{g}^{[\lambda]}$ is graded by a group $\Gamma \subset \mathbf{k}/\mathbb{Z}$. Assume further that the gradation on $\mathfrak{g}$ agrees with the form $(\cdot\,|\,\cdot)$ in the following way:

$$(\mathfrak{g}^{[\lambda]} | \mathfrak{g}^{[\mu]}) = 0 \quad \text{unless } \lambda + \mu = 0. \tag{11}$$

Consider the Lie algebra $\hat{L}_\Gamma = \mathfrak{g} \otimes \mathbf{k}[\Gamma + \mathbb{Z}] \oplus \mathbf{k}\mathsf{c}$, where $\mathbf{k}[\Gamma + \mathbb{Z}]$ is the group algebra of $\Gamma + \mathbb{Z}$. We will write $a(n) = a \otimes n$ for $a \in \mathfrak{g},\ n \in \Gamma + \mathbb{Z}$. The brackets in $\hat{L}_\Gamma$ are defined by

$$[a(m), b(n)] = [a,b](m+n) + \delta_{m+n,0}\, m (a^{[m]} | b^{[n]}) \mathsf{c}, \quad \mathsf{c} \in Z(\hat{L}_\Gamma),$$

where $a^{[m]}$ and $b^{[n]}$ are projections of a and b onto $\mathfrak{g}^{[m]}$ and $\mathfrak{g}^{[n]}$, respectively. The twisted affine Lie algebra $L_\Gamma \subset \hat{L}_\Gamma$ is the subalgebra of $\hat{L}_\Gamma$ spanned by c and all elements of the form $a(n)$ for $a \in \mathfrak{g}^{[n]}$. The grading on $\mathfrak{g}$ induces a grading on L_Γ by setting $\deg a(n) = \deg a,\ \deg \mathsf{c} = 0$. From now on, if $a \otimes n \in \hat{L}_\Gamma \setminus L_\Gamma$, then we set $a(n) = 0$.

For any $a \in \mathfrak{g}$ consider series $\tilde{a} = \sum_{n \in \Gamma + \mathbb{Z}} a(n) z^{-n-1} \in L_\Gamma\{z\}$. These series are local of order 2, and together with the series $\mathsf{c} = \mathsf{c}\, z^0$ they generate a conformal algebra $\mathfrak{L} \subset L_\Gamma\{z\}$. The products between $\tilde{a}$'s are

$$\tilde{a} \boxed{0} \tilde{b} = \widetilde{[a,b]}, \quad \tilde{a} \boxed{1} \tilde{b} = (a|b)\mathsf{c}. \tag{12}$$

The affine conformal algebra $\mathfrak{L}$ is a homogeneous subalgebra of $L_\Gamma\{z\}$ so that for $a \in \mathfrak{g}^{[\lambda]}$ we have $\tilde{a} \in L_\Gamma\{z\}^{[\lambda]}$. It is independent on the Γ-grading of $\mathfrak{g}$. It is easy to see that $L_\Gamma = \operatorname{Coeff}_\Gamma \mathfrak{L}$.

For each equivalence class $[\lambda] \in \Gamma \subset \mathbf{k}/\mathbb{Z}$ choose a representative $\lambda \in \mathbf{k}$ such that the representative of $\mathbb{Z}$ is 0 and if $[\lambda] + [\mu] = 0$, then either $\lambda = \mu = 0$ or $\lambda + \mu = 1$. For example, if $\mathbf{k} \subseteq \mathbb{R}$, then we can take $0 \leqslant \lambda < 1$. A very important special case is when Γ is a finite cyclic group of order p, then one can take the set of representatives of Γ in $\mathbf{k}$ to be $\{0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}\}$.

For a homogeneous element $a \in \mathfrak{g}^{[\lambda]}$ consider the series $\tau_a = z^\lambda \tilde{a} \in L_\Gamma[[z^{\pm 1}]]$. Clearly, the series τ_a are pairwise local of order 2, so together with c they generate a conformal algebra $\mathfrak{L}_\Gamma \subset L_\Gamma[[z^{\pm 1}]]$, called a twisted affine conformal algebra. We calculate the non-zero products of these series, using (6) and (12). Here $a, b \in \mathfrak{g}$, $\deg a = [\lambda]$, $\deg b = [\mu]$:

$$\tau_a \boxed{0}\, \tau_b = z^{\lambda+\mu}\, \widetilde{[a,b]} + \lambda z^{\lambda+\mu-1}(a|b)\mathsf{c}$$

$$= \begin{cases} z\ \tau_{[a,b]} + \lambda(a|b)\mathsf{c} & \text{if } \lambda+\mu \geqslant 1, \\ \tau_{[a,b]} & \text{if } \lambda+\mu < 1, \end{cases}$$

$$\tau_a \boxed{1}\, \tau_b = z^{\lambda+\mu}\,(a|b)\mathsf{c} = \begin{cases} (a|b)\mathsf{c} & \text{if } \lambda = \mu = 0, \\ z(a|b)\mathsf{c} & \text{if } \lambda+\mu = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is not difficult to see that L_Γ is isomorphic to the subalgebra of Coeff $\mathfrak{L}_\Gamma$ consisting of all elements of the form $a(0)$, i.e., the embedding $\mathfrak{L}_\Gamma \to L_\Gamma[[z^{\pm 1}]]$ is the universal one among realizations of $\mathfrak{L}_\Gamma$ by formal series that agree with the multiplication by z, see Section 2.1.

Remark. While in general we cannot guarantee that the twisted representation map $\rho : \mathfrak{L} \to L_\Gamma\{w\}$ is injective, in the particular case when $\mathfrak{L}$ is an affine conformal algebra we do know that ρ is an isomorphism.

Remark. In principal, one can use the construction of Section 2.1 to get twisted representations of a vertex algebra. Namely, let $\mathfrak{A} = \bigoplus_{[\lambda] \in \Gamma} \mathfrak{A}^{[\lambda]}$ be a graded vertex algebra, and let $\pi : \mathfrak{A} \to \mathrm{vo}(V)$ be its representation. Let $\mathfrak{A}_\Gamma \subset \mathrm{vo}\{V\}$ be the vertex algebra generated by the generalized vertex operators $z^n \pi(a)$ for $a \in \mathfrak{A}^{[n]}$. Then $\mathfrak{A}_\Gamma$ will be closed under the multiplication by $z^{\pm 1}$. If we have a representation $\eta : \mathfrak{A}_\Gamma \to \mathrm{vo}(U)$ such that $\eta(z^k a) = w^k \eta(a)$ for $k \in \mathbb{Z}$, then the map $a \mapsto \eta(z^\lambda a) w^{-\lambda}$ for $a \in \mathfrak{A}^{[\lambda]}$ defines a twisted representation of $\mathfrak{A}$ on U. However, in contrast with the conformal case, it is not clear how to construct such a representation η of $\mathfrak{A}_\Gamma$. On the other hand, if $\mathfrak{A}$ is generated by a conformal algebra $\mathfrak{L} \subset \mathfrak{A}$, then applying the construction of Section 2.1 to $\mathfrak{L}$ we can get a twisted representation of possibly some other enveloping vertex algebra $\mathfrak{B} \supset \mathfrak{L}$.

3. Lattice vertex algebras

In this section apply the technique developed in Section 1 to lattice vertex algebras. We assume here that the ground field is $\mathbb{C}$ and the group Γ is the cyclic group of order $p \geqslant 0$. We identify $\Gamma \subset \mathbb{C}/\mathbb{Z}$ with the set $[0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p}]$.

We remark that most of the constructions below can be done in a much more general setting.

3.1. Representation theory of Heisenberg algebras

Let $\mathfrak{h}$ be a $\mathbf{k}$-vector space of dimension $l<\infty$ equipped with a non-degenerate bilinear form $(\cdot\,|\,\cdot)$. Assume that $\mathfrak{h}=\bigoplus_{\lambda\in\Gamma}\mathfrak{h}^{[\lambda]}$ is graded by Γ and that the grading agrees with the form in sense of (11). Recall that the grading induces an automorphism $\sigma:\mathfrak{h}\to\mathfrak{h}$ such that $\sigma|_{\mathfrak{h}^{[\lambda]}}=\exp(2\pi i\,\lambda)$. In fact the existence of an automorphism $\sigma:\mathfrak{h}\to\mathfrak{h}$ of order p is equivalent to the existence of the above grading of $\mathfrak{h}$ by the group $\Gamma=\mathbb{Z}/p\mathbb{Z}$. Note that σ preserves the norm on $\mathfrak{h}$. For $h\in\mathfrak{h}$ we denote by $h^{[\lambda]}$ the projection of h onto $\mathfrak{h}^{[\lambda]}$.

View $\mathfrak{h}$ as an Abelian Lie algebra, and let

$$H=H_\Gamma=\mathrm{Span}\{a(n)\,|\,a\in\mathfrak{h}^{[n]}\}\oplus\mathbb{C}\mathsf{c}$$

be the corresponding (twisted, unless $p=1$) affine Lie algebra, see Section 2.2. It is usually called a Heisenberg Lie algebra. As in Section 2.2, for an element $a\in\mathfrak{h}$ consider formal series $\tilde{a}=\sum_{n\in\Gamma+\mathbb{Z}}a(n)z^{-n-1}\in H\{z\}$. These series, together with c, span over $\mathbb{C}[D]$ a copy of conformal Heisenberg algebra $\mathfrak{H}\subset H\{z\}$, so that $H=\mathrm{Coeff}_\Gamma\,\mathfrak{H}$. The grading on $\mathfrak{h}$ lifts to a grading on H and $\mathfrak{H}$ and the automorphism σ lifts to automorphisms of H and $\mathfrak{H}$. Recall also that there is another grading on H given by setting $\deg a(n)=-n\in\frac{1}{p}\mathbb{Z}$.

We note that $\mathfrak{h}^{[0]}\subset Z(H)$ so that $H=H'\oplus\mathfrak{h}^{[0]}$, where

$$H'=\mathrm{Span}\{a(n)\,|\,a\in\mathfrak{h},n\in\mathbb{C}^\times\}\oplus\mathbb{C}\mathsf{c}.$$

Let $H_\pm=\mathrm{Span}\{a(n)\,|\,n\gtrless 0\}\subset H'$. We have $H'=H_-\oplus\mathbb{C}\mathsf{c}\oplus H_+$.

