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Biorthogonal polynomials associated with reflection groups and a formula of Macdonald

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Abstract

Dunkl operators are differential-difference operators on $\mathbb{R}^N$ which generalize partial derivatives. They lead to generalizations of Laplace operators, Fourier transforms, heat semigroups, Hermite polynomials, and so on. In this paper we introduce two systems of biorthogonal polynomials with respect to Dunkl's Gaussian distributions in a canonical way. These systems, called Appell systems, admit many properties known from classical Hermite polynomials, and turn out to be useful for the analysis of Dunkl's Gaussian distributions. In particular, these polynomials lead to a new proof of a generalized formula of Macdonald due to Dunkl. The ideas for this paper are taken from recent works on non-Gaussian white noise analysis and from the umbral calculus. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Dunkl operators are differential-difference operators on $\mathbb{R}^N$ related to finite reflection groups. They can be regarded as a generalization of partial derivatives and play a major role in the description of Calogero–Moser–Sutherland models of quantum many-body systems on the line. Dunkl operators lead to generalizations of exponential functions, Fourier transforms, Laplace operators, and Gaussian distributions on $\mathbb{R}^N$. The corresponding basic theory is developed in [6–8, 10, 13, 16, 17] and will

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be briefly reviewed in Section 2 below (with references to proofs). A more detailed approach to the Dunkl theory will be given in [18].

In this paper, we study systems of biorthogonal polynomials with respect to generalized Gaussian distributions of the form $w(x)\exp\{-c|x|^2\}$, where w is a homogeneous weight function with certain reflection group symmetries. Generalized Gaussians of such type were first considered by Macdonald and Mehta (see [12]) and replace the classical Gaussian in the Dunkl theory. We mention that systems of orthogonal polynomials related to Dunkl's Gaussian distributions on $\mathbb{R}^N$, called generalized Hermite polynomials, have been studied in various contexts during the last years, see e.g [2, 5, 16, 19] and references therein. In the case of the symmetric group, the generalized Hermite polynomials play a role in quantum many-body systems of Calogero–Moser–Sutherland type and are closely related to Jack polynomials.

The biorthogonal systems of this paper are introduced in a canonical way; the method is as follows: In Section 3 we define the so-called modified moment functions m_ν ($\nu\in\mathbb{Z}_+^N$) which form generalizations of the monomials $x^\nu := x_1^{\nu_1}\cdots x_N^{\nu_N}$ on $\mathbb{R}^N$ such that each m_ν is a homogeneous polynomial of degree $|\nu| = \nu_1+\cdots+\nu_N$. These functions lead to generalized moments of Dunkl's Gaussian distributions and are very closely related to the classical moments of the classical Gaussian distributions on $\mathbb{R}^N$. Based on these modified moments, we introduce two systems $R_\nu(t,x)$ and $S_\nu(t,x)$ ($\nu\in\mathbb{Z}_+^N$, $x\in\mathbb{R}^N$, $t\in\mathbb{R}$) of polynomials in $N+1$ variables via two generating functions involving Dunkl's kernel K and Dunkl's Gaussian distribution. We show that for all $t>0$, the so-called Appell systems $(R_\nu(t,.))_{\nu\in\mathbb{Z}_+^N}$ and $(S_\nu(t,.))_{\nu\in\mathbb{Z}_+^N}$ form biorthogonal polynomials with respect to Dunkl's Gaussian distribution with variance parameter t. We also show that $R_\nu(t,x)=\mathrm{e}^{-t\Delta_k}m_\nu(x)$ and $S_\nu(t,x)=\mathrm{e}^{-t\Delta_k}x^\nu$ hold where Δ_k denotes Dunkl's Laplacian. At the end of this paper, we present some applications of these results including a Rodriguez-type formula and a new proof of a generalized Macdonald-formula given originally in [7]. We point out that the methods of this paper can be extended to other distributions and that there exist natural applications in martingale theory (which are well known for classical Hermite polynomials and Brownian motions); for details see [18]. The main purpose of this paper is to give a short introduction to the Appell systems $(R_\nu(t,.))_{\nu\in\mathbb{Z}_+^N}$ and $(S_\nu(t,.))_{\nu\in\mathbb{Z}_+^N}$ without a deeper probabilistic background.

2. Dunkl operators and Dunkl transform

In this section we collect some basic notations and facts from the Dunkl theory which will be important later on. General references here are [6–8, 10, 13]; for a background on reflection groups and root systems see, for instance, [9].

2.1. Basic notions

For $\alpha\in\mathbb{R}^N\setminus\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha\subset\mathbb{R}^N$ orthogonal to α, i.e., $\sigma_\alpha(x)=x-(2\langle\alpha,x\rangle/|\alpha|^2)\alpha$, where $\langle .,.\rangle$ is the Euclidean scalar product on $\mathbb{R}^N$ and $|x| := \sqrt{\langle x,x\rangle}$.

A finite set $R\subset\mathbb{R}^N\setminus\{0\}$ is called a root system if $R\cap\mathbb{R}\cdot\alpha=\{\pm\alpha\}$ and $\sigma_\alpha R=R$ for all $\alpha\in R$. For a given root system R the reflections σ_α ($\alpha\in R$) generate a group W, the reflection group associated with R. This group is finite, and all reflections in W correspond to suitable pairs of roots; see [9]. Now fix $\beta\in\mathbb{R}^N\setminus\bigcup_{\alpha\in R}H_\alpha$ and a positive subsystem $R_+=\{\alpha\in R:\langle\alpha,\beta\rangle>0\}$; then for each $\alpha\in R$

either $\alpha \in R_+$ or $-\alpha \in R_+$. We assume from now on, with no loss of generality, that the root system R is normalized, i.e, that $|\alpha| = \sqrt{2}$ for all $\alpha \in R$.

