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A local homology theory for linearly compact modules

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Abstract

We introduce a local homology theory for linearly compact modules which is in some sense dual to the local cohomology theory of A. Grothendieck [A. Grothendieck, Local Cohomology, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin/Tokyo/New York, 1967. [10]]. Some basic properties such as the noetherianness, the vanishing and non-vanishing of local homology modules of linearly compact modules are proved. A duality theory between local homology and local cohomology modules of linearly compact modules is developed by using Matlis duality and Macdonald duality. As consequences of the duality theorem we obtain some generalizations of well-known results in the theory of local cohomology for semi-discrete linearly compact modules.
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1. Introduction

Although the theory of local cohomology has been enveloped rapidly for the last 40 years and proved to be a very important tool in algebraic geometry and commutative algebra, not so much is known about the theory of local homology. First, E. Matlis in [17,18] studied the left derived functors $L^I_\bullet(-)$ of the I-adic completion functor $\Lambda_I(-) = \varprojlim_t (R/I^t \otimes_R -)$, where the ideal I was generated by a regular sequence in a local noetherian ring R and proved some duality

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between this functor and the local cohomology functor by using a duality which is called today the Matlis dual functor.

Next, Simon in [24] suggested to investigate the module $L_i^I(M)$ when I is an arbitrary ideal of a noetherian ring. Later, J.P.C. Greenlees and J.P. May [9] use the homotopy colimit, or telescope, of the cochain of Koszul complexes to define so-called local homology groups of a module M by

$$H_\bullet^I(M) = H_\bullet\big(\mathrm{Hom}\big(\mathrm{Tel}\, K^\bullet\big(\underline{x}^t\big), M\big)\big),$$

where $\underline{x}$ is a finitely generated system of I and they showed, under some condition on $\underline{x}$ which are automatically satisfied when R is noetherian, that the left derived functors $L_\bullet^I(-)$ of the I-adic completion can be computed in terms of these local homology groups. Then came the work of L. Alonso Tarrío, A. Jeremias López and J. Lipman [1], they gave in that paper a sheafified derived-category generalization of Greenlees–May results for a quasi-compact separated scheme. Note that a strong connection between local cohomology and local homology was shown in [1,8] and [9]. Recently in [6], we defined the ith *local homology module* $H_i^I(M)$ of an R-module M with respect to the ideal I by

$$H_i^I(M) = \varprojlim_t \mathrm{Tor}_i^R\big(R/I^t, M\big).$$

We also proved in [6] many basic properties of local homology modules and that $H_i^I(M) \cong L_i^I(M)$ when M is artinian. Hence we can say that there exists, on the noetherian local ring, a theory for the left derived functors $L_\bullet^I(-)$ of the I-adic completion (as the local homology functors) on the category of artinian modules parallel to the theory of local cohomology functors on the category of noetherian modules. However, while the local cohomology functors $H_I^\bullet(-)$ are still defined as the right derived functors of the I-torsion functor $\Gamma_I(-) = \varinjlim_t \mathrm{Hom}_R(R/I^t, -)$ for not finitely generated modules, our definition of local homology module above may not coincide with $L_i^I(M)$ in this case. One of the most important reasons is that, even if the ring R is noetherian, the I-adic completion functor $\Lambda_I(-)$ is neither left nor right exact on the category of all R-modules. Fortunately, it was shown by results of C.U. Jensen in [11] that the inverse limit functors and therefore the local homology functors still have good behavior on the category of linearly compact modules. The purpose of this paper is towards a local homology theory for linearly compact modules. It should be mentioned that the concept of linearly compact spaces was first introduced by Lefschetz [14] for vector spaces of infinite dimension and it was then generalized for modules by D. Zelinsky [29] and I.G. Macdonald [16]. It was also studied by other authors such as H. Leptin [15], C.U. Jensen [11], H. Zöschinger [30,31], ... The class of linearly compact modules is very large, it contains many important classes of modules such as the class of artinian modules, or the class of finitely generated modules over a complete ring.