Now let M be a restricted H-module, i.e. such that for any $u\in M$ we have $h(n)u=0$ for $n\gg 0$. Assume that c acts on M by the identity. Then the vertex operators $\tilde{h}\in\mathrm{vo}\{M\}$ generate a vertex algebra $\mathfrak{V}_0\subset\mathrm{vo}\{M\}$. It is an *enveloping vertex algebra* of the conformal Heisenberg algebra $\mathfrak{H}$. The algebra $\mathfrak{V}_0$ is a module over the Heisenberg algebra H by $h(n)x=\tilde{h}\,\boxed{n}\,x$. It is well known that $\mathfrak{V}_0$ is the unique enveloping vertex algebra of $\mathfrak{H}$ such that $\mathsf{c}=\mathbb{1}$. As a module over H, the vertex algebra $\mathfrak{V}_0$ is isomorphic to the so-called canonical relations representation $M(1)=U(H_-)\mathbb{1}$, which is generated by a single element $\mathbb{1}$ such that $H_+\mathbb{1}=0$.

The vertex algebra $\mathfrak{V}_0$ is in fact a vertex operator algebra: it is graded so that $\deg\tilde{h}=1$ for $h\in\mathfrak{h}$ and it contains a Virasoro element $\upsilon=\frac{1}{2}\sum_{i=1}^{\ell}\tilde{\alpha}_i\,\boxed{-1}\,\tilde{\beta}_i$, where $(\alpha_1,\ldots,\alpha_\ell)$ and $(\beta_1,\ldots,\beta_\ell)$ are dual bases of $\mathfrak{h}$, i.e. such that $(\alpha_i|\beta_j)=\delta_{ij}$. We have $\upsilon\,\boxed{0}\,u=Du$ for all $v\in\mathfrak{V}_0$, $\upsilon\,\boxed{1}\,u=(\deg u)u$ for all homogeneous $u\in\mathfrak{V}_0$, $\upsilon\,\boxed{2}\,\upsilon=0$ and $\upsilon\,\boxed{3}\,\upsilon=\frac{1}{2}\mathbb{1}$.

An H-module V is called $\mathfrak{h}^{[0]}$-diagonalizable if it can be decomposed into a direct sum of subspaces $V=\bigoplus_{\xi\in(\mathfrak{h}^{[0]})^*}V_\xi$, so that for $h\in\mathfrak{h}^{[0]}$ one has $h|_{V_\xi}=\xi(h)$. Recall [14] that an H-module V belongs to the category $\mathcal{O}$ if V is $\mathfrak{h}^{[0]}$-diagonalizable and for any $v\in V$ there is $n\in\frac{1}{p}\mathbb{Z}$ such that for any $x\in U(H)$ of $\deg x\geqslant n$ we have $xv=0$. Clearly, any module from the category $\mathcal{O}$ is restricted.

For the future reference, we cite here a result from the representation theory of Heisenberg algebras.

Lemma 1. *Let V be a module over the Heisenberg Lie algebra H. Let $\Omega=\{v\in V\mid H_+v=0\}\subset V$ be the vacuum subspace of V. Then the following conditions are equivalent*:

(i) $V\cong M(1)\otimes\Omega$ *and* Ω *is* $\mathfrak{h}^{[0]}$*-diagonalizable*;
(ii) $V=U(H)\Omega$ *and* Ω *is* $\mathfrak{h}^{[0]}$*-diagonalizable*;
(iii) $V\in\mathcal{O}$ *and* V *is completely reducible over* H;
(iv) $V\in\mathcal{O}$ *and there is a grading* $V=\bigoplus_{n\in 1/p\,\mathbb{Z}}V_n$ *on* V *such that* $\deg a(n)=-n$ *and* $\upsilon(1)|_{V_n}=n$, *where*

$$\upsilon(1)=\frac{1}{2}\sum_{i=1}^{l}\left(\sum_{s<0}\alpha_i(s)\beta_i(-s)+\sum_{s\geqslant 0}\beta_i(-s)\alpha_i(s)\right)\in\overline{U(H)}$$

is the first coefficient of the Virasoro element $\upsilon\in\mathfrak{V}_0$.

Clearly, the vacuum space $\Omega\subset V$ is stable under the action of $\mathfrak{h}^{[0]}$. Condition (iv) means that V is a module over the vertex operator algebra $\mathfrak{V}_0$, see Section 1.7. We remark that if in (iv) we assume that the grading on V is bounded from below, that is, there is $n_0\in\frac{1}{p}\mathbb{Z}$ such that $V_n=0$ for $n<n_0$, then the condition $V\in\mathcal{O}$ becomes obsolete.

3.2. Fock spaces

Let $\Lambda\subset\mathfrak{h}$ be a lattice in $\mathfrak{h}$ of rank l. Assume that Λ agrees with the Γ-grading on $\mathfrak{h}$ in the sense that $\operatorname{rk}\Lambda^{[\lambda]}=\dim\mathfrak{h}^{[\lambda]}$ for $[\lambda]\in\Gamma$. Consider a central extension

$$1\to\Phi\to\hat{\Lambda}\to\Lambda\to 1$$

of Λ by the multiplicative group $\Phi\cong\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$. Let $e:\Lambda\to\hat{\Lambda}$ be a section $\varepsilon:\Lambda\times\Lambda\to Z$ be the corresponding 2-cocycle, so that $e(\alpha)e(\beta)=\varepsilon(\alpha,\beta)e(\alpha+\beta)$ for $\alpha,\beta\in\Lambda$. Denote by R the quotient of the group algebra $\mathbb{C}[\hat{\Lambda}]$ obtained by the identification of Φ with $\mathbb{C}^\times\subset\mathbb{C}$.

The key object of the construction is a vector space V, which is a module over the associative algebra R and a restricted module over the twisted Heisenberg algebra H such that

$$[h(n),e(\alpha)]=\delta_{n,0}(h|\alpha)e(\alpha)\quad\text{for } h\in\mathfrak{h},\ \ n\in\Gamma+\mathbb{Z},\tag{13}$$

and $\mathsf{c} = \mathrm{id}$. Formula (13) mean that H acts on R by derivations, so we can form a skew tensor product algebra $U(H)\tilde{\otimes}R$ such that V is a $U(H)\tilde{\otimes}R$-module.

We assume that $V = \bigoplus_{\xi\in(\mathfrak{h}^{[0]})^*} V_\xi$ is $\mathfrak{h}^{[0]}$-diagonalizable. The bilinear form on Λ induces a homomorphism $v : \Lambda \to (\mathfrak{h}^{[0]})^*$ by $v(\alpha)\beta = (\alpha|\beta)$ for $\alpha\in\Lambda$, $\beta\in\Lambda^{[0]}$. The relation $[\beta(0), e(\alpha)] = (\alpha|\beta)e(\alpha)$ is equivalent to the fact that $e(\alpha)V_\xi \subseteq V_{\xi+v(\alpha)}$.

Such a module V is sometimes referred to as a *Fock space*. In Section 3.4 we will make V to be a module over certain vertex algebra, under some additional assumptions.

A standard way to construct a Fock space V is as follows. Let $\Omega = \bigoplus_{\xi\in(\mathfrak{h}^{[0]})^*} \Omega_\xi$ be a $\mathfrak{h}^{[0]}$-diagonalizable module over R and $\mathfrak{h}^{[0]}$ such that $[h, e(\alpha)] = (h|\alpha)e(\alpha)$ for $h\in\mathfrak{h}^{[0]}$. In other words, Ω is a module over a certain skew tensor product algebra $U(\mathfrak{h}^{[0]})\tilde{\otimes}R$. As before, we have $e(\alpha)\Omega_\xi \subseteq \Omega_{\xi+v(\alpha)}$. Take now a restricted H-module M such that $\mathfrak{h}^{[0]}$ acts on M by 0. Let $V = M\otimes\Omega$. Then $V = \bigoplus_{\xi\in(\mathfrak{h}^{[0]})^*} V_\xi$, $V_\xi = M\otimes\Omega_\xi$, is also a restricted $\mathfrak{h}^{[0]}$-diagonalizable H-module and also it is a module over R such that relation (13) hold.

For example, assume that V satisfies the conditions of Lemma 1 as a module over H. Then $V = M(1)\otimes\Omega$ can be obtained by the above construction for $M = M(1)$. The vacuum space $\Omega \simeq \mathbb{1}\otimes\Omega \subset V$ becomes a module over $U(\mathfrak{h}^{[0]})\tilde{\otimes}R$.

3.3. Twisted vertex operators

Now we are ready to define the main ingredient of this construction—the vertex operator $X_\alpha \in \mathrm{vo}\{V\}$ for $\alpha\in\Lambda$. Let $h' = h - h^{[0]}$ for $h\in\mathfrak{h}$. We set

$$X_\alpha(z) = e(\alpha)E_-(\alpha, z)E_+(\alpha, z)z^{\alpha(0)}z^{-(\alpha'|\alpha')/2}, \tag{14}$$

where

$$E_\pm(\alpha, z) = \exp \sum_{n\in\Gamma+\mathbb{Z}, n\gtrless 0} -\frac{\alpha(n)}{n} z^{-n}.$$

Note that $\alpha(0) = \alpha^{[0]}$ is the projection of α onto $\mathfrak{h}^{[0]}$, so that $z^{\alpha(0)}|_{V_\xi} = z^{\xi(\alpha^{[0]})}$.

Remark. In fact, we need that V is $\mathfrak{h}^{[0]}$-diagonalizable only for the expression $z^{\alpha(0)}$ to make sense. We can instead interpret $z^{\alpha(0)} = \exp(\alpha(0)\log z)$, and then this expression is well defined under somewhat weaker assumptions, for example it is enough to require that $\alpha(0)$ acts locally finite dimensionally.