A multiplicity function k on a root system R is defined as a W-invariant function $k : R \to \mathbb{C}$. If one regards k as function on the corresponding reflections, this W-invariance just means that k is constant on the conjugacy classes of reflections in W. In this paper, we always assume that $k \geqslant 0$ (i.e. all values of k are nonnegative), though several results may be extended to larger ranges of k. For abbreviation, we introduce

$$\gamma := \gamma(k) := \sum_{\alpha \in R_+} k(\alpha) \quad \text{and} \quad c_k := \left(\int_{\mathbb{R}^N} e^{-|x|^2} w_k(x)\, dx \right)^{-1}, \tag{2.1}$$

where w_k is the weight function

$$w_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}. \tag{2.2}$$

Note that w_k is W-invariant and homogeneous of degree 2γ. We shall use the following further abbreviations: $\mathscr{P} = \mathbb{C}[\mathbb{R}^N]$ denotes the algebra of polynomial functions on $\mathbb{R}^N$ and $\mathscr{P}_n$ $(n \in \mathbb{Z}_+)$ the subspace of homogeneous polynomials of degree n. We use the standard multi-index notations, i.e. for multi-indices ν, $\rho \in \mathbb{Z}_+^N$ we write

$$|\nu| := \nu_1 + \cdots + \nu_N, \qquad \nu! := \nu_1! \cdot \nu_2! \cdots \nu_N!, \qquad \binom{\nu}{\rho} := \binom{\nu_1}{\rho_1}\binom{\nu_2}{\rho_2}\cdots\binom{\nu_N}{\rho_N},$$

as well as

$$x^\nu := x_1^{\nu_1} \cdots x_N^{\nu_N} \quad \text{and} \quad A^\nu := A_1^{\nu_1} \cdots A_N^{\nu_N}$$

for $x \in \mathbb{R}^N$ and any family $A = (A_1, \dots, A_N)$ of commuting operators on $\mathscr{P}$. Finally, the partial ordering $\leqslant$ on $\mathbb{Z}_+^N$ is defined by $\rho \leqslant \nu :\Leftrightarrow \rho_i \leqslant \nu_i$ for $i = 1, \dots, N$.

2.2. Dunkl operators

The Dunkl operators T_i $(i = 1, \dots, N)$ on $\mathbb{R}^N$ associated with the finite reflection group W and multiplicity function k are given by

$$T_i f(x) := \partial_i f(x) + \sum_{\alpha \in R_+} k(\alpha)\, \alpha_i \cdot \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad f \in C^1(\mathbb{R}^N), \tag{2.3}$$

here ∂_i denotes the ith partial derivative. In case $k = 0$, the T_i reduce to the corresponding partial derivatives. The most important basic properties of the T_i are as follows (see [6]):

(1) The set $\{T_i\}$ generates a commutative algebra of differential-difference operators on $\mathscr{P}$.

(2) Each T_i is homogeneous of degree -1 on $\mathscr{P}$, i.e., $T_i p \in \mathscr{P}_{n-1}$ for $p \in \mathscr{P}_n$.

(3) (*Product rule*:) $T_i(fg) = (T_i f)g + f(T_i g)$ for $i = 1, \dots, N$ and $f, g \in C^1(\mathbb{R}^N)$ such that g is W-invariant.

A major tool in this paper is a generalized exponential kernel $K(x, y)$ on $\mathbb{R}^N \times \mathbb{R}^N$, which replaces the usual exponential function $e^{\langle x, y \rangle}$. It was introduced in [7] by means of an intertwining isomorphism V of $\mathscr{P}$ which is characterized by the properties

$$V(\mathscr{P}_n)=\mathscr{P}_n, \qquad V|_{\mathscr{P}_0}=id, \quad \text{and} \quad T_iV=V\partial_i \quad (i=1,\ldots,N). \tag{2.4}$$

Let $B=\{x\in\mathbb{R}^N: |x|\leqslant 1\}$. Then V extends to a bounded linear operator on the algebra

$$A:=\left\{f:B\to\mathbb{C}: f=\sum_{n=0}^{\infty}f_n \quad \text{with } f_n\in\mathscr{P}_n \quad \text{and} \quad \|f\|_A:=\sum_{n=0}^{\infty}\|f_n\|_{\infty,B}<\infty\right\}$$

by $V(\sum_{n=0}^{\infty}f_n):=\sum_{n=0}^{\infty}Vf_n$. The Dunkl kernel K is now defined by

$$K(x,y):=V(e^{\langle\cdot,y\rangle})(x) \quad (x,y\in\mathbb{R}^N). \tag{2.5}$$

K has a holomorphic extension to $\mathbb{C}^N\times\mathbb{C}^N$ and is symmetric in its arguments. We also note that for $y\in\mathbb{R}^N$, the function $x\mapsto K(x,y)$ may be characterized as the unique analytic solution of $T_if=y_if$ $(i=1,\ldots,N)$ with $f(0)=1$; see [13].

Example 2.1. (1) If $k=0$, then $K(z,w)=e^{\langle z,w\rangle}:=e^{\sum_{j=1}^N z_jw_j}$ for all $z,w\in\mathbb{C}^N$.

(2) If $N=1$ and $W=\mathbb{Z}_2$, then the multiplicity function is a single parameter $k\geqslant 0$, and for $k>0$ the intertwining operator is given explicitly by

$$V_kf(x)=\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k)}\int_{-1}^{1}f(xt)(1-t)^{k-1}(1+t)^k\,dt,$$

see [7]. The associated Dunkl kernel can be written as

$$K(z,w)=j_{k-1/2}(izw)+\frac{zw}{2k+1}j_{k+1/2}(izw), \quad z,w\in\mathbb{C},$$

where for $\alpha\geqslant-\frac{1}{2}$, j_α is the normalized spherical Bessel function

$$j_\alpha(z)=2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha}=\Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}.$$

For details and related material see [8, 15, 18, 19], and references cited there.

For later references, we next list some further known properties of K.

Theorem 2.2. *Let $z,w\in\mathbb{C}^N$, $x,y\in\mathbb{R}^N$, $\lambda\in\mathbb{C}$, and $g\in W$. Then*

(1) $K(z,0)=1$, $K(\lambda z,w)=K(z,\lambda w)$, $K(z,w)=K(w,z)$, $K(-ix,y)=\overline{K(ix,y)}$, *and* $K(g(x),g(y))=K(x,y)$.

(2) $T_j^xK(x,y)=y_jK(x,y)$ *for* $j=1,\ldots,N$, *where the superscript x indicates that the operators act with respect to the x-variable.*

(3) *For each $x\in\mathbb{R}^N$ there is a unique probability measure $\mu_x\in M^1(\mathbb{R}^N)$ with supp $\mu_x\subset\{z\in\mathbb{R}^N: |z|\leqslant|x|\}$ such that*

$$K(x,y)=\int_{\mathbb{R}^N}e^{\langle z,y\rangle}\,d\mu_x(z) \quad \textit{for all } y\in\mathbb{C}^N.$$

In particular, $K(x,y)>0$ for all $x,y\in\mathbb{R}^N$.