The organization of our paper is as follows. In Section 2 we recall the concepts of linearly compact and semi-discrete linearly compact modules by using the terminology of Macdonald [16] and their basic facts. For any R-module N and a linearly compact R-module M we show that there exists uniquely a topology induced by a free resolution of N for $\mathrm{Ext}_R^i(N, M)$, and in addition, if N is finitely generated, for $\mathrm{Tor}_i^R(N, M)$; moreover these modules are linearly compact.

In Section 3 we present some basic properties of local homology modules of linearly compact modules such as the local homology functor $H_i^I(-)$ is closed in the category of linearly compact modules (Proposition 3.3). Proposition 3.5 shows that our definition of local homology modules can be identified with the definition of local homology modules of J.P.C. Greenlees and J.P. May [9, 2.4] in the category of linearly compact modules.

In Section 4 we study the vanishing and non-vanishing of local homology modules. Let M be a linearly compact R-module with $\operatorname{Ndim} M = d$, then $H_i^I(M) = 0$ for all $i > d$ (Theorem 4.8). It was proved in [6, 4.8, 4.10] that $\operatorname{Ndim} M = \max\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\}$ if M is an artinian module over a local ring $(R, \mathfrak{m})$, where $\operatorname{Ndim} M$ is the noetherian dimension defined by N.R. Roberts [21] (see also [13]). Unfortunately, as in a personal communication of H. Zöschinger, he gave us the existence of semi-discrete linearly compact modules K of noetherian dimension 1 such that $H_i^{\mathfrak{m}}(K) = 0$ for all non-negative integers i. However, we can prove in Theorem 4.10 that the above equality still holds for semi-discrete linearly compact modules with $\operatorname{Ndim} M \neq 1$, moreover $\operatorname{Ndim} \Gamma_{\mathfrak{m}}(M) = \max\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\}$ if $\Gamma_{\mathfrak{m}}(M) \neq 0$.

In Section 5 we show that local homology modules $H_i^{\mathfrak{m}}(M)$ of a semi-discrete linearly compact module M over a noetherian local ring $(R, \mathfrak{m})$ are noetherian modules on the $\mathfrak{m}$-adic completion $\widehat{R}$ of R (Theorem 5.2). On the other hand, for any ideal I, $H_d^I(M)$ is a noetherian $\Lambda_I(R)$-module provided M is a semi-discrete linearly compact R-module with the noetherian dimension $\operatorname{Ndim} M = d$ (Theorem 5.3).

Section 6 is devoted to study duality. In this section $(R, \mathfrak{m})$ is a noetherian local ring and the topology on R is the $\mathfrak{m}$-adic topology. Let $E(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$ and M a Hausdorff linearly topologized R-module. Then the *Macdonald dual* M^* of M is defined by $M^* = \operatorname{Hom}(M, E(R/\mathfrak{m}))$ the set of continuous homomorphisms of R-modules. Note by Macdonald [16, 5.8] that M is a semi-discrete module if and only if $D(M) = M^*$, where $D(M) = \operatorname{Hom}(M, E(R/\mathfrak{m}))$ is the Matlis dual of M. The main result of this section is Theorem 6.4 which gives a duality between local cohomology modules and local homology modules.

In the last section, based on the duality Theorem 6.4 and the properties of local homology modules in previous sections we can extend some well-known properties of local cohomology of finitely generated modules for semi-discrete linearly compact modules.

Throughout this paper, R is a commutative noetherian ring with non-zero identity; the terminology "isomorphism" means "algebraic isomorphism" and "topological isomorphism" means "algebraic isomorphism with the homomorphisms (and its inverse) are continuous." For basic properties of commutative algebra and homological algebra we refer the reader to the books [4] and [25].