Proposition 3 (Lepowsky [19]). *Let $h\in\mathfrak{h}$, $n\in\Gamma+\mathbb{Z}$, $\alpha,\beta\in\Lambda$. We have*

(a) $[h(n), X_\alpha(z)] = (\alpha|h)z^n X_\alpha(z)$;
(b) $\tilde{h}\,\boxed{n}\,X_\alpha = 0$ *if* $1\leqslant n\in\mathbb{Z}_+$ *and* $\tilde{h}\,\boxed{0}\,X_\alpha = (\alpha|h)X_\alpha$;

(c) $DX_\alpha = \tilde{\alpha}\,\boxed{-1}\,X_\alpha$;

(d) $\upsilon\,\boxed{0}\,X_\alpha = DX_\alpha$, $\upsilon\,\boxed{1}\,X_\alpha = \frac{1}{2}(\alpha|\alpha)X_\alpha$;

(e) $X_\alpha(w)X_\beta(z) = \varepsilon(\alpha,\beta)\,X_{\alpha,\beta}(w,z)\iota_{w,z}\prod_{s=0}^{p-1}(w^{1/p}-\omega^s z^{1/p})^{(\sigma^{-s}\alpha|\beta)}$, *where* $\omega = \exp\frac{2\pi i}{p}$ *is the primitive pth root of unity and*

$$X_{\alpha,\beta}(w,z) = e(\alpha+\beta)E_-(\alpha,w)E_-(\beta,z)E_+(\alpha,w)E_+(\beta,z)$$

$$\times\, w^{\alpha(0)}z^{\beta(0)}w^{-(\alpha'|\alpha')/2}z^{-(\beta'|\beta')/2}.$$

We have $X_{\alpha,\beta}(w,z) = X_{\beta,\alpha}(z,w)$ and $X_{\alpha,\beta}(z,z) = X_{\alpha+\beta}(z)z^{(\alpha'|\beta')}$. The notation $\iota_{w,z}$ in (e) is a short for $\iota_{w^{1/p},z^{1/p}}$, see (2).

Proof. (a) Let $0\neq n\in\Gamma+\mathbb{Z}$. Then

$$\left[h(n),\exp\left(-\frac{\alpha(m)}{m}z^{-m}\right)\right] = \delta_{n,-m}\exp\left(-\frac{\alpha(m)}{m}z^{-m}\right)(\alpha|h)z^n,$$

and $h(n)$ commutes with all the rest of the factors in (14). Also, $h(0)$ commutes with all the factors in (14) except $e(\alpha)$, whose commutators are given by (13), so (a) follows.

(b) It follows that $[h(n),X_\alpha(m)] = X_\alpha(m+n)$ for every $m,n\in\Gamma+\mathbb{Z}$. Hence we have for $n\in\mathbb{Z}_+$

$$(\tilde{h}\,\boxed{n}\,X_\alpha)(m) = \sum_s(-1)^s\binom{n}{s}[h(n-s),X_\alpha(m+s)]$$

$$= (\alpha|h)\sum_s(-1)^s\binom{n}{s}X_\alpha(m+n) = \begin{cases}0 & \text{if } n>0,\\ (\alpha|h)X_\alpha(m) & \text{if } n=0.\end{cases}$$

(c) Using that $(\alpha'|\alpha')/2 = \sum_{\lambda\in\Gamma}\lambda(\alpha^{[\lambda]}|\alpha)$, we get

$$DX_\alpha(z) = \sum_{n<0}\alpha(n)z^{-n-1}X_\alpha(z) + X_\alpha(z)\sum_{n>0}\alpha(n)z^{-n-1}$$

$$+\; X_\alpha(z)\alpha(0)z^{-1} - \sum_{\lambda\in\Gamma}\lambda(\alpha^{[\lambda]}|\alpha)z^{-1}X_\alpha(z)$$

$$= :\tilde{\alpha}X_\alpha: - \sum_{\lambda\in\Gamma}\lambda(\alpha^{[\lambda]}|\alpha)z^{-1}X_\alpha(z).$$

On the other hand, set $\tilde{\alpha}(z) = \sum_\lambda z^{-\lambda}\tilde{\alpha}^{[\lambda]}(z)$. The locality of $\tilde{\alpha}$ and X_α is 1, hence, using (6),

$$\begin{aligned}\tilde{\alpha}\boxed{-1}X_\alpha &= \sum_\lambda (z^{-\lambda}\tilde{\alpha}^{[\lambda]})\boxed{-1}X_\alpha\\ &= \sum_\lambda (z^{-\lambda}(\tilde{\alpha}^{[\lambda]}\boxed{-1}X_\alpha) - \lambda z^{-\lambda-1}(\tilde{\alpha}^{[\lambda]}\boxed{0}X_\alpha))\\ &= :\tilde{\alpha}X_\alpha: - \sum_{\lambda\in\Gamma}\lambda(\alpha^{[\lambda]}|\alpha)z^{-1}X_\alpha(z).\end{aligned}$$

(d) Let us calculate $\upsilon\boxed{0}X_\alpha$ by the associativity formula (V3). We have, using (b) and (c):

$$\begin{aligned}\upsilon\boxed{0}X_\alpha &= \frac{1}{2}\sum_i\left(\sum_{s<0}\alpha_i(s)\beta_i(-s-1) + \sum_{s\geqslant 0}\beta_i(-s-1)\alpha_i(s)\right)X_\alpha\\ &= \frac{1}{2}\sum_i(\alpha_i(-1)(\beta_i|\alpha) + \beta_i(-1)(\alpha_i|\alpha))X_\alpha\\ &= \alpha(-1)X_\alpha = DX_\alpha.\end{aligned}$$

The other relation is proved in the same way.

(e) Let us first calculate $S = E_+(\alpha, w)E_-(\beta, z)E_+^{-1}(\alpha, w)$. Since

$$\begin{aligned}&\exp\left(-\frac{\alpha(n)}{n}w^{-n}\right)E_-(\beta, z)\exp\left(\frac{\alpha(n)}{n}w^{-n}\right)\\ &\quad= \exp\left(\frac{w^{-n}}{n}\operatorname{ad}\alpha(n)\right)E_-(\beta, z)\\ &\quad= (\alpha^{[n]}|\beta)\frac{z^n w^{-n}}{n}E_-(\beta, z),\end{aligned}$$

we have

$$\begin{aligned}S &= \exp\sum_{0<n\in\Gamma+\mathbb{Z}} -(\alpha^{[n]}|\beta)\frac{z^n w^{-n}}{n}\\ &= \exp\sum_{\lambda\in\{\frac{1}{p},\dots,\frac{p-1}{p},1\}} -(\alpha^{[\lambda]}|\beta)\sum_{n\in\lambda+\mathbb{Z}_+}\frac{z^n w^{-n}}{n}.\end{aligned}$$

Denote $y = (z/w)^{1/p}$ and let $\lambda = \frac{q}{p},\ 1\leqslant q\leqslant p$. Then

$$\sum_{n\in\lambda+\mathbb{Z}_+}\frac{z^n w^{-n}}{n} = \int\frac{(z/w)^{\lambda-1}}{1-(z/w)}d(z/w) = p\int\frac{y^{q-1}}{1-y^p}dy.$$

Using the elementary fraction decomposition

$$\frac{y^r}{1-y^p} = \frac{1}{p}\sum_{s=0}^{p-1} \frac{\omega^{-sr}}{1-\omega^s y},$$

we get that

$$\sum_{n\in\lambda+\mathbb{Z}_+} \frac{z^n w^{-n}}{n} = -\sum_{s=0}^{p-1} \ln(1-\omega^s y)^{\omega^{-sq}},$$

so that

$$\begin{aligned} S &= \prod_{s,q=0}^{p-1} (1-\omega^s y)^{\omega^{-sq}(\alpha^{(q/p)}|\beta)} = \prod_{s=0}^{p-1} (1-\omega^s y)^{\sum_q \omega^{-sq}(\alpha^{(q/p)}|\beta)} \\ &= \prod_{s=0}^{p-1} (1-\omega^s y)^{(\sigma^{-s}\alpha|\beta)}. \end{aligned}$$

The rest of non-trivial commutation relations between the factors in (14) are

$$e(\alpha)\varepsilon(\beta) = \varepsilon(\alpha,\beta)e(\alpha+\beta), \quad w^{\alpha(0)}e(\beta) = e(\beta)w^{\alpha(0)}w^{(\alpha^{[0]}|\beta)},$$

so we finally get, using that $\alpha^{[0]} = \frac{1}{p}\sum_{s=0}^{p-1}\sigma^s\alpha$,

$$\begin{aligned} X_\alpha(w)X_\beta(z) &= \varepsilon(\alpha,\beta)X_{\alpha,\beta}(w,z)w^{(\alpha^{[0]}|\beta)}S \\ &= \varepsilon(\alpha,\beta)X_{\alpha,\beta}(w,z)\iota_{w,z}\prod_{s=0}^{p-1}(w^{1/p}-\omega^s z^{1/p})^{(\sigma^{-s}\alpha|\beta)}. \qquad \square \end{aligned}$$

Statement (b) implies that X_α generates a highest weight module over the Heisenberg Lie algebra H, where the action is given as usual by $h(n)u = \tilde{h}\boxed{n}\,u$. This module is irreducible and is, in fact, a module over the Heisenberg vertex algebra $\mathfrak{B}_0 = U(H)\mathbb{1}$.

The Virasoro element $\upsilon \in \mathfrak{B}_0$ gives the operator

$$D = \upsilon(0) = \sum_i \left(\sum_{s<0} \alpha_i(s)\beta_i(-s-1) + \sum_{s\geqslant 0} \beta_i(-s-1)\alpha_i(s) \right). \tag{15}$$

By (d) the vertex operators X_α are of weight 0 with respect to this D, see Section 1.6.

3.4. Lattice vertex algebras and their generalized representations

In this section, we construct the vertex superalgebra $\mathfrak{V}_\Lambda$, known as the lattice vertex superalgebra.