(4) *For all multi-indices* $\nu \in \mathbb{Z}_+^N$,

$$|\partial_z^\nu K(x,z)| \leqslant |x|^{|\nu|} e^{|x| \cdot |\mathrm{Re}\, z|}.$$

In particular $|K(z,w)| \leqslant e^{|z||w|}$ *and* $|K(\mathrm{i}x, y)| \leqslant 1$.

Proof. Parts (1) and (2) follow from the characterizations of K in Section 2.2; cf. [7, 8]. Part (3) is shown in [17], and Part (4) is a consequence of Part (3). □

The generalized exponential function K gives rise to an integral transform, called the Dunkl transform on $\mathbb{R}^N$, which was introduced in [8]. To emphasize the similarity to the classical Fourier transform, we use the following notion:

2.3. The Dunkl transform

The Dunkl transform associated with W and $k \geqslant 0$ is given by

$$\widehat{\ }: L^1(\mathbb{R}^n, w_k(x)\,\mathrm{d}x) \to C_b(\mathbb{R}^N); \qquad \widehat{f}(y) := \int_{\mathbb{R}^N} f(x) K(-\mathrm{i}y, x)\, w_k(x)\,\mathrm{d}x \quad (y \in \mathbb{R}^N).$$

The Dunkl transform of a function $f \in L^1(\mathbb{R}^N, w_k(x)\,\mathrm{d}x)$ satisfies $\|\widehat{f}\|_\infty \leqslant \|f\|_{1, w_k(x)\,\mathrm{d}x}$ by Theorem 2.2(4). Moreover, according to [8, 10, 18] many results from classical Fourier analysis on $\mathbb{R}^N$ have analogues for the Dunkl transform, like the L^1-inversion theorem, the lemma of Riemann–Lebesgue, and Plancherel's formula.

We next extend the Dunkl transform to measures. We denote the Banach space of all $\mathbb{C}$-valued bounded Borel measures on $\mathbb{R}^N$ by $M_b(\mathbb{R}^N)$, while $M^1(\mathbb{R}^N)$ is the subspace consisting of all probability measures on $\mathbb{R}^N$.

The Dunkl transform of a measure $\mu \in M_b(\mathbb{R}^N)$ is given by $\widehat{\mu}(y) := \int_{\mathbb{R}^N} K(-\mathrm{i}y, x)\,\mathrm{d}\mu(x)$ $(y \in \mathbb{R}^N)$; it satisfies $\widehat{\mu} \in C_b(\mathbb{R}^N)$ with $\|\widehat{\mu}\|_\infty \leqslant \|\mu\|$; cf. Theorem 2.2(4). Moreover,
(1) If $\mu \in M_b(\mathbb{R}^N)$ and $f \in L^1(\mathbb{R}^N, w_k(x)\,\mathrm{d}x)$, then

$$\int_{\mathbb{R}^N} \widehat{\mu}(x) f(x) w_k(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \widehat{f}\,\mathrm{d}\mu;$$

(2) If $\mu \in M_b(\mathbb{R}^N)$ satisfies $\widehat{\mu} = 0$, then $\mu = 0$.

In fact, Part (1) follows from Fubini's theorem, and Part (2) follows from Part (1) and the fact that $(L^1(\mathbb{R}^N, w_k(x)\,\mathrm{d}x))^\wedge$ is $\|\,.\,\|_\infty$-dense in $C_0(\mathbb{R}^N)$; see [10].

We next turn to Dunkl's Laplace operator and the associated heat semigroup:

2.4. The generalized Laplacian

The generalized Laplacian Δ_k associated with W and $k \geqslant 0$ is defined by

$$\Delta_k f := \sum_{j=1}^N T_j^2 f, \quad f \in C^2(\mathbb{R}^N). \tag{2.6}$$

It is shown in [16] that Δ_k is a closable linear operator on $C_0(\mathbb{R}^N)$ and that its closure (again denoted by Δ_k) generates a positive, strongly continuous contraction semigroup $(e^{t\Delta_k})_{t\geqslant 0}$ on $C_0(\mathbb{R}^N)$. This semigroup is given explicitly in terms of the following generalized heat kernels.

2.5. Generalized heat kernels

The generalized heat kernel Γ_k is defined by

$$\Gamma_k(x,y,t) := \frac{c_k}{(4t)^{\gamma+N/2}} e^{-(|x|^2+|y|^2)/4t} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) \quad (x,y\in\mathbb{R}^N,\ t>0), \tag{2.7}$$

where c_k is given in (2.1). The kernel Γ_k has the following properties (see Lemma 4.5 in [16]): Let $x,y,z\in\mathbb{R}^N$ and $t>0$. Then

(1) $\Gamma_k(x,y,t)=\Gamma_k(y,x,t)=c_k^2/(4^{\gamma+N/2})\int_{\mathbb{R}^N} e^{-t|\xi|^2}K(ix,\xi)K(-iy,\xi)\,w_k(\xi)\,d\xi$, and by the inversion formula for the Dunkl transform, $\Gamma_k(x,.,t)^\wedge(z)=e^{-t|z|^2}\cdot K(-ix,z)$.

(2) $\int_{\mathbb{R}^N}\Gamma_k(x,y,t)\,w_k(x)\,dx=1$ and $|\Gamma_k(x,y,t)|\leqslant M_k/(t^{\gamma+N/2})e^{-(|x|-|y|)^2/4t}$.

Moreover, the integral operators

$$H(t)f(x) := \int_{\mathbb{R}^N}\Gamma_k(x,y,t)f(y)\,w_k(y)\,dy \quad \text{for } t>0 \quad \text{and} \quad H(0)f:=f \tag{2.8}$$

are related to the semigroup $(e^{t\Delta_k})_{t\geqslant 0}$ by the fact that for all $f\in C_0(\mathbb{R}^N)\cup\mathscr{P}$

$$e^{t\Delta_k}f=H(t)f \quad (t\geqslant 0). \tag{2.9}$$

For $f\in C_0(\mathbb{R}^N)$ this is shown in [16]. Moreover, by Proposition 2.1 of [8] we have

$$p(x)=\int_{\mathbb{R}^N}\Gamma_k(x,y,\tfrac{1}{2})\,(e^{-\Delta_k/2}p)(y)w_k(y)\,dy \quad \text{for} \quad p\in\mathscr{P}. \tag{2.10}$$

This proves (2.9) for $t=\frac{1}{2}$, as $e^{-\Delta_k/2}$ is the inverse of $e^{\Delta_k/2}$ on $\mathscr{P}$. The general case $t>0$ follows by renormalization (see Lemma 2.1 of [16]).