2. Linearly compact modules

First we recall the concept of linearly compact modules by using the terminology of I.G. Macdonald [16] and some of their basic properties. Let M be a topological R-module. A *nucleus* of M is a neighborhood of the zero element of M, and a *nuclear base* of M is a base for the nuclei of M. If N is a submodule of M which contains a nucleus then N is open (and therefore closed) in M and M/N is discrete. M is Hausdorff if and only if the intersection of all the nuclei of M is 0. M is said to be *linearly topologized* if M has a nuclear base $\mathcal{M}$ consisting of submodules.

Definition 2.1. A Hausdorff linearly topologized R-module M is said to be *linearly compact* if M has the following property: if $\mathcal{F}$ is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in $\mathcal{F}$ have a non-empty intersection.

It should be noted that an artinian R-module is linearly compact with the discrete topology (see [16, 3.10]).

Remark 2.2. Let M be an R-module. If $\mathcal{M}$ is a family of submodules of M satisfying the conditions:

(i) For all $N_1, N_2 \in \mathcal{M}$ there is an $N_3 \in \mathcal{M}$ such that $N_3 \subseteq N_1 \cap N_2$.
(ii) For an element $x \in M$ and $N \in \mathcal{M}$ there is a nucleus U of R such that $Ux \subseteq N$, then $\mathcal{M}$ is a base of a linear topology on M (see [16, 2.1]).

The following properties of linearly compact modules are often used in this paper.

Lemma 2.3. *(See [16, §3].)*

(i) *Let M be a Hausdorff linearly topologized R-module, N a closed submodule of M. Then M is linearly compact if and only if N and M/N are linearly compact.*
(ii) *Let $f : M \to N$ be a continuous homomorphism of Hausdorff linearly topologized R-modules. If M is linearly compact, then $f(M)$ is linearly compact and therefore f is a closed map.*
(iii) *If $\{M_i\}_{i\in I}$ is a family of linearly compact R-modules, then $\prod_{i\in I} M_i$ is linearly compact with the product topology.*
(iv) *The inverse limit of a system of linearly compact R-modules and continuous homomorphisms is linearly compact with the obvious topology.*

Lemma 2.4. *(See [11, 7.1].) Let $\{M_t\}$ be an inverse system of linearly compact modules with continuous homomorphisms. Then $\varprojlim_t^i M_t = 0$ for all $i > 0$. Therefore, if*

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

is a short exact sequence of inverse systems of R-modules, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact.

Let M be a linearly compact R-module and F a free R-module with a base $\{e_i\}_{i\in J}$. We can define the topology on $\operatorname{Hom}_R(F, M)$ as the product topology via the isomorphism $\operatorname{Hom}_R(F, M) \cong M^J$, where $M^J = \prod_{i\in J} M_i$ with $M_i = M$ for all $i \in J$. Then $\operatorname{Hom}_R(F, M)$ is a linearly compact R-module by 2.3(iii). Moreover, if $h : F \to F'$ is a homomorphism of free R-modules, the induced homomorphism $h^* : \operatorname{Hom}_R(F', M) \to \operatorname{Hom}_R(F, M)$ is continuous by [11, 7.4]. Let now

$$\mathbf{F}_\bullet : \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

a free resolution of an R-module N. Then $\operatorname{Ext}^i_R(N, M)$ is a linearly topologized R-module with the quotient topology of $\operatorname{Hom}(F_i, M)$. This topology on $\operatorname{Ext}^i_R(N, M)$ is called the topology induced by the free resolution $\mathbf{F}_\bullet$ of N.

Lemma 2.5. *Let M be a linearly compact R-module and N an R-module. Then for all $i \geqslant 0$, $\operatorname{Ext}^i_R(N, M)$ is a linearly compact R-module with the topology induced by a free resolution of N and this topology is independent of the choice of free resolutions of N. Moreover, if $f : N \to N'$ is* *a homomorphism of R-modules, then the induced homomorphism* $\mathrm{Ext}^i_R(N', M) \to \mathrm{Ext}^i_R(N, M)$ *is continuous.*