We start with the remark that only minimal modification is needed to transfer all the definitions and results of Sections 1 and 2 to the realm of superalgebras. All commutators must be interpreted as supercommutators, formula (3) must change to

$$(\alpha \boxed{n} \beta)(z) = \mathrm{Res}_w(\alpha(w)\beta(z)\iota_{w,z}(w-z)^n - (-1)^{p(\alpha)p(\beta)}\beta(z)\alpha(w)\iota_{z,w}(w-z)^n),$$

etc. Notably, in the definition of vertex operator superalgebra the eigenvalues of the grading derivation $\upsilon(1)$ are allowed to be half-integer, and the vertex algebra $\mathfrak{A} = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} \mathfrak{A}_n$ is graded by $\frac{1}{2}\mathbb{Z}$ so that the even and odd parts of $\mathfrak{A}$ are, respectively, $\mathfrak{A}^{\bar{0}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{A}_n$, $\mathfrak{A}^{\bar{1}} = \bigoplus_{n\in\mathbb{Z}+1/2} \mathfrak{A}_n$. A reader who is averse to supermathematics can assume that the lattice Λ is even, i.e. $(\alpha|\alpha)\in 2\mathbb{Z}$ for all $\alpha\in\Lambda$, see (i) below.

Here, we make the following assumptions on the data introduced in Section 3.3.

(i) For any $\alpha,\beta\in\Lambda$ the numbers $m_s = (\sigma^{-s}\alpha|\beta)$, $0\leqslant s\leqslant p-1$, are integer, in other words $\Lambda\subset\mathfrak{h}$ is an integer lattice and the automorphism $\sigma:\mathfrak{h}\to\mathfrak{h}$ preserves the dual lattice $\Lambda' = \{\alpha\in\mathfrak{h} \mid (\alpha|\Lambda)\subset\mathbb{Z}\}\supseteq\Lambda$.

(ii) The cocycle $\varepsilon:\Lambda\times\Lambda\to\mathbb{C}^\times$ is such that the corresponding commutator map is

$$C(\alpha,\beta) = \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1} = (-1)^{(\alpha|\alpha)(\beta|\beta)+p(\alpha^{[0]}|\beta^{[0]})}\omega^{-\sum_{s=1}^{p-1} sm_s}. \tag{16}$$

Note that $p(\alpha^{[0]}|\beta^{[0]}) = \sum_{s=0}^{p-1} m_s\in\mathbb{Z}$.

The cocycle ε satisfying (16) can be easily constructed in the following way. Let $\alpha_1,\dots,\alpha_l$ be a $\mathbb{Z}$-basis of Λ. Define ε first for $\alpha,\beta\in\{\alpha_1,\dots,\alpha_l\}$ such that (16) holds. This is possible since $C(\alpha,\beta) = C(\beta,\alpha)^{-1}$ and $C(\alpha,\alpha)=1$. Then, since $C:\Lambda\times\Lambda\to\mathbb{C}^\times$ is bimultiplicative, the identity (16) will continue to hold for the bimultiplicative extension of ε to the whole Λ. Note that for $p=1$ or 2 we have $C(\alpha,\beta) = (-1)^{(\alpha|\alpha)(\beta|\beta)}(-1)^{(\alpha|\beta)}$.

Theorem 2. *Under assumptions* (i) *and* (ii) *above, the vertex operators $X_\alpha, X_\beta\in\mathrm{vo}\{V\}$ are local of order*

$$N(\alpha,\beta) = \max\{-m_s \mid m_s<0, 0\leqslant s\leqslant p-1\}\cup\{0\}.$$

They generate the lattice vertex superalgebra $\mathfrak{V} = \mathfrak{V}_\Lambda\subset\mathrm{vo}\{V\}$, which does not depend on the Γ-grading of $\mathfrak{h}$. The products of the generators are given by

$$X_\alpha \boxed{-(\alpha|\beta)-n-1} X_\beta = \varkappa(\alpha,\beta)\frac{1}{n!}(D-\beta(-1))^{(n)}X_{\alpha+\beta} \tag{17}$$

for $n \geqslant 0$, *and* $X_\alpha \boxed{n} X_\beta = 0$ *if* $n \geqslant -(\alpha|\beta)$. *Here*

$$\varkappa(\alpha,\beta) = \varepsilon(\alpha,\beta) p^{-(\alpha|\beta)} \prod_{s=1}^{p-1} (1-\omega^s)^{m_s}. \tag{18}$$

In particular,

$$X_\alpha \boxed{-(\alpha|\beta)-1} X_\beta = \varkappa(\alpha,\beta) X_{\alpha+\beta}, \quad X_\alpha \boxed{-(\alpha|\alpha)-2} X_{-\alpha} = \varkappa(\alpha,\alpha)^{-1}\tilde{\alpha}.$$

In the case when $p=1$ this is just the usual construction of lattice vertex algebras, see e.g. [12,15].

The lattice vertex algebra $\mathfrak{V} = \oplus_{\alpha\in\lambda} \mathfrak{V}_\alpha$ is graded by the lattice Λ. The subspace $\mathfrak{V}_0$ is the Heisenberg vertex subalgebra of $\mathfrak{V}$, and the rest of $\mathfrak{V}_\alpha$'s are irreducible modules over $\mathfrak{V}_0$.

The even and odd parts of $\mathfrak{V}$ are

$$\mathfrak{V}^{\bar{0}} = \bigoplus_{\substack{\alpha\in\Lambda:\\ (\alpha|\alpha)\in 2\mathbb{Z}}} \mathfrak{V}_\alpha, \quad \mathfrak{V}^{\bar{1}} = \bigoplus_{\substack{\alpha\in\Lambda:\\ (\alpha|\alpha)\in 2\mathbb{Z}+1}} \mathfrak{V}_\alpha.$$

We also note that $\mathfrak{V}$ is a simple vertex algebra.

Let us calculate the commutator map of the cocycle $\varkappa : \Lambda\times\Lambda \to \mathbb{C}^\times$ given by (18):

$$\begin{aligned}\varkappa(\alpha,\beta)\varkappa(\beta,\alpha)^{-1} &= C(\alpha,\beta)\prod_{s=1}^{p-1}(1-\omega^s)^{m_s - m_{p-s}}\\ &= (-1)^{(\alpha|\alpha)(\beta|\beta)+(\alpha|\beta)}\prod_{s=1}^{p-1}(-\omega^{-s})^{m_s}\prod_{s=1}^{p-1}\left(\frac{1-\omega^s}{1-\omega^{-s}}\right)^{m_s}\\ &= (-1)^{(\alpha|\alpha)(\beta|\beta)}(-1)^{(\alpha|\beta)}.\end{aligned}$$

It follows that for different automorphisms σ all resulting cocycles $\varkappa$ are cohomological and define certain class $\varkappa \in H^2(\Lambda, \mathbb{C}^\times)$. Therefore, the vertex algebra $\mathfrak{V}_\Lambda \subset \mathrm{vo}\{V\}$ generated by X_α's is indeed independent on σ.

Remark. If $(\alpha|\beta)$ is not an integer, then the vertex operators X_α and X_β are not local. However, they are local in a generalized sense, and they generate a generalized vertex algebra [6], mentioned at the end of Section 1.7. One can define products $X_\alpha \boxed{n} X_\beta$ for $n \equiv -(\alpha|\beta) \bmod \mathbb{Z}$.

3.5. Proof of Theorem 2

The assertion about locality follows from Proposition 3(e) and the formula

$$(w-z)=\prod_{s=0}^{p-1}(w^{1/p}-\omega^s z^{1/p}).$$

Denote by $\mathscr{P}(m)=\{\boldsymbol{r}=(r_1,r_2,\dots)\,|\,r_i\geqslant 0,\sum_{i\geqslant 1} ir_i=m\}$ the set of partitions of $m\in\mathbb{Z}$. Some standard combinatorial argument shows that (17) can be rewritten as

$$X_\alpha \boxed{n} X_\beta=\varkappa(\alpha,\beta)\sum_{\boldsymbol{r}\in\mathscr{P}(-(\alpha|\beta)-n-1)}\prod_{j\geqslant 1}\left(\frac{\alpha(-j)}{j!}\right)^{r_j}X_{\alpha+\beta}, \tag{19}$$

where $n<-(\alpha|\beta)$.

Consider the operator $\delta=\iota_{w,z}-\iota_{z,w}:\mathbb{C}(w,z)\to\mathbb{C}[[w^{\pm 1},z^{\pm 1}]]$, see (2). It is easy to see that $\delta(g)=0$ if and only if $g\in\mathbb{C}[w^{\pm 1},z^{\pm 1}]$. In this case g commutes with δ: $\delta(gf)=g\delta(f)$. We will also make use of the following formula. For any formal power series $f(w,z)$ in the variables w,z one has

$$\operatorname{Res}_w(f(w,z)\delta(w-z)^{-k-1})=\frac{1}{k!}\frac{\partial^k}{\partial w^k}\bigg|_{w=z}f(w,z)$$

whenever both sides make sense.

Let $n=-(\alpha|\beta)-k-1$. We calculate the product $X_\alpha \boxed{n} X_\beta$ using (8) and Proposition 3(e):

$$\begin{aligned}(X_\alpha \boxed{n} X_\beta)(z)&=\operatorname{Res}_w\Bigg((X_\alpha(w)X_\beta(z)\iota_{w,z}(w-z)^n\\&\quad-(-1)^{(\alpha|\alpha)(\beta|\beta)}X_\beta(z)X_\alpha(w)\iota_{z,w}(w-z)^n)\\&\quad\times F(N+m+k)\prod_{s=1}^{p-1}(w^{1/p}-\omega^s z^{1/p})^{N-n}\Bigg)\\&=\varepsilon(\alpha,\beta)\operatorname{Res}_w\Bigg(X_{\alpha,\beta}(w,z)F(N+m+k)\\&\quad\times\prod_{s=1}^{p-1}(w^{1/p}-\omega^s z^{1/p})^{N-n}\delta\prod_{s=0}^{p-1}(w^{1/p}-\omega^s z^{1/p})^{m_s+n}\Bigg)\\&=\varepsilon(\alpha,\beta)\operatorname{Res}_{w^{1/p}}\Bigg(X_{\alpha,\beta}(w,z)w^{p-1/p}F(N+m+k)\\&\quad\times\prod_{s=1}^{p-1}(w^{1/p}-\omega^s z^{1/p})^{m_s+N}\delta(w^{1/p}-z^{1/p})^{-k-1}\Bigg)\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon(\alpha,\beta)\frac{1}{k!}\frac{\partial^k}{(\partial w^{1/p})^k}\bigg|_{w^{1/p}=z^{1/p}} \\
&\quad\times \left(X_{\alpha,\beta}(w,z) w^{(p-1)/p} F(N+m+k) \prod_{s=1}^{p-1} (w^{1/p}-\omega^s z^{1/p})^{m_s+N} \right). \qquad (20)
\end{aligned}
$$

We use here that $\mathrm{Res}_w = \mathrm{Res}_{w^{1/p}}\, w^{(p-1)/p}$. Note that $N+m_s \geqslant 0$ for all $0 \leqslant s \leqslant p-1$.