We next turn to a probabilistic interpretation of the generalized heat kernels:

2.6. k-Gaussian semigroups

The k-Gaussian distribution $P_t^\Gamma(x,.)\in M^1(\mathbb{R}^N)$ with "center" $x\in\mathbb{R}^N$ and "variance parameter" $t>0$ is given by

$$P_t^\Gamma(x,A) := \int_A \Gamma_k(x,y,t)w_k(y)\,dy \quad (A\subset\mathbb{R}^N \text{ a Borel set}). \tag{2.11}$$

In particular,

$$P_t^\Gamma(0,A)=\frac{c_k}{(4t)^{\gamma+N/2}}\int_A e^{-|y|^2/4t}\,w_k(y)\,dy.$$

It follows readily from the statements of Section 2.5 that the k-Gaussian distributions $P_t^\Gamma(x,.)$ $(t>0)$ have the following properties:

(1) The Dunkl transforms of the probability measures $P_t^\Gamma(x,.)$ $(t\geq 0,\ x\in\mathbb{R}^N)$ satisfy

$$P_t^\Gamma(0,.)^\wedge(y)=\mathrm{e}^{-t|y|^2}\quad\text{and}\quad P_t^\Gamma(x,.)^\wedge(y)=K(-\mathrm{i}x,y)\cdot P_t^\Gamma(0,.)^\wedge(y)\quad\text{for } y\in\mathbb{R}^N.$$

(2) For each $t>0$, P_t^Γ is a Markov kernel on $\mathbb{R}^N$, and $(P_t^\Gamma)_{t\geq 0}$ forms a semigroup of Markov kernels on $\mathbb{R}^N$; see Section 3.5 of [18] for details.

3. Moment functions

In this section we first introduce homogeneous polynomials m_ν which generalize the monomials x^ν on $\mathbb{R}^N$. These monomials will be called generalized moment functions and will be used to define generalized moments of k-Gaussian distributions. Our approach is motivated by similar notions in [4] and references there in the setting of probability measures on hypergroups. In the sequel, a reflection group W with root system R and multiplicity function $k\geq 0$ is fixed.

3.1. Modified moment functions

As the Dunkl kernel K is analytic on $\mathbb{C}^{N\times N}$, there exist unique analytic coefficient functions m_ν $(\nu\in\mathbb{Z}_+^N)$ on $\mathbb{C}^N$ with

$$K(x,y)=\sum_{\nu\in\mathbb{Z}_+^N}\frac{m_\nu(x)}{\nu!}y^\nu\quad(x,y\in\mathbb{C}^N).\tag{3.1}$$

The restriction of m_ν to $\mathbb{R}^N$ is called the νth moment function. It is given explicitly by

$$m_\nu(x)=(\partial_y^\nu K(x,y))|_{y=0}=V(x^\nu),\tag{3.2}$$

where the first equation is clear by (3.1) and the second one follows from

$$\partial_y^\nu K(x,y))|_{y=0}=\partial_y^\nu(V_x\mathrm{e}^{\langle x,y\rangle})|_{y=0}=V_x(\partial_y^\nu\mathrm{e}^{\langle x,y\rangle}|_{y=0})=V(x^\nu);$$

see Section 2.2. In particular, the homogeneity of V ensures that $m_\nu\in\mathscr{P}_{|\nu|}$. Moreover, for each $n\in\mathbb{Z}_+$, the moment functions $(m_\nu)_{|\nu|=n}$ form a basis of the space $\mathscr{P}_n$.

We have the following Taylor-type formula involving the moment functions m_ν:

Proposition 3.1. *Let $f:\mathbb{C}^N\to\mathbb{C}$ be analytic in a neighborhood of* 0. *Then*

$$f(y)=\sum_{n=0}^\infty\sum_{|\nu|=n}\frac{m_\nu(y)}{\nu!}T^\nu f(0),$$

where the series $\sum_{n=0}^\infty$ converges absolutely and uniformly in a neighborhood of 0.

Proof. Assume first that $f\in\mathscr{P}$. As $V\mathscr{P}_n=\mathscr{P}_n$, we have $\partial^\nu f(0)=V\partial^\nu f(0)=T^\nu Vf(0)$. Thus,

$$f(y)=\sum_\nu\frac{y^\nu}{\nu!}T^\nu Vf(0)\quad\text{and}\quad(V^{-1}f)(y)=\sum_\nu\frac{y^\nu}{\nu!}T^\nu f(0),$$

which gives

$$f(y) = \sum_{\nu} \frac{m_\nu(y)}{\nu!} T^\nu f(0).$$

The assertion now follows from the corresponding results for the classical case. □

Using the modified moment functions m_ν, we next introduce the modified moments of the k-Gaussian measures $P_t^\Gamma(x,.)$.

3.2. Modified moments of k-Gaussian measures

For $t>0$, $x\in\mathbb{R}^N$ and $\nu\in\mathbb{Z}_+^N$, the νth modified moment of $P_t^\Gamma(x,.)$ is defined by

$$m_\nu(P_t^\Gamma(x,.)) := \int_{\mathbb{R}^N} m_\nu \, \mathrm{d}P_t^\Gamma(x,.). \tag{3.3}$$

These modified moments are closely related to the classical moments of the classical normal distributions on $\mathbb{R}^N$:

Lemma 3.2. *Let* $\nu\in\mathbb{Z}_+^N$, $t>0$, *and* $x\in\mathbb{R}^N$. *Then*:

(1) $m_\nu(P_t^\Gamma(x,.)) = \mathrm{i}^{|\nu|}\partial_y^\nu P_t^\Gamma(x,.)^\wedge(y)|_{y=0} = \mathrm{i}^{|\nu|}\partial_y^\nu(K(x,-\mathrm{i}y)\cdot \mathrm{e}^{-t|y|^2})|_{y=0}$;

(2) $m_\nu(P_t^\Gamma(x,.)) = \sum_{\rho\leqslant\nu} \binom{\nu}{\rho} m_\rho(P_t^\Gamma(0,.)) m_{\nu-\rho}(x)$;

(3) $m_\nu(P_t^\Gamma(0,.)) = \begin{cases} \frac{(2\mu)!}{\mu!} t^{|\mu|} & \textit{if } \nu = 2\mu \textit{ for some } \mu\in\mathbb{Z}_+^N, \\ 0 & \textit{otherwise.} \end{cases}$

Proof. (1) The first equation is obtained from (3.2) and inductive use of the dominated convergence theorem (which is applicable by Theorem 2.2(4)). The second assertion is clear.