Proof. Let $\mathbf{F}_\bullet$ be a free resolution of N. It follows as above that $\mathrm{Hom}_R(\mathbf{F}_\bullet, M)$ is a complex of linearly compact modules with continuous homomorphisms. Therefore $\mathrm{Ext}^i_R(N, M) = H^i(\mathrm{Hom}_R(\mathbf{F}_\bullet, M))$ is linearly compact by 2.3(i), (ii). Let now $\mathbf{G}_\bullet$ be a second free resolution of N. Then we get a quasi-isomorphism of complexes $\varphi_\bullet : \mathbf{F}_\bullet \to \mathbf{G}_\bullet$ lifting the identity map of N. Therefore the induced homomorphism

$$\bar{\varphi}_i : H^i\big(\mathrm{Hom}_R(\mathbf{F}_\bullet, M)\big) \longrightarrow H^i\big(\mathrm{Hom}_R(\mathbf{G}_\bullet, M)\big)$$

is a topological isomorphism by [11, 7.4] and 2.3(i), (ii) for all i. Similarly we can prove for the last statement. □

Let N be a finitely generated R-module and

$$\mathbf{F}_\bullet = \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

a free resolution of N with the finitely generated free modules. As above, we can define for a linearly compact module M a topology on $\mathrm{Tor}^R_i(N, M)$ induced from the product topology of $F_i \otimes_R M$. Then by an argument analogous to that used for the proof of Lemma 2.5, we get the following lemma.

Lemma 2.6. *Let N be a finitely generated R-module and M a linearly compact R-module. Then $\mathrm{Tor}^R_i(N, M)$ is a linearly compact R-module with the topology induced by a free resolution of N (consisting of finitely generated free modules) and this topology is independent of the choice of free resolutions of N. Moreover, if $f : N \to N'$ is a homomorphism of finitely generated R-modules, then the induced homomorphism $\psi_{i,M} : \mathrm{Tor}^R_i(N, M) \to \mathrm{Tor}^R_i(N', M)$ is continuous.*

The next result is often used in the sequel.

Lemma 2.7. *Let N be a finitely generated R-module and $\{M_t\}$ an inverse system of linearly compact R-modules with continuous homomorphisms. Then for all $i \geqslant 0$, $\{\mathrm{Tor}^R_i(N, M_t)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover, we have*

$$\mathrm{Tor}^R_i\Big(N, \varprojlim_t M_t\Big) \cong \varprojlim_t \mathrm{Tor}^R_i(N, M_t).$$

Proof. Let $\mathbf{F}_\bullet$ be a free resolution of N with finitely generated free R-modules. Since $\{M_t\}$ is an inverse system of linearly compact modules with continuous homomorphisms, $\{F_i \otimes_R M_t\}$ forms an inverse system of linearly compact modules with continuous homomorphisms for all $i \geqslant 0$ by 2.3(iii). Then $\{\mathrm{Tor}^R_i(N, M_t)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Moreover

$$\mathbf{F}_\bullet \otimes_R \varprojlim_t M_t \cong \varprojlim_t (\mathbf{F}_\bullet \otimes_R M_t),$$

since the inverse limit commutes with the direct product and

$$H_i\Big(\varprojlim_t(\mathbf{F}_\bullet \otimes_R M_t)\Big) \cong \varprojlim_t H_i(\mathbf{F}_\bullet \otimes_R M_t)$$

by 2.4 and [19, 6.1, Theorem 1]. This finishes the proof. □

A Hausdorff linearly topologized R-module M is called *semi-discrete* if every submodule of M is closed. Thus a discrete R-module is semi-discrete. The class of semi-discrete linearly compact modules contains all artinian modules.