Set

$$
B = w^{(p-1)/p} F(N+m+k) \prod_{s=1}^{p-1} (w^{1/p} - \omega^s z^{1/p})^{m_s+N}.
$$

Recall that there is an operator $D: V \to V$ given by (15), such that the weights of all vertex operators X_α are 0, see Section 1.6. Using (9) and the formula $\prod_{s=1}^{p-1}(1-\omega^s) = p$, we calculate

$$
B|_{w^{1/p}=z^{1/p}} = p^{-m-k} \prod_{s=1}^{p-1} (1-\omega^s)^{m_s} z^{-(\alpha'|\beta')} z^{k\,(1-p)/p},
$$

hence $\mathrm{wt}\, B|_{w^{1/p}=z^{1/p}} = -(\alpha'|\beta') - k\frac{p-1}{p}$. Since B is a Laurent polynomial in $w^{1/p}$ and $z^{1/p}$, we get that

$$
\mathrm{wt}\, \frac{\partial^i}{(\partial w^{1/p})^i}\bigg|_{w^{1/p}=z^{1/p}} B = -(\alpha'|\beta') - k\frac{p-1}{p} - \frac{i}{p}.
$$

Let us now calculate the derivative of $X_{\alpha,\beta}$, as in the proof of Proposition 3(c):

$$
\begin{aligned}
\frac{\partial}{\partial w^{1/p}} X_{\alpha,\beta}(w,z) &= p w^{(p-1)/p} \frac{\partial}{\partial w} F_{\alpha,\beta}(w,z) \\
&= p w^{(p-1)/p} \left(:\tilde{\alpha} X_{\alpha,\beta}: - \frac{1}{2}(\alpha'|\alpha') w^{-1} X_{\alpha,\beta} \right) \\
&= p w^{(p-1)/p} \left(\tilde{\alpha}\boxed{-1} X_{\alpha,\beta} + \frac{1}{2}(\alpha'|\beta') w^{-1} X_{\alpha,\beta} \right).
\end{aligned}
$$

Iterating this formula and using the fact that $X_{\alpha,\beta}(z,z) = X_{\alpha+\beta}(z) z^{(\alpha'|\beta')}$, we get

$$
\begin{aligned}
&\frac{1}{i!}\frac{\partial^i}{(\partial w^{1/p})^i}\bigg|_{w^{1/p}=z^{1/p}} X_{\alpha,\beta} \\
&\quad = p^i z^{(\alpha'|\beta')} z^{i\,[(p-1)/p]} \sum_{(r_1,r_2,\dots)\in\mathscr{P}(i)} \prod_{j\geqslant 1} \left(\frac{\alpha(-j)}{j!}\right)^{r_j} X_{\alpha+\beta} \\
&\qquad + \text{a sum of terms of the weight less than } i\tfrac{p-1}{p} + (\alpha'|\beta'). \qquad (21)
\end{aligned}
$$

Now expand (20) using the Leibniz rule and (21), and note that the only term of weight 0 in this expansion is the biggest weight term in

$$\varepsilon(\alpha,\beta)\left(\frac{1}{i!}\frac{\partial^i}{(\partial w^{\frac{1}{p}})^i}\bigg|_{w^{1/p}=z^{1/p}} X_{\alpha,\beta}\right)B,$$

which is precisely the right-hand side of formula (19). This finishes the proof of Theorem 2. $\square$

Remark. One could have used the theorem of Li and Xu [24] on characterization of lattice vertex algebras to show that the vertex algebra generated by the vertex operators $X_\alpha \in \mathrm{vo}\{V\}$ is isomorphic to the lattice vertex algebra. They consider only the case of an even lattice, but one can easily generalize their result to the case of a non-trivial odd part. It is easy then to deduce the product formula (19) but this proof would not give the explicit formula (18) for the cocycle $\varkappa$.

Dong and Lepowsky [7] have constructed all twisted modules of $\mathfrak{V}_\Lambda$ for the case when Λ is even, see Section 1.7 for the definition. They used, however, completely different techniques, in particular, they didn't get the cocycle $\varkappa$ explicitly. Some special cases were also considered in [5,9]. We will study twisted modules of $\mathfrak{V}_\Lambda$ in Section 3.7.

Corollary 1. *Let $U \subset V$ be a $U(H)\,\tilde{\otimes}\,R$-submodule of V. Then U is a submodule over the vertex algebra $\mathfrak{V}_\Lambda$.*

3.6. Do we get all generalized modules of $\mathfrak{V}_\Lambda$?

In this section, we show that in fact the construction of Theorem 2 exhausts all reasonable modules of $\mathfrak{V}_\Lambda$. A similar argument was also used in [4,5,21,24].

Let V be a generalized module over the lattice vertex algebra $\mathfrak{V}_\Lambda$ which is twisted as a module over the Heisenberg algebra $\mathfrak{V}_0$. This means that for any homogeneous $h \in \mathfrak{h}^{[\lambda]}$ the corresponding vertex operator $\tilde{h} \in \mathrm{vo}\{V\}^{[\lambda]}$ is also homogeneous. Then the V is a module over the twisted Heisenberg algebra H, such that c acts as the identity, see Section 3.1.

Assume that $V = \oplus_{\xi \in (\mathfrak{h}^{[0]})^*} V_\xi$ is $\mathfrak{h}^{[0]}$-diagonalizable, though as in the remark preceding Proposition 3, we note that this assumption can be relaxed. We will show that $\mathfrak{V}_\Lambda$-module V can be obtained by the construction of Section 3.4.

For $\alpha \in \Lambda$ define

$$e(\alpha) = E_-(\alpha, z)^{-1} X_\alpha(z) E_+(\alpha, z)^{-1} z^{-\alpha(0)} z^{(\alpha'|\alpha')/2}.$$

Using the same calculations as in the proof of (a) and (c) of Proposition 3 we see that $\frac{d}{dz}e(\alpha) = 0$ and $[h(n), e(\alpha)] = \delta_{n,0}(h|\alpha)e(\alpha)$, so that relations (13) hold. Now we want to show that $e(\alpha): V \to V$ generate an action of the central extension $\hat{\Lambda}$ of the lattice Λ corresponding to a cocycle $\varepsilon: \Lambda \times \Lambda \to \mathbb{C}^\times$ satisfying (16). It will follow that V is a module over the skew tensor product $U(H) \tilde{\otimes} R$ introduced in Section 3.2.

Modifying slightly the proof of Proposition 3(e) we get that

$$X_\alpha(w)X_\beta(z) = e(\alpha)e(\beta)X'_{\alpha,\beta}(w,z) \prod_{s=0}^{p-1} (w^{1/p} - \omega^s z^{1/p})^{(\sigma^{-s}\alpha|\beta)},$$

where

$$X'_{\alpha,\beta}(w,z) = E_-(\alpha,w)E_-(\beta,z)E_+(\alpha,w)E_+(\beta,z)w^{\alpha(0)}z^{\beta(0)}w^{-(\alpha'|\alpha')/2}z^{-(\beta'|\beta')/2}$$

is symmetric, $X'_{\alpha,\beta}(w,z) = X'_{\beta,\alpha}(z,w)$, and also $e(\alpha+\beta)X'_{\alpha,\beta}(z,z) = X_{\alpha+\beta}(z)z^{(\alpha'|\beta')}$.

Take N sufficiently large. Then, since X_α and X_β must be local, we get

$$\begin{aligned} 0 &= (X_\alpha(w)X_\beta(z) - (-1)^{(\alpha|\alpha)(\beta|\beta)}X_\beta(z)X_\alpha(w))(w-z)^N \\ &= (e(\alpha)e(\beta) - C(\alpha,\beta)e(\beta)e(\alpha))X'_{\alpha,\beta}(w,z)\prod_{s=0}^{p-1} (w^{1/p} - \omega^s z^{1/p})^{n_s} \end{aligned}$$

for some $n_s \in \mathbb{Z}_+$ and $C(\alpha,\beta)$ as in (16). But this can only happen if

$$e(\alpha)e(\beta) = C(\alpha,\beta)e(\beta)e(\alpha),$$

because otherwise we get $X'_{\alpha,\beta}(w,z)\,(w-z)^n = 0$ for $n = \max_s n_s$, and this is impossible since $X'_{\alpha,\beta}$ is regular at $w = z$.

Now we can apply calculations (20) to the product $X_\alpha \boxed{n} X_\beta$ for $n = -(\alpha|\beta) - 1$. Similar to (20) we get that

$$e(\alpha+\beta)X_\alpha \boxed{n} X_\beta = e(\alpha)e(\beta)p^{-(\alpha|\beta)} \prod_{s=1}^{p-1} (1-\omega^s)^{(\sigma^{-s}\alpha|\beta)} X_{\alpha+\beta}.$$

But we know that $X_\alpha \boxed{n} X_\beta = \varkappa(\alpha,\beta)X_{\alpha+\beta}$ in $\mathfrak{V}_\Lambda$, therefore

$$e(\alpha)e(\beta)X_{\alpha+\beta} = \varepsilon(\alpha,\beta)e(\alpha+\beta)X_{\alpha+\beta},$$

hence $e(\alpha)e(\beta) = \varepsilon(\alpha,\beta)e(\alpha+\beta)$.