(2) By Part (1), Eq. (3.2) and the Leibniz rule for partial derivatives of products we obtain

$$\begin{aligned} m_\nu(P_t^\Gamma(x,.)) &= \mathrm{i}^{|\nu|} \sum_{\rho\in\mathbb{Z}_+^N,\rho\leqslant\nu} \binom{\nu}{\rho} \partial_y^\rho(\mathrm{e}^{-t|y|^2})|_{y=0} \partial_y^{\nu-\rho} K(x,-\mathrm{i}y)|_{y=0} \\ &= \sum_{\rho\leqslant\nu} \binom{\nu}{\rho} m_\rho(P_t^\Gamma(0,.)) m_{\nu-\rho}(x). \end{aligned} \tag{3.4}$$

(3) This follows from Part (1) for $x=0$ and the power series of the exponential function. □

4. Appell systems for k-Gaussian semigroups

Based on the moment functions of the previous section and certain generating functions, we now construct two systems $(R_\nu)_{\nu\in\mathbb{Z}_+^N}$ and $(S_\nu)_{\nu\in\mathbb{Z}_+^N}$ of polynomials on $\mathbb{R}\times\mathbb{R}^N$ associated with the

k-Gaussian semigroup $(P_t^\Gamma)_{t\geqslant 0}$. These systems, called Appell characters and cocharacters, satisfy several algebraic relations, the most important being the biorthogonality established in Theorem 4.3 below. Among other results, we present a new proof for a generalization of a formula of Macdonald [11] due to Dunkl [7]. Our approach and notations are motivated by related concepts in non-Gaussian white-noise-analysis (see [1, 3]) and the umbral calculus in [14].

4.1. Appell characters

As the Dunkl kernel K is analytic, we have a power series expansion of the form

$$\frac{K(x,-\mathrm{i}y)}{P_t(0,.)^\wedge(y)} = K(x,-\mathrm{i}y)\mathrm{e}^{t|y|^2} = \sum_{\nu\in\mathbb{Z}_+^N} \frac{(-\mathrm{i}y)^\nu}{\nu!} R_\nu(t,x) \quad \text{for } t\geqslant 0 \text{ and } x,\ y\in\mathbb{R}^N \tag{4.1}$$

with certain functions R_ν on $[0,\infty[\times\mathbb{R}^N$. Analogous to (3.4), these are given by

$$R_\nu(t,x) = \mathrm{i}^{|\nu|}\partial_y^\nu(K(x,-\mathrm{i}y)\mathrm{e}^{t|y|^2})|_{y=0} = \sum_{\rho\leqslant\nu}\binom{\nu}{\rho} a_{\nu-\rho}(t)m_\rho(x), \tag{4.2}$$

where

$$a_\lambda(t) := \mathrm{i}^{|\lambda|}\partial_y^\lambda(\mathrm{e}^{t|y|^2})|_{y=0} = \begin{cases} \frac{(2\mu)!}{\mu!}(-t)^{|\mu|} & \text{if } \lambda=2\mu \text{ for } \mu\in\mathbb{Z}_+^N,\\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

In particular, the R_ν are real polynomials in the $(N+1)$ variables (t,x) of degree $|\nu|$, and after analytic continuation, $R_\nu(t,.)$ is a polynomial of degree $|\nu|$ for each $t\in\mathbb{R}$. The polynomials R_ν are called the *Appell characters* associated with the semigroup $(P_t^\Gamma)_{t\geqslant 0}$.

We next collect some properties and examples of Appell characters.

Lemma 4.1. *In the setting of Section* 4.1, *the following holds for all* $\nu\in\mathbb{Z}_+^N$:

(1) *Inversion formula*: *For all* $x\in\mathbb{R}^N$ *and* $t\in\mathbb{R}$,

$$m_\nu(x) = \sum_{\rho\in\mathbb{Z}_+^N,\,\rho\leqslant\nu}\binom{\nu}{\rho} a_{\nu-\rho}(-t)R_\rho(t,x).$$

(2) *For all* $t\in\mathbb{R}$ *and* $n\in\mathbb{Z}_+$, *the family* $(R_\nu(t,.))_{\nu\in\mathbb{Z}_+^N,|\nu|\leqslant n}$ *is a basis of the space* $\bigoplus_{j=0}^n\mathscr{P}_j$ *of all polynomials of degree at most n.*

(3) *For* $x\in\mathbb{R}^N$ *and* $t\geqslant 0$,

$$\int_{\mathbb{R}^N} R_\nu(t,y)\,\mathrm{d}P_t^\Gamma(x,.)(y) = m_\nu(x).$$

(4) *For* $t>0$ *and* $x\in\mathbb{R}^N$, $R_\nu(t,x) = \sqrt{t}^{|\nu|}R_\nu(1,x/\sqrt{t})$.

(5) *For all* $x\in\mathbb{R}^N$, $t\in\mathbb{R}$ *and* $j\in\{1,\ldots,N\}$,

$$T_jR_{\nu+e_j}(t,x) = (\nu_j+1)R_\nu(t,x),$$

here the Dunkl operator T_j *acts with respect to the variable x and* $e_j\in\mathbb{Z}_+^N$ *is the jth unit vector.*

Proof. (1) Using (4.1) and (4.3), we obtain

$$K(x,-\mathrm{i}y) = \mathrm{e}^{-t|y|^2}\cdot(\mathrm{e}^{t|y|^2}K(x,-\mathrm{i}y)) = \left(\sum_{\lambda\in\mathbb{Z}_+}\frac{a_\lambda(-t)}{\lambda!}(-\mathrm{i}y)^\lambda\right)\left(\sum_{\rho\in\mathbb{Z}_+}\frac{(-\mathrm{i}y)^\rho}{\rho!}R_\rho(t,x)\right)$$

$$= \sum_{\nu\in\mathbb{Z}_+}\left(\sum_{\rho\leqslant\nu}\binom{\nu}{\rho}a_{\nu-\rho}(-t)R_\rho(t,x)\right)\frac{(-\mathrm{i}y)^\nu}{\nu!}.$$

A comparision of this expansion with Eq. (3.1) leads to Part (1).