Moreover, a finitely generated module M over a complete local ring is a semi-discrete linearly compact modules. Indeed, since $R = \varprojlim_t R/\mathfrak{m}^t$ and each $R/\mathfrak{m}^t$ is artinian, R is semi-discrete linearly compact with the $\mathfrak{m}$-adic topology. Since M is a finitely generated R-module, we have an open continuous epimorphism $R^t \to M$. Therefore M is semi-discrete and linearly compact (see [16, 7.3]). Thus the class of semi-discrete linearly compact modules contains also all finitely generated modules over a complete ring. It should be mentioned here that our notions of linearly compact and semi-discrete modules follow Macdonald's definitions in [16]. Therefore the notion of linearly compact modules defined by H. Zöschinger in [31] is different to our notion of linearly compact modules, but it is coincident with the terminology of semi-discrete linearly compact modules in this paper.

Denote by $L(M)$ the sum of all artinian submodules of M, we have the following properties of semi-discrete linearly compact modules.

Lemma 2.8. *(See [31, 1(L5)].) Let M be a semi-discrete linearly compact R-module. Then $L(M)$ is an artinian module.*

We now recall the concept of *co-associated primes* of a module (see [5,27,31]). A prime ideal $\mathfrak{p}$ is called *co-associated* to a non-zero R-module M if there is an artinian homomorphic image L of M with $\mathfrak{p} = \operatorname{Ann}_R L$. The set of all co-associated primes to M is denoted by $\operatorname{Coass}_R(M)$. M is called $\mathfrak{p}$-*coprimary* if $\operatorname{Coass}_R(M) = \{\mathfrak{p}\}$. A module is called *sum-irreducible* if it cannot be written as a sum of two proper submodules. A sum-irreducible module M is $\mathfrak{p}$-coprimary, where $\mathfrak{p} = \{x \in R/xM \neq M\}$ (see [5, 2]).

Lemma 2.9. *(See [31, 1(L3), (L4)].) Let M be a semi-discrete linearly compact R-module. Then M can be written as a finite sum of sum-irreducible modules and therefore the set* $\operatorname{Coass}(M)$ *is finite.*

3. Local homology modules of linearly compact modules

Let I be an ideal of R, the *ith local homology* module $H_i^I(M)$ of an R-module M with respect to I is defined by (see [6, 3.1])

$$H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R\big(R/I^t, M\big).$$

It is clear that $H_0^I(M) \cong \Lambda_I(M)$, in which $\Lambda_I(M) = \varprojlim_t M/I^tM$ the I-adic completion of M.

Remark 3.1.

(i) As $I^t \operatorname{Tor}_i^R(R/I^t, M) = 0$, $\operatorname{Tor}_i^R(M/I^tM, N)$ has a natural structure as a module over the ring R/I^t for all $t > 0$. Then $H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M)$ has a natural structure as a module over the ring $\Lambda_I(R) = \varprojlim_t R/I^t$.
(ii) If M is a finitely generated R-module, then $H_i^I(M) = 0$ for all $i > 0$ (see [6, 3.2(ii)]).

Lemma 3.2. *(See [6, §3], [26].) Let I be an ideal generated by elements $x_1, x_2, \dots, x_r$ and $H_i(\underline{x}(t), M)$ the ith Koszul homology module of M with respect to the sequence $\underline{x}(t) = (x_1^t, \dots, x_r^t)$. Then for all $i \geqslant 0$,*

(i) *$H_i^I(M) \cong \varprojlim_t H_i(\underline{x}(t), M)$,*
(ii) *$H_i^I(M)$ is I-separated, it means that $\bigcap_{t>0} I^t H_i^I(M) = 0$.*

Let M be a linearly compact R-module. Then $\operatorname{Tor}_i^R(R/I^t, M)$ is also a linearly compact R-module by the topology defined as in 2.6, so we have an induced topology on the local homology module $H_i^I(M)$.

Proposition 3.3. *Let M be a linearly compact R-module. Then for all $i \geqslant 0$, $H_i^I(M)$ is a linearly compact R-module.*

Proof. It follows from 2.6 that $\{\operatorname{Tor}_i^R(R/I^t, M)\}_t$ forms an inverse system of linearly compact modules with continuous homomorphisms. Hence $H_i^I(M)$ is also a linearly compact R-module by 2.3(iv). □

The following proposition shows that local homology modules can be commuted with inverse limits of inverse systems of linearly compact R-modules with continuous homomorphisms.