So we have proved the following

Theorem 3. *Let V be a generalized module over the lattice vertex superalgebra $\mathfrak{V}_\Lambda$ such that $\tilde{h} \in \mathrm{vo}\{V\}^{[\lambda]}$ for $h \in \mathfrak{h}^{[\lambda]}$. Assume that V is $\mathfrak{h}^{[0]}$-diagonalizable. Then there is a unique action of the extended lattice $\hat{\Lambda}$ on V such that for every $\alpha \in \Lambda$ the highest weights vectors $X_\alpha \in \mathfrak{V}_\alpha$ act on V by the vertex operators* (14).

It follows from Theorem 3 that a $\mathfrak{V}_\Lambda$-submodule $U \subset V$ is a $U(H)\tilde{\otimes} R$-submodule of V. Combining this with Corollary 1, we get the following statement.

Corollary 2. *Assume V is an irreducible module over $U(H)\tilde{\otimes} R$, satisfying the assumptions of Theorem 2. Then V is an irreducible module over the vertex algebra $\mathfrak{V}_\Lambda$. Conversely, let V be a module over $\mathfrak{V}_\Lambda$, satisfying the assumptions of Theorem 3. Then V is an irreducible $U(H)\tilde{\otimes} R$-module.*

3.7. Twisted modules over lattice vertex superalgebras

In this section, we study twisted modules over the lattice vertex algebra $\mathfrak{V} = \mathfrak{V}_\lambda$, see Section 1.7 for the definition. As in Section 3.3, we fix a pth primitive root of unity $\omega = \exp(\frac{2\pi i}{p})$.

Assume that the automorphism $\sigma : \mathfrak{h} \to \mathfrak{h}$ preserves the lattice Λ. Then σ induces an automorphism $\sigma : \mathfrak{V}_0 \to \mathfrak{V}_0$ of the Heisenberg vertex algebra, see Section 3.1. Let $\hat{\sigma} : \mathfrak{V} \to \mathfrak{V}$ be an extension of this automorphism to the whole vertex algebra $\mathfrak{V}$. It is easy to see that $\hat{\sigma} X_\alpha = \varphi(\alpha) X_{\sigma\alpha}$ for some 1-cocycle $\varphi : \Lambda \to \mathbb{C}^\times$ such that

$$d\varphi(\alpha, \beta) = \frac{\varphi(\alpha+\beta)}{\varphi(\alpha)\varphi(\beta)} = \frac{\varkappa(\sigma\alpha, \sigma\beta)}{\varkappa(\alpha, \beta)}.$$

Since σ preserves the norm $(\cdot \,|\, \cdot)$ on $\mathfrak{h}$ and

$$\varkappa(\alpha, \beta)\varkappa(\beta, \alpha)^{-1} = (-1)^{(\alpha|\alpha)(\beta|\beta)}(-1)^{(\alpha|\beta)},$$

we see that the cocycle $\frac{\varkappa(\sigma\alpha,\sigma\beta)}{\varkappa(\alpha,\beta)}$ is indeed symmetric and therefore equal to $d\varphi_0$ for

$$\varphi_0(\alpha) = \left(\frac{\varkappa(\sigma\alpha, \sigma\alpha)}{\varkappa(\alpha, \alpha)}\right)^{\frac{1}{2}}.$$

Formula (18) implies that

$$\frac{\varkappa(\sigma\alpha, \sigma\beta)}{\varkappa(\alpha, \beta)} = \frac{\varepsilon(\sigma\alpha, \sigma\beta)}{\varepsilon(\alpha, \beta)},$$

therefore the map $e(\alpha) \mapsto \varphi(\alpha) e(\sigma\alpha)$ defines an automorphism of the algebra $R = \mathbb{C}[\hat{\Lambda}]/\langle \Phi \equiv \mathbb{C}^\times \rangle$. And vice versa, any such automorphism of R defines an automorphism of $\mathfrak{V}$.

Remark. In general, when the order of σ is p, the order of $\hat{\sigma} : \mathfrak{V} \to \mathfrak{V}$ may not be equal to p, in fact it may not be finite at all. However, the extension corresponding to φ_0 is of order p. Indeed,

$$\hat{\sigma}^p X_\alpha = \varphi_0(\alpha)\varphi_0(\sigma\alpha)\cdots\varphi_0(\sigma^{p-1}\alpha)X_\alpha$$

$$= \left(\frac{\varkappa(\sigma\alpha,\sigma\alpha)\varkappa(\sigma^2\alpha,\sigma^2\alpha)\cdots\varkappa(\alpha,\alpha)}{\varkappa(\alpha,\alpha)\varkappa(\sigma\alpha,\sigma\alpha)\cdots\varkappa(\sigma^{p-1}\alpha,\sigma^{p-1}\alpha)}\right)^{\frac{1}{2}} X_\alpha = X_\alpha.$$

Now we show how the vertex algebra $\mathfrak{V}$ is decomposed into a sum of the root spaces of $\hat{\sigma}$. Consider the action of $\hat{\sigma}$ on a linear span $\mathrm{Span}\{X_\alpha, X_{\sigma\alpha}, \ldots, X_{\sigma^{p-1}\alpha}\}$ of the $\hat{\sigma}$-orbit of X_α. It is easy to see that the eigenvalues of $\hat{\sigma}$ on this space are all the different roots

$$\mu(\alpha) = \sqrt[p]{\varphi(\alpha)\varphi(\sigma\alpha)\cdots\varphi(\sigma^{p-1}\alpha)}. \tag{22}$$

Fix such a root $\mu = \mu(\alpha)$. The eigenvector corresponding to the eigenvalue $\mu\omega^j$ for some $0 \leqslant j \leqslant p-1$ is

$$Y_j = \sum_{s=0}^{p-1} \omega^{-js} k_s X_{\sigma^s\alpha}, \tag{23}$$

where $k_s = \mu^{-s}\varphi(\alpha)\varphi(\sigma\alpha)\cdots\varphi(\sigma^{s-1}\alpha)$ for $s>0$ and $k_0 = 1$.

So we deduce that $\mathfrak{V} = \bigoplus_{\mu\in\mathbb{C}^\times} \mathfrak{V}_\mu$ is decomposed into a direct sum of the root spaces of $\hat{\sigma}$ such that $\hat{\sigma}|_{\mathfrak{V}_\mu} = \mu$.

Let $V = \bigoplus_{\xi\in(\mathfrak{h}^{[0]})^*} V_\xi$ be a generalized $\mathfrak{h}^{[0]}$-diagonalizable $\mathfrak{V}$-module. Denote by $\Xi = \{\xi \in (\mathfrak{h}^{[0]})^* \mid V_\xi \neq 0\}$ the set of weights of V. By Theorem 3 there are operators $e(\alpha): V \to V$, $\alpha\in\Lambda$, that define an action of R on V, satisfying the commutation relations (13).

Theorem 4. *The $\mathfrak{V}$-module V is $\hat{\sigma}$-twisted if and only if for every $\alpha\in\Lambda$ there is a number $\mu(\alpha)\in\mathbb{C}^\times$, given by* (22), *such that*

(i) $e(\sigma^s\alpha) = k_s^{-1}e(\alpha)$ *for every* $0\leqslant s\leqslant p-1$ *and*
(ii) $\xi(\alpha(0)) \equiv \frac{(\alpha'|\alpha')}{2} - \lambda \bmod \mathbb{Z}$ *for every* $\xi\in\Xi$,

where p is the length of σ-orbit of α, k_s are given by (23) *and* $\lambda = \frac{1}{2\pi i}\ln\mu(\alpha)$.

Condition (i) means that $e(\sigma\alpha) = \mu(\alpha)\varphi(\alpha)^{-1}e(\alpha)$ for every $\alpha\in\Lambda$, where the root $\mu(\alpha)$ is the same for all α's in a same σ-orbit. We also note that it is enough to impose (i) and (ii) only for α running over an integer basis of Λ.

Proof. Fix a root $\mu = \mu(\alpha)$ satisfying (22), and let $\lambda = \frac{1}{2\pi i}\ln\mu$. Let $\pi: \mathfrak{V}\to \mathrm{vo}\{V\}$ be the representation map. Then V is a twisted $\mathfrak{V}$-module if and only if

$\pi(\mathfrak{V}_\mu) \subset \mathrm{vo}\{V\}^{[\lambda]}$. It is enough to require that $Y_j \in \mathrm{vo}\{V\}^{[(j/p)+\lambda]}$ where Y_j is (the image under π of) the eigenvector of $\hat{\sigma}$ given by (23). Moreover, it is enough to require this only for a finite set of generators of $\mathfrak{V}$, for example for all the Y_j's corresponding to the σ-orbits of an integer basis of Λ.

Set $\mathring{X}_\alpha(z) = E_-(\alpha, z)E_+(\alpha, z) \subset \mathrm{vo}\{V\}$ so that (see Section 3.3)

$$X_\alpha = e(\alpha)\mathring{X}_\alpha z^{\alpha(0)} z^{-(\alpha'|\alpha')/2}.$$

Set also $\mathring{Y}_j = \sum_{s=0}^{p-1} \omega^{-js} k_s e(\sigma^s\alpha) X_{\sigma^s(\alpha)}$ and then

$$Y_j = \mathring{Y}_j z^{\alpha(0)} z^{-(\alpha'|\alpha')/2}.$$

It follows that the field $Y_j \in \mathrm{vo}\{V\}$ is homogeneous if and only if $\mathring{Y}_j$ is homogeneous and the values $\xi(\alpha(0))$ are the same modulo $\mathbb{Z}$ for all $\xi \in \Xi$.