(2) This follows from Part (1) of this lemma, and the fact that $(m_\nu)_{|\nu|=j}$ is a basis of $\mathscr{P}_j$.

(3) Comparison of formula (4.3) and Lemma 3.2(3) shows that $m_\lambda(P_t^\Gamma(0,.)) = a_\lambda(-t)$. Hence, by (4.2) and Lemma 3.2(2),

$$\int_{\mathbb{R}^N} R_\nu(t,y)\,\mathrm{d}P_t^\Gamma(x,.)(y) = \sum_{\rho\leqslant\nu}\binom{\nu}{\rho}a_{\nu-\rho}(t)\int_{\mathbb{R}^N} m_\rho(y)\,\mathrm{d}P_t^\Gamma(x,.)(y)$$

$$= \sum_{\rho\leqslant\nu}\binom{\nu}{\rho}a_{\nu-\rho}(t)\left(\sum_{\lambda\leqslant\rho}\binom{\nu}{\rho}m_\lambda(P_t^\Gamma(0,.))m_{\rho-\lambda}(x)\right)$$

$$= \sum_{\rho\leqslant\nu}\binom{\nu}{\rho}a_{\nu-\rho}(t)R_\rho(-t,x).$$

The assertion now follows from the inversion formula of Part (1).

(4) This is a consequence of the homogeneity of the moment functions m_ν and of (4.2) and (4.3).

(5) Note first that by (3.2) and the intertwining property of V,

$$T_j m_{\nu+e_j} = (\nu_j+1)m_\nu \quad (j=1,\ldots,N,\ \nu\in\mathbb{Z}_+^N).$$

This, together with identity (4.2) and Proposition 3.1, yields

$$T_jR_{\nu+e_j}(t,x) = \sum_{\rho\leqslant\nu+e_j}\binom{\nu+e_j}{\rho}T_jm_\rho(x)a_{\nu+e_j-\rho}(t) = \sum_{\rho\leqslant\nu}\binom{\nu+e_j}{\rho+e_j}(\rho_j+1)m_\rho(x)a_{\nu-\rho}(t)$$

$$= (\nu_j+1)\cdot\sum_{\rho\leqslant\nu}\binom{\nu}{\rho}m_\rho(x)a_{\nu-\rho}(t) = (\nu_j+1)R_\nu(t,x). \qquad \square$$

Example 4.2. (1) In the classical case $k=0$ with $m_\nu(x) := x^\nu$, Eq. (4.3) leads to

$$R_\nu(t,x) = \sqrt{t}^{|\nu|}\tilde{H}_\nu\left(\frac{x}{2\sqrt{t}}\right) \quad (x\in\mathbb{R}^N,\ \nu\in\mathbb{Z}_+^N,\ t\in\mathbb{R}\setminus\{0\}), \tag{4.4}$$

where the $\tilde{H}_\nu$ are the classical, N-variable Hermite polynomials defined by

$$\tilde{H}_\nu(x) = \prod_{i=1}^N H_{\nu_i}(x_i) \quad \text{with } H_n(y) = \sum_{j=0}^{\lfloor n/2\rfloor}\frac{(-1)^j n!}{j!(n-2j)!}(2y)^{n-2j};$$

c.f. Section 5.5 of [20] for the one-dimensional case.

(2) If $N=1$, $W=\mathbb{Z}_2$ and $k\geqslant 0$, then Example 2.1(2) easily leads to an explicit formula for the moment functions m_n. Using then (4.2), we finally obtain

$$R_{2n}(t,x)=(-1)^n 2^{2n} n!\, t^n L_n^{(k-1/2)}(x^2/4t)$$

and

$$R_{2n+1}(t,x)=(-1)^n 2^{2n+1} n!\, t^n x L_n^{(k+1/2)}(x^2/4t)$$

for $n\in\mathbb{Z}_+$; here the $L_n^{(\alpha)}$ are the Laguerre polynomials (see Section 5.1 of [20]) given by

$$L_n^{(\alpha)}(x)=\frac{1}{n!}x^{-\alpha}e^x\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x})=\sum_{j=0}^{n}\binom{n+\alpha}{n-j}\frac{(-x)^j}{j!}.$$

The polynomials $(R_n)_{n\geqslant 0}$ are called generalized Hermite polynomials and studied e.g. in [19]. For each $t>0$ the polynomials $(R_n(t,.))_{n\geqslant 0}$ are orthogonal with respect to the k-Gaussian measure

$$dP_t^{\Gamma}(0,.)(x)=\frac{\Gamma(k+1/2)}{(4t)^{k+1/2}}\,|x|^{2k}e^{-x^2/4t}\,dx.$$

An uninformed reader might suggest from these examples that k-Gaussian Appell characters are always orthogonal with repect to $P_t^{\Gamma}(0,.)$ for $t>0$. This is, however, not correct in general for the S_N- and B_N-cases; see Section 8 of [18]. To overcome this problem, we introduce the so-called Appell cocharacters, which turn out to form biorthogonal systems for the Appell characters.

4.2. Appell cocharacters

The noncentered k-Gaussian measures $P_t^{\Gamma}(x,.)$ admit $P_t^{\Gamma}(0,.)$-densities $\theta_t(x,.)$ for $t>0$, $x\in\mathbb{R}^N$. These densities are given by

$$\theta_t(x,y):=\frac{dP_t^{\Gamma}(x,.)(y)}{dP_t^{\Gamma}(0,.)(y)}=e^{-|x|^2/4t}K(x,y/2t)=\sum_{n=0}^{\infty}\sum_{|\nu|=n}\frac{m_\nu(x)}{\nu!}S_\nu(t,y), \tag{4.5}$$

where in view of Proposition 3.1 the coefficients S_ν are given by

$$S_\nu(t,y)=T_x^{\nu}(e^{-|x|^2/4t}K(x,y/2t))|_{x=0}.$$

We mention that the function θ_t (with $t=1$) appears also as the generating function of the generalized Hermite polynomials associated with W and k in [16]. By Proposition 3.8 of [16], the convergence of the series $\sum_{n=0}^{\infty}$ in (4.5) is locally uniform on $\mathbb{C}^N\times\mathbb{C}^N$. Just as the R_ν, the functions S_ν are polynomials of degree $|\nu|$; they are called the *Appell cocharacters* of the k-Gaussian semigroup $(P_t^{\Gamma})_{t\geqslant 0}$.