Proposition 3.4. *Let $\{M_s\}$ be an inverse system of linearly compact R-modules with the continuous homomorphisms. Then*

$$H_i^I\Big(\varprojlim_s M_s\Big) \cong \varprojlim_s H_i^I(M_s).$$

Proof. Note that inverse limits are commuted. Therefore

$$\begin{aligned} H_i^I\Big(\varprojlim_s M_s\Big) &= \varprojlim_t \operatorname{Tor}_i^R\Big(R/I^t, \varprojlim_s M_s\Big) \\ &\cong \varprojlim_t \varprojlim_s \operatorname{Tor}_i^R\big(R/I^t, M_s\big) \\ &\cong \varprojlim_s \varprojlim_t \operatorname{Tor}_i^R\big(R/I^t, M_s\big) = \varprojlim_s H_i^I(M_s) \end{aligned}$$

by 2.7. □

Let L_i^I be the ith left derived functor of the I-adic completion functor Λ_I. The next result shows that in case M is linearly compact, the local homology module $H_i^I(M)$ is isomorphic to the module $L_i^I(M)$, thus our definition of local homology modules can be identified with the definition of J.P.C. Greenlees and J.P. May (see [9, 2.4]).

Proposition 3.5. *Let M be a linearly compact R-module. Then*

$$H_i^I(M) \cong L_i^I(M)$$

for all $i \geqslant 0$.

Proof. For all $i \geqslant 0$ we have a short exact sequence by [9, 1.1],

$$0 \longrightarrow \varprojlim_t{}^1 \operatorname{Tor}_{i+1}^R\big(R/I^t, M\big) \longrightarrow L_i^I(M) \longrightarrow H_i^I(M) \longrightarrow 0.$$

Moreover, it follows from 2.6 that $\{\operatorname{Tor}_{i+1}^R(R/I^t, M)\}$ forms an inverse system of linearly compact modules with continuous homomorphisms. Hence, by 2.4

$$\varprojlim_t{}^1 \operatorname{Tor}_{i+1}^R\big(R/I^t, M\big) = 0$$

and the conclusion follows. □

Remark 3.6. There are possibly other modules than linearly compact one's which satisfy the isomorphism of 3.5. For example, let J be any injective R-module and M an arbitrary R-module, write $N = \operatorname{Hom}_R(M, J)$. Then, following [24, 5.6] we have

$$\operatorname{Hom}_R\big(H_I^i(M), J\big) \cong L_i^I(N).$$

On the other hand, with the same method that used in the proof of [6, 3.3(ii)] we can show that

$$\operatorname{Hom}_R\big(H_I^i(M), J\big) \cong H_i^I(N).$$

Thus $H_i^I(N) \cong L_i^I(N)$, in particular $H_i^I(J) \cong L_i^I(J)$. But, in general an injective module is not a linearly compact module.

The following corollary is an immediate consequence of 3.5.

Corollary 3.7. *Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a short exact sequence of linearly compact modules. Then we have a long exact sequence of local homology modules

$$\begin{aligned} \cdots \longrightarrow H_i^I(M') \longrightarrow H_i^I(M) \longrightarrow H_i^I(M'') \longrightarrow \cdots \\ \longrightarrow H_0^I(M') \longrightarrow H_0^I(M) \longrightarrow H_0^I(M'') \longrightarrow 0. \end{aligned}$$

The following theorem gives us a characterization of I-separated modules.

Theorem 3.8. *Let M be a linearly compact R-module. The following statements are equivalent*:

(i) *M is I-separated, it means that $\bigcap_{t>0} I^t M = 0$.*
(ii) *M is complete with respect to the I-adic topology, it means that $\Lambda_I(M) \cong M$.*
(iii) *$H_0^I(M) \cong M$, $H_i^I(M) = 0$ for all $i > 0$.*

To prove Theorem 3.8, we need the two auxiliary lemmas. The first lemma shows that local homology modules $H_i^I(M)$ are Λ_I-acyclic for all $i > 0$.