Assume that $\mathring{Y}_j$ is homogeneous. Since $\mathring{X}_\alpha(z) \in \oplus_{q=0}^{p-1} \mathrm{vo}\{V\}^{[q/p]}$, we have $\mathring{Y}_j \in \mathrm{vo}\{V\}^{[(q+j)/p]}$ for some $0 \leqslant q \leqslant p-1$. Let τ be the automorphism of $\oplus_{q=0}^{p-1} \mathrm{vo}\{V\}^{[q/p]}$ such that $\tau|_{\mathrm{vo}\{V\}^{[q/p]}} = \omega^q$. It is easy to see that $\mathring{X}_{\sigma\alpha} = \tau\mathring{X}_\alpha$. Take some $m \equiv -\frac{r}{p} \bmod \mathbb{Z}$. Denote by x_s the coefficient of z^m in $\mathring{X}_{\sigma^s\alpha}$. It follows that $x_s = \omega^{rs}x_0$. If $r \not\equiv q+j \bmod p$, then the coefficient of z^m in $\mathring{Y}_j$ is equal to 0. This gives us the following system of linear equations:

$$\sum_{s=0}^{p-1} \omega^{(r-j)s}\, k_s\, e(\sigma^s\alpha) = 0 \quad \text{for } 0 \leqslant r \leqslant p-1, \;\; r \not\equiv q+j \bmod p.$$

Since $k_0 = 1$, the solution of this system is $e(\sigma^s\alpha) = \omega^{qs}k_s^{-1}e(\alpha)$. Taking $\mu\omega^q$ instead of μ we get exactly (i).

Condition (ii) follows from the fact that in order to have $\deg Y_j = \frac{j}{p} + \lambda$ we must have

$$\xi(\alpha(0)) \equiv \frac{(\alpha'|\alpha')}{2} - \lambda - \frac{q}{p} \equiv \frac{(\alpha'|\alpha')}{2} - \frac{1}{2\pi i}\ln(\mu\omega^q) \bmod \mathbb{Z}$$

for all $\xi \in \Xi$. $\square$

It follows that sometimes the lattice vertex algebra $\mathfrak{V} = \mathfrak{V}_\Lambda$ has no twisted representations that agree with an automorphism $\sigma : \Lambda \to \Lambda$. Recall that $C : \Lambda \times \Lambda \to \mathbb{C}^\times$ is the commutator map given by (16).

Corollary 3. *If for some $\alpha \in \Lambda$ and $0 \leqslant j \leqslant p-1$ we have $C(\alpha, \sigma^j\alpha) \neq 1$, then there are no non-trivial twisted modules of $\mathfrak{V}_\Lambda$ corresponding to either of the extensions $\hat{\sigma} : \mathfrak{V}_\Lambda \to \mathfrak{V}_\Lambda$ of σ.*

Proof. Set $\beta = \sigma^j\alpha$. If $C(\alpha,\beta) \neq 1$, then $e(\alpha)e(\beta) \neq e(\beta)e(\alpha)$. Yet by Theorem 4, we have $e(\alpha)e(\beta)^{-1} \in \mathbb{C}^\times$. Therefore, R acts by 0 on every twisted $\mathfrak{V}_\Lambda$-module, hence by Theorem 3, so does $\mathfrak{V}_\Lambda$. $\square$

3.8. Semisimplicity of twisted representations

Consider the category $\mathcal{O}_{\hat\sigma}$ of $\mathfrak{h}^{[0]}$-diagonalizable $\hat\sigma$-twisted modules V over $\mathfrak{V}_\Lambda$ such that V satisfies the conditions of Lemma 1 as a module over the Heisenberg algebra H. In particular, V must be a module over the vertex operator algebra $\mathfrak{V}_\Lambda$. Note that if $V = \oplus_n V_n$ is a $\hat\sigma$-twisted module over the vertex operator algebra $\mathfrak{V}_\Lambda$ such that $V_n = 0$ for $n \ll 0$, then $V \in \mathcal{O}_{\hat\sigma}$. The latter modules appear in the representation theory of vertex algebras quite often, in particular in connection with the Zhu theory [8,29].

Note that by Corollary 3, the category $\mathcal{O}_{\hat\sigma}$ can sometimes be trivial.

Theorem 5. *The category $\mathcal{O}_{\hat\sigma}$ is semisimple with finitely many isomorphism classes of simple objects.*

In order to prove this theorem we have to study the quotient algebra A of $R = \mathbb{C}[\hat\Lambda]/\langle \Phi \equiv \mathbb{C}^\times \rangle$ modulo the ideal generated by relations (i) of Theorem 4.

More precisely, let $\Pi \subset \Lambda$ be a finite set of vectors, closed under σ and spanning Λ over $\mathbb{Z}$. For $\alpha \in \Pi$ choose the roots $\mu(\alpha)$, given by (22), such that $\mu(\alpha) = \mu(\beta)$ if α and β lie in the same σ-orbit of Π. For $x \in R$ denote by $\bar x$ its image in A. Then the algebra A is generated by the set $\{\overline{e(\alpha)} \mid \alpha \in \Pi\}$ subject to relations $\overline{e(\sigma\alpha)} = \mu(\alpha)\phi(\alpha)^{-1}\overline{e(\alpha)}$ for all $\alpha \in \Pi$. Hence A depends on the cocycle $\varphi : \Lambda \to \mathbb{C}^\times$, which determines the extension $\hat\sigma$, and also on the choice of roots $\mu(\alpha)$ for every σ-orbit of Π.

Note that relations (i) of Theorem 4 belong in fact to the group $\hat\Lambda$. Let G be the quotient group of $\hat\Lambda$ modulo the normal subgroup generated by these relations. Then $A = \mathbb{C}[G]/\langle \Phi \equiv \mathbb{C}^\times \rangle$.

Let $v : \Lambda \to (\mathfrak{h}^{[0]})^*$ be the map given by $v(\alpha)h = (\alpha|h)$. Then $v(\Lambda)$ is a sublattice of the dual lattice to $\Lambda^{[0]}$. The algebra R is graded by $v(\Lambda)$ by setting $\deg e(\alpha) = v(\alpha)$. We observe that if $\alpha, \beta \in \Lambda$ belong to the same σ-orbit, then $v(\alpha) = v(\beta)$. It follows that relations (i) of Theorem 4 are in the kernel of the composition map $\hat\Lambda \to \Lambda \xrightarrow{v} v(\Lambda)$, which therefore induces a homomorphism $G \to v(\Lambda)$, and this makes $\mathbb{C}[G]$ and A graded by $v(\Lambda)$ as well.

Proposition 4. *The algebra A is $v(\Lambda)$-graded semisimple.*

Recall that a graded algebra is called graded semisimple if it is semisimple as a left graded module over itself. Like in a non-graded case, an algebra is graded semisimple if and only if it is decomposed into a direct product of graded simple algebras, the latter by a graded version of the classical Wedderburn theorem are isomorphic to matrix algebras over graded division algebras. If an algebra A is graded semisimple, then the category of graded R-modules is semisimple, i.e. every module is completely reducible. The simple graded A-modules are the simple homogeneous ideals of A with a possible shift of degrees.

Similar to the non-graded case, the graded Jacobson radical $J_{\mathrm{gr}}(A)$ of a graded algebra A is defined to be the intersection of all maximal graded left (or right) ideals. It is a standard exercise to prove that $J_{\mathrm{gr}}(A)$ is in fact a double sided ideal that acts by 0 on any simple graded A-module. If $J_{\mathrm{gr}}(A) = 0$ that A is called semiprimitive.

The algebra A is called graded (left) Artinian if all strictly decreasing chains of graded left ideals are finite. In this case $J_{\mathrm{gr}}(A)$ is an intersection of finitely many left maximal ideas. If A is graded Artinian and $J_{\mathrm{gr}}(A) = 0$ then A is graded semisimple.

For the non-graded case all this is a classical theory presented in most graduate algebra textbooks, like Jacobson's *Basic Algebra* [13]. For the graded case the references are much more scarce, see, however, [25].

We will use the following rather obvious fact: A homomorphic image of a graded semisimple algebra modulo a homogeneous ideal is again a graded semisimple algebra.

Proof of Proposition 4. Let $\Pi = \bigsqcup_{j=1}^{n} \Pi_j$ be the decomposition of Π into a disjoint union of σ-orbits. Choose some $\alpha_j \in \Pi_j$ for every $1 \leqslant j \leqslant n$. Let $x_j = \overline{e(\alpha_j)} \in A$. Then the set $x_1^{\pm 1}, \dots, x_n^{\pm 1}$ is a set of generators of A. We note that $x_i x_j = c_{ij} x_j x_i$ for $c_{ij} = C(\alpha_i, \alpha_j) \in \mathbb{C}^\times$, where C is the commutator map given by (16).

Let $\xi_j = \deg \alpha_j = v(\alpha_j) \in v(\Lambda)$. It is easy to see that $\xi_j = 0$ if and only if $\sum_{\alpha \in \Pi_j} \alpha = 0$. We can assume that $\xi_1, \dots, \xi_m \neq 0$ for some $m \leqslant n$ and $\xi_{m+1} = \dots = \xi_n = 0$. Clearly, ξ_j's span $v(\Lambda)$ over $\mathbb{Z}$.

Assume also that our choice of Π yields the minimal possible value of m. We claim that in this case $\xi_1, \dots, \xi_m$ is a $\mathbb{Z}$-basis of $v(\Lambda)$. Indeed, otherwise there is an invertible matrix $M \in SL(m, \mathbb{Z})$ such that the mth column of $(\xi_1 \ \cdots \ \xi_m)M$ is 0. Let α'_j be the jth column of $(\alpha_1 \ \cdots \ \alpha_m)M$ for $1 \leqslant j \leqslant m$, and let Π' be the closure of $\{\alpha'_1, \dots, \alpha'_m\} \cup \Pi_{m+1} \cup \dots \cup \Pi_n$ with respect to the action of σ. Then Π' is a generating set of Λ, closed under σ and, since $v(\alpha'_m) = 0$ the number of σ-orbits in Π' with non-zero degree is at most $m - 1$.

Suppose there is a linear relation between elements of Π of the form

$$\sum_{j=1}^{n} \sum_{s=0}^{p-1} r_{js} \sigma^s \alpha_j = 0, \quad r_{js} \in \mathbb{Z}.$$

In R this relation becomes $\prod_{j,s} e(\sigma^s \alpha_j)^{r_{js}} = \theta'$, where $\theta' \in \mathbb{C}^\times$ is a product of some values of the cocycle ε. In A this relation becomes

$$\prod_{j=1}^{n} x_j^{r_j} = \theta, \quad r_j = \sum_{s=0}^{p-1} r_{js}$$

for some other constant $\theta \in \mathbb{C}^\times$.