Using the homogeneity of m_ν, we obtain the following analogue of Lemma 4.1(4):

$$S_\nu(t,y)=\left(\frac{1}{\sqrt{t}}\right)^{|\nu|}S_\nu(1,y/\sqrt{t})\quad(t>0). \tag{4.6}$$

As announced, Appell characters and cocharacters have the following biorthogonality property:

Theorem 4.3. *Let $t>0$, $\nu,\rho\in\mathbb{Z}_+^N$, and $p\in\mathscr{P}$ a polynomial of degree less than $|\nu|$. Then:*

(1) $\displaystyle\int_{\mathbb{R}^N} R_\nu(t,y)\,S_\rho(t,y)\,\mathrm{d}P_t^\Gamma(0,.)(y)=\nu!\,\delta_{\nu,\rho}.$

As a consequence, the family $(S_\nu(t,.))_{|\nu|\leqslant n}$ is a basis of $\bigoplus_{j=0}^n\mathscr{P}_j$ for each $n\in\mathbb{Z}_+$.

(2) $\displaystyle\int_{\mathbb{R}^N} p(y)\,S_\nu(t,y)\,\mathrm{d}P_t^\Gamma(0,.)(y)=\int_{\mathbb{R}^N} p(y)\,R_\nu(t,y)\,\mathrm{d}P_t^\Gamma(0,.)(y)=0.$

Proof. We use the definition of θ_t and Lemma 4.1(3) and conclude that for $x\in\mathbb{R}^N$,

$$m_\nu(x)=\int_{\mathbb{R}^N} R_\nu(t,y)\theta_t(x,y)\,\mathrm{d}P_t^\Gamma(0,.)(y)=\int_{\mathbb{R}^N}\sum_{n=0}^\infty\sum_{|\rho|=n} R_\nu(t,y)S_\rho(t,y)\frac{m_\rho(x)}{\rho!}\,\mathrm{d}P_t^\Gamma(0,.)(y)$$

$$=\sum_{n=0}^\infty\sum_{|\rho|=n}\frac{m_\rho(x)}{\rho!}\int_{\mathbb{R}^N} R_\nu(t,y)S_\rho(t,y)\,\mathrm{d}P_t^\Gamma(0,.)(y), \tag{4.7}$$

where we still have to justify that the summation can be interchanged with the integration. For this, we restrict our attention to the case $t=\frac14$, as the general case then follows by renormalization (see Lemma 4.1(4) and formula (4.6).) We follow the proof of Proposition 3.8 of [16] and decompose $\theta_{1/4}(x,y)$ into its x-homogeneous parts:

$$\theta_{1/4}(x,y)=\sum_{n=0}^\infty L_n(y,x)\qquad\text{with}\quad L_n(y,x)=\sum_{|\rho|=n}\frac{m_\rho(x)}{\rho!}S_\rho(1/4,y).$$

The estimations of Theorem 2.2(4) imply that

$$|L_{2n}(y,x)|\leqslant\frac{|x|^{2n}}{n!}(1+2|y|^2)^n\quad(n\in\mathbb{Z}_+)$$

and a similar estimations holds if the degree is odd; for details see the proof of 3.8 in [16]. Therefore,

$$\sum_{n=0}^\infty\int_{\mathbb{R}^N}|L_n(y,x)|R_\nu(1/4,y)\,\mathrm{d}P_{1/4}^\Gamma(0,.)(y)<\infty.$$

The dominated convergence theorem now justifies the last step in (4.7) for $t=\frac14$. The biorthogonality is now a consequence of (4.7), and, in view of Lemma 4.1(2), it is also clear that $(S_\nu(t,.))_{|\nu|\leqslant n}$ is a basis of $\bigoplus_{j=0}^n\mathscr{P}_j$ for $n\in\mathbb{Z}_+$. Finally, Part (2) follows from Part (1), Lemma 4.1(2), and Section 4.2. □

The following result also reflects the dual nature of Appell characters and cocharacters.

Proposition 4.4. *Let $t\in\mathbb{R}$, $x\in\mathbb{R}^N$ and $\nu\in\mathbb{Z}_+^N$. Then*

$$R_\nu(t,x)=\mathrm{e}^{-t\Delta_k}m_\nu(x)\quad\textit{and}\quad S_\nu(t,x)=\left(\frac{1}{2t}\right)^{|\nu|}\mathrm{e}^{-t\Delta_k}x^\nu.$$

Proof. In view of Section 2.5, Lemma 4.1(3) just says that $e^{t\Delta_k}R_\nu(t,x) = m_\nu(x)$ for $t \geq 0$. This shows the first part for $t \geq 0$. As both sides are polynomials in t (write $e^{-t\Delta_k}$ as terminating power series of operators acting on polynomials!), the first equation holds generally. Let Δ_k^y be the k-Laplacian acting on the variable y, and let V_x be the intertwining operator acting on the variable x. Then

$$e^{t\Delta_k^y}\left(e^{-|x|^2/4t}K\left(x,\frac{y}{2t}\right)\right) = e^{-|x|^2/4t}e^{|x|^2/4t}K\left(x,\frac{y}{2t}\right) = K\left(x,\frac{y}{2t}\right) = V_x(e^{\langle x, y/2t\rangle}).$$

Now, consider on both sides the homogeneous part W_n of degree n in the variable x. Using the left-hand side, we obtain from (4.5) that

$$W_n = e^{t\Delta_k^y}\left(\sum_{|\nu|=n}\frac{m_\nu(x)}{\nu!}S_\nu(y,t)\right) = \sum_{|\nu|=n}\frac{m_\nu(x)}{\nu!}e^{t\Delta_k^y}S_\nu(y,t).$$

Using the right-hand side, we conclude from the identity $V_x(x^\nu) = m_\nu(x)$ that

$$W_n = V_x\left(\sum_{|\nu|=n}\frac{x^\nu}{\nu!}(y/2t)^\nu\right) = \sum_{|\nu|=n}\frac{m_\nu(x)}{\nu!}(y/2t)^\nu.$$

A comparison of the corresponding coefficients leads to the second equation. □

We now combine Theorem 4.3 and Proposition 4.4 to rediscover a generalization of a formula of Macdonald [11] due to Dunkl [7]. However, our proof is completely different from [7]. We need the following notation: For a multiplicity function $k \geq 0$, we introduce the bilinear form

$$[p,q]_k := (p(T)q)(0) \quad \text{for } p,q \in \mathscr{P}. \tag{4.8}$$

Notice that $[p,q]_k = 0$ for $p \in \mathscr{P}_n$, $q \in \mathscr{P}_m$ with $n \neq m$.