Lemma 3.9. *Let M be a linearly compact R-module. Then for all $j \geqslant 0$,*

$$H_i^I\big(H_j^I(M)\big) \cong \begin{cases} H_j^I(M), & i = 0, \\ 0, & i > 0. \end{cases}$$

Proof. It follows from 2.6 that $\{\operatorname{Tor}_j^R(R/I^t, M)\}_t$ forms an inverse system of linearly compact R-modules with the continuous homomorphisms. Then we have by 3.4 and 3.2(i),

$$\begin{aligned} H_i^I\big(H_j^I(M)\big) &= H_i^I\Big(\varprojlim_t \operatorname{Tor}_j^R\big(R/I^t, M\big)\Big) \\ &\cong \varprojlim_t H_i^I\big(\operatorname{Tor}_j^R\big(R/I^t, M\big)\big) \\ &\cong \varprojlim_t \varprojlim_s H_i\big(\underline{x}(s), \operatorname{Tor}_j^R\big(R/I^t, M\big)\big), \end{aligned}$$

in which $\underline{x} = (x_1, \ldots, x_r)$ is a system of generators of I and $\underline{x}(s) = (x_1^s, \ldots, x_r^s)$. Since $\underline{x}(s) \operatorname{Tor}_j^R(R/I^t, M) = 0$ for all $s \geqslant t$, we get

$$\varprojlim_s H_i\big(\underline{x}(s), \operatorname{Tor}_j^R\big(R/I^t, M\big)\big) \cong \begin{cases} \operatorname{Tor}_j^R(R/I^t, M), & i = 0, \\ 0, & i > 0. \end{cases}$$

This finishes the proof. $\square$

Lemma 3.10. *Let M be a linearly compact R-module. Then*

$$H_i^I\Big(\bigcap_{t>0} I^t M\Big) \cong \begin{cases} 0, & i = 0, \\ H_i^I(M), & i > 0. \end{cases}$$

Proof. From the short exact sequence of linearly compact R-modules

$$0 \longrightarrow I^t M \longrightarrow M \longrightarrow M/I^t M \longrightarrow 0$$

for all $t > 0$ we derive by 2.4 a short exact sequence of linearly compact R-modules

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \longrightarrow \Lambda_I(M) \longrightarrow 0.$$

Hence we get a long exact sequence of local homology modules

$$\cdots \longrightarrow H^I_{i+1}\big(\Lambda_I(M)\big) \longrightarrow H^I_i\Big(\bigcap_{t>0} I^t M\Big) \longrightarrow H^I_i(M) \longrightarrow H^I_i\big(\Lambda_I(M)\big) \longrightarrow \cdots$$
$$\longrightarrow H^I_1\big(\Lambda_I(M)\big) \longrightarrow H^I_0\Big(\bigcap_{t>0} I^t M\Big) \longrightarrow H^I_0(M) \longrightarrow H^I_0\big(\Lambda_I(M)\big) \longrightarrow 0.$$

The lemma now follows from 3.9. □

Proof of Theorem 3.8. (ii) ⇔ (i) is clear from the short exact sequence

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \longrightarrow \Lambda_I(M) \longrightarrow 0.$$

(i) ⇒ (iii). We have $H^I_0(M) \cong \Lambda_I(M) \cong M$. Combining 3.10 with (i) gives $H^I_i(M) \cong H^I_i(\bigcap_{t>0} I^t M) = 0$ for all $i > 0$.
(iii) ⇒ (ii) is trivial. □

From Theorem 3.8 we have the following criterion for a finitely generated module over a local noetherian ring to be linearly compact.