For $m+1 \leqslant j \leqslant n$ we have $\sum_s \sigma^s \alpha_j = 0$, hence in A we get the relation $x_j^p = \theta_j \in \mathbb{C}^\times$. By the change of variables $x_j \mapsto \theta_j^{-1/p} x_j$ we can make $\theta_j = 1$.

A look at formula (16) for the commutator map C shows that if $\sum_{s=0}^p \sigma^s \alpha = 0$ for some $\alpha \in \Lambda$, then $C(\alpha, \beta)^p = 1$ for any other $\beta \in \Lambda$.

Summing up, we get that A is a homomorphic image of the algebra

$$B = \mathbf{k}\left\langle x_1^{\pm 1}, \dots, x_n^{\pm 1} \;\middle|\; \begin{array}{c} x_i x_j = c_{ij} x_j x_i \text{ for } 1 \leqslant i,j \leqslant n \\ x_j^p = 1 \text{ for } m+1 \leqslant j \leqslant n \end{array} \right\rangle,$$

where $c_{ij} = c_{ji}^{-1} \in \mathbb{C}^\times$ for $1 \leqslant i,j \leqslant n$ and $c_{ij}^p = 1$ if either $i > m$ or $j > m$. The $v(\Lambda)$-grading on B is defined by $\deg x_j = 0$ for $m+1 \leqslant j \leqslant n$ and $\deg x_j = \xi_j$ for $1 \leqslant j \leqslant m$ where $\xi_1, \dots, \xi_m$ is a $\mathbb{Z}$-basis of $v(\Lambda)$. It is enough to show that B is $v(\Lambda)$-graded semisimple.

Let $\xi = k_1\xi_1 + \cdots + k_m\xi_m$, $k_i \in \mathbb{Z}$, be an arbitrary vector in $v(\Lambda)$. Denote by B_ξ the homogeneous component of B of degree ξ. Note that the element $x^\xi = x_1^{k_1} \cdots x_m^{k_m} \in B_\xi$ is invertible. For a graded subspace $I \subset B$ denote by I_ξ the homogeneous component of I of degree ξ.

The component B_0 is the homomorphic image of the group algebra of finite group

$$G_0 = \left\langle x_k, c_{ij} \;\middle|\; \begin{array}{c} m+1 \leqslant i,j,k \leqslant n, \;\; x_k^p = c_{ij}^p = 1, \;\; c_{ij} = c_{ji}^{-1}, \\ x_i x_j = c_{ij} x_j x_i, \;\; x_k c_{ij} = c_{ij} x_k \end{array} \right\rangle.$$

By Maschke's theorem (see e.g. [13]), $\mathbb{C}[G_0]$ and hence B_0 is a finite-dimensional semisimple algebra.

We claim that if $I, J \subset B$ are two graded left ideals such that $I_0 = J_0$, then $I = J$. Indeed, if $b \in I_\xi$, then $x^{-\xi} b \in I_0 = J_0$, hence $b = x^\xi x^{-\xi} b \in J_\xi$. It follows that B is graded (left) Artinian.

Let $I \subset B$ be a graded left ideal. Then I is graded maximal if and only if I_0 is a maximal ideal of B_0. Therefore, since $J(B_0) = 0$, we must have $J_{\text{gr}}(B) = 0$, and that finishes the proof. $\square$

Remark (Passman [26]). The argument above shows in fact that if $B = B_0 * G$ is a crossed product of an algebra B_0 with a group G, then B is G-graded Artinian if and only if B_0 is Artinian, and also $J_{\text{gr}}(B) = 0$ if and only if $J(B) = 0$. The same is true if B is a strongly G-graded algebra. See e.g. [18,26] for definitions and further results.

Proof of Theorem 5. Let $V \in \mathcal{O}_{\hat{\sigma}}$. By (i) of Lemma 1 and Theorem 4 we have $V = M(1) \otimes \Omega$, where Ω is $v(\Lambda)$-graded module over A. Hence it follows from Proposition 4 that Ω is decomposed into a direct sum of graded irreducible A-modules, and Corollary 2 implies that $\mathfrak{V}$ is decomposed into a direct sum of irreducible $\mathfrak{V}_\Lambda$-modules.

A $\mathfrak{V}$-module $V = M(1) \otimes \Omega \in \mathcal{O}_{\hat{\sigma}}$ is simple if and only if Ω is a simple $v(\Lambda)$-graded A-module. Such Ω must be isomorphic to a simple homogeneous ideal of A up to a shift of weights. The weights of Ω are restricted by (ii) of Theorem 4. We claim that a simple object of $\mathcal{O}_{\hat{\sigma}}$ is determined up to an isomorphism by a choice of roots $\mu(\alpha_j)$, given by (22), for each generating σ-orbit $\Pi_j \ni \alpha_j$ of Λ, a choice of simple homogeneous ideal Ω of A and an equivalence class $\eta \in (\Lambda^{[0]})'/\nu(\Lambda)$.

Indeed, assume all these choices are made. Since the extension $\hat{\sigma}$ is fixed, the choice of $\mu(\alpha_j)$'s determines the $\nu(\Lambda)$-graded semisimple algebra A. The set $\Xi = \{\xi \in (\mathfrak{h}^{[0]})^* \mid \Omega_\xi \neq 0\}$ is an equivalence class in $(\mathfrak{h}^{[0]})^*/\nu(\Lambda)$. Let $\xi \in (\mathfrak{h}^{[0]})^*$ be such that $\xi(\alpha(0)) = \frac{1}{2}(\alpha'|\alpha') - \frac{1}{2\pi i}\ln \mu(\alpha)$ for $\alpha \in \Lambda$. By Theorem 4 (ii) we have that $\Xi \equiv \xi \bmod (\Lambda^{[0]})'$, so now we further specify $\Xi \equiv \xi + \eta \bmod \nu(\Lambda)$.

It follows that there are at most finitely many isomorphism classes of simple objects in $\mathcal{O}_{\hat{\sigma}}$. $\square$

3.9. Examples

Example 0. If $\sigma = \mathrm{Id}$, then $\nu(\Lambda) \cong \Lambda$, since the form is non-degenerate, and $A = R$ is a Λ-graded division algebra. The automorphism $\widehat{\mathrm{id}} : \mathfrak{V}_\Lambda \to \mathfrak{V}_\Lambda$ is defined by choosing an arbitrary values $\mu_j = \varphi(\alpha_j) \in \mathbb{C}^\times$ for a basis $\alpha_1, \ldots, \alpha_l$ of Λ. The simple objects of category $\mathcal{O}_{\widehat{\mathrm{id}}}$ are parametrized by Λ'/Λ, which agrees with the result of Dong [4]. Let $\xi \in \mathfrak{h}^*$ be the functional defined by $\xi(\alpha_j) = \frac{1}{2\pi i}\ln \mu_j$. Then the simple object of category $\mathcal{O}_{\widehat{\mathrm{id}}}$ corresponding to an equivalence class $\eta \in \Lambda'/\Lambda$ is $M(1) \otimes \mathbb{C}[\Lambda]$, where $\mathbb{C}[\Lambda]$ is graded such that its set of weights is equal to $\eta + \xi + \Lambda$.

Example 1. A more interesting example is when $\sigma = -1$. In this case take $\Pi = \{\pm\alpha_1, \ldots, \pm\alpha_l\}$. The grading is trivial, since $\Lambda^{[0]} = 0$. One can choose the cocycle ε so that $\varepsilon(\alpha_j, \alpha_j) = 1$ and then $\varphi(-\alpha_j) = \varphi(\alpha_j)^{-1}$, hence (22) gives $\mu_j = \pm 1$ for all $1 \leqslant j \leqslant l$. Set $x_j = \overline{e(\alpha_j)} \in A$, then

$$A = \mathbb{C}[x_1, \ldots, x_l]/\langle x_i x_j = c_{ij} x_j x_i, x_j^2 = \mu_j \phi(\alpha_j) \rangle,$$

where $c_{ij} = (-1)^{(\alpha_i|\alpha_i)(\alpha_j|\alpha_j)}(-1)^{(\alpha_i|\alpha_j)}$. So A is a semisimple algebra of dimension 2^l for any choice of φ and μ_j. A simple module $V \in \mathcal{O}_{\hat{\sigma}}$ is decomposed $V = M(1) \otimes \Omega$, where Ω isomorphic to a simple ideal of A. This case was studied in [5,9,12].

We remark that $\widehat{-1}$ is an isomorphism of any lattice Λ. When Λ is the Leech lattice, a certain $\widehat{-1}$-twisted module over $\mathfrak{V}_\Lambda$ is used to construct the Moonshine vertex algebra $V^\natural$, see [12].

Example 2. Take $\Lambda = \mathbb{Z}\alpha + \mathbb{Z}\beta$ and let σ be the rotation $\alpha \mapsto \beta,\ \beta \mapsto -\alpha$. Then we must have $(\alpha|\beta) = 0,\ (\alpha|\alpha) = (\beta|\beta)$. As in the previous example, $\Lambda^{[0]} = 0$, so the grading is trivial. Formula (16) yields $C = 1$, therefore the cocycle ε is trivial and hence $\varphi : \Lambda \to \mathbb{C}^\times$ is a character. Set $\varphi(\alpha) = \varphi_1,\ \varphi(\beta) = \varphi_2$. The lattice Λ is generated by a single orbit $\{\alpha, \beta, -\alpha, -\beta\}$, and for that orbit $\mu = \sqrt[4]{1}$ by (22). Set $x = \overline{e(\alpha)}, y = \overline{e(\beta)} \in A$. Then relations (i) of Theorem 4 give $y = \mu\varphi_1^{-1}x,\ x^{-1} = \mu\varphi_2^{-1}y$, therefore $A = \mathbb{C}[x]/\langle x^2 = \mu^2\varphi_1\varphi_2 \rangle$. It follows that there are exactly 2 irreducible $\hat{\sigma}$-twisted $\mathfrak{V}_A$-modules for every extension $\hat{\sigma} : \mathfrak{V}_A \to \mathfrak{V}_A$.

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