Corollary 4.5. *For all $p,q \in \mathscr{P}_n$ and $t > 0$,*

$$[p,q]_k = \frac{1}{(2t)^n}\int_{\mathbb{R}^N} e^{-t\Delta_k}(p)\, e^{-t\Delta_k}(q)\, dP_t^\Gamma(0,.).$$

In particular, $[.,.]_k$ is a scalar product on $\mathscr{P}$.

Proof. Let $t > 0$ and $\nu, \rho \in \mathbb{Z}_+^N$. Then, by Theorem 4.3(1) and Proposition 4.4,

$$\frac{1}{(2t)^{|\nu|}}\int_{\mathbb{R}^N} e^{-t\Delta_k}(x^\nu)\, e^{-t\Delta_k}(m_\rho)\, dP_t^\Gamma(0,.) = \nu! \cdot \delta_{\nu,\rho}.$$

On the other hand, as V acts on $\mathscr{P}$ in a homogeneous way,

$$[x^\nu, m_\rho]_k = (T^\nu m_\rho)(0) = (T^\nu V x^\rho)(0) = (V\partial^\nu x^\rho)(0) = \nu! \cdot \delta_{\nu,\rho}. \tag{4.9}$$

This yields the first statement. The second statement is clear. □

We give a further application of Theorem 4.3 for $t=\frac{1}{2}$. For this, we use the adjoint operator T_j^* of the Dunkl operator T_j $(j=1,\dots,N)$ in $L^2(\mathbb{R}^N, \mathrm{d}P_{1/2}^\Gamma(0,.))$, which is given by

$$T_j^* f(x) = x_j f(x) - T_j f(x) = -\mathrm{e}^{|x|^2/2} T_j(\mathrm{e}^{-|x|^2/2} f(x)) \quad (f\in\mathscr{P}); \tag{4.10}$$

see Lemma 3.7 of [7]. (The second equation is a consequence of the product rule of Section 2.2).

Corollary 4.6. *For all* $\nu\in\mathbb{Z}_+^N$, $j=1,\dots,N$, $x\in\mathbb{R}^N$ *and* $t>0$,
(1) $S_{\nu+e_j}(\frac{1}{2},x) = T_j^* S_\nu(\frac{1}{2},x)$;
(2) *Rodriguez formula*: $S_\nu(t,x) = (-1)^{|\nu|}\mathrm{e}^{|x|^2/4t}\cdot T^\nu(\mathrm{e}^{-|x|^2/4t})$.

Proof. Using Theorem 4.3(1) and Lemma 4.1(5), we obtain for all $\rho\in\mathbb{Z}_+^N$ that

$$\begin{aligned}\int_{\mathbb{R}^N} R_{\rho+e_j}\cdot T_j^* S_\nu\,\mathrm{d}P_{1/2}^\Gamma(0,.) &= \int_{\mathbb{R}^N} T_j R_{\rho+e_j} S_\nu\,\mathrm{d}P_{1/2}^\Gamma(0,.) = (\rho_j+1)\int_{\mathbb{R}^N} R_\rho S_\nu\,\mathrm{d}P_{1/2}^\Gamma(0,.)\\ &= \delta_{\rho,\nu}\cdot(\rho+e_j)! = \int_{\mathbb{R}^N} R_{\rho+e_j} S_{\nu+e_j}\,\mathrm{d}P_{1/2}^\Gamma(0,.).\end{aligned}$$

As $\mathscr{P}$ is dense in $L^2(\mathbb{R}^N,\mathrm{d}P_{1/2}^\Gamma(0,.))$, this implies Part (1). Part (2) for $t=\frac{1}{2}$ now follows from (4.10), and the general case is a consequence of formula (4.6). □

Remark 4.7. In Example 4.2 (i.e., in the classical case $k=0$ and in the one-dimensional case), Proposition 4.4 implies that the systems $(S_\nu(t,.))_{\nu\in\mathbb{Z}_+^N}$ and $(R_\nu(t,.))_{\nu\in\mathbb{Z}_+^N}$ coincide – up to multiplicative factors which depend on t and ν. Recall also that in these cases they are orthogonal with respect to $P_t^\Gamma(0,.)$, and can be considered as generalized Hermite polynomials. In general, however, both systems fail to be orthogonal. In any case, Corollary 4.5 allows to introduce orthogonal polynomials with respect to $P_t^\Gamma(0,.)$. For this, one has to choose an orthogonal basis $(\varphi_\nu)_{\nu\in\mathbb{Z}_+^N}\subset\mathscr{P}$ with respect to the scalar product $[.,.]_k$ in (4.8) with $\varphi_\nu\in\mathscr{P}_{|\nu|}$ for $\nu\in\mathbb{Z}_+^N$. (Note that by (4.9), $\mathscr{P}_n\perp\mathscr{P}_m$ for $n\neq m$.) It is then clear from Corollary 4.5 that $(H_\nu := \mathrm{e}^{-t\Delta_k}\varphi_\nu)_{\nu\in\mathbb{Z}_+^N}$ forms a system of orthogonal polynomials with respect to $P_t^\Gamma(0,.)$. These generalized Hermite polynomials are studied (for $t=\frac{1}{4}$) in [16], and their relations to the Appell systems $(R_\nu)_{\nu\in\mathbb{Z}_+^N}$ and $(S_\nu)_{\nu\in\mathbb{Z}_+^N}$ are studied in [18]. We also refer to related investigations in [2, 5] and references given there.

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