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a local noetherian ring and M a finitely generated R-module. Then M is a linearly compact R-module if and only if M is complete with respect to the $\mathfrak{m}$-adic topology.*

Proof. Since M is a finitely generated R-module, M is $\mathfrak{m}$-separated. Thus, if M is a linearly compact R-module, $\Lambda_{\mathfrak{m}}(M) \cong M$ by 3.8. Conversely, if M is complete in $\mathfrak{m}$-adic topology, we have $M \cong \varprojlim_t M/\mathfrak{m}^t M$. Therefore M is a linearly compact R-module by 2.3(iv), as $M/\mathfrak{m}^t M$ are artinian R-modules for all $t > 0$. □

4. Vanishing and non-vanishing of local homology modules

Recall that $L(M)$ is the sum of all artinian submodules of M and $\mathrm{Soc}(M)$ the socle of M is the sum of all simple submodules of M. The I-torsion functor Γ_I is defined by $\Gamma_I(M) = \bigcup_{t>0}(0 :_M I^t)$. To prove the vanishing and non-vanishing theorems of local homology modules, we need the following lemmas.

Lemma 4.1. *Let M be a semi-discrete linearly compact R-module. Then $H^I_0(M) = 0$ if and only if $xM = M$ for some $x \in I$.*

Proof. By [6, 2.5], $H^I_0(M) = 0$ if and only if $IM = M$. Hence the result follows from 2.9 and [5, 2.9]. □

Lemma 4.2. *Let M be a semi-discrete linearly compact R-module and* $\mathrm{Soc}(M) = 0$. *Then*

$$H^I_i(M) = 0$$

for all $i > 0$.

Proof. Combining 3.10 with 4.1 we may assume, by replacing M with $\bigcap_{t>0} I^t M$, that there is an $x \in I$ such that $xM = M$. As $\operatorname{Soc}(M) = 0$, it follows from [31, 1.6(b)] that $0 :_M x = 0$. Thus we have an isomorphism $M \overset{x}{\cong} M$. It induces an isomorphism

$$H_i^I(M) \overset{x}{\cong} H_i^I(M)$$

for all $i > 0$. By 3.2(ii), we have

$$H_i^I(M) = xH_i^I(M) = \bigcap_{t>0} x^t H_i^I(M) = 0$$

for all $i > 0$. □

Lemma 4.3. *Let M be a semi-discrete linearly compact R-module. Then there are only finitely many distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ of R such that*

$$L(M) = \bigoplus_{j=1}^{n} \Gamma_{\mathfrak{m}_j}(M).$$

Proof. By 2.8, $L(M)$ is an artinian R-module. Thus, by virtue of [23, 1.4] there are finitely many distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ of R such that

$$L(M) = \bigoplus_{j=1}^{n} \Gamma_{\mathfrak{m}_j}\big(L(M)\big) \subseteq \bigoplus_{j=1}^{n} \Gamma_{\mathfrak{m}_j}(M).$$

Therefore it remains to show that $\Gamma_{\mathfrak{m}}(M)$ is artinian for any maximal ideal $\mathfrak{m}$ of R. Indeed, there is from [31, Theorem] a short exact sequence $0 \to N \to M \to A \to 0$, where N is finitely generated and A is artinian. Then we have an exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}}(N) \longrightarrow \Gamma_{\mathfrak{m}}(M) \longrightarrow \Gamma_{\mathfrak{m}}(A).$$

Obviously, $\Gamma_{\mathfrak{m}}(A)$ is an artinian R-module, $\Gamma_{\mathfrak{m}}(N)$ is a finitely generated R-module annihilated by a power of $\mathfrak{m}$, and hence it is of finite length. So $\Gamma_{\mathfrak{m}}(M)$ is an artinian R-module as required. □

Lemma 4.4. *Let M be a semi-discrete linearly compact R-module. Then there are only finitely many distinct maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$ of R such that*

$$H_i^I(M) \cong \bigoplus_{j=1}^{n} H_i^I\big(\Gamma_{\mathfrak{m}_j}(M)\big)$$

for all $i > 0$, and the following sequence is exact

$$0 \longrightarrow \bigoplus_{j=1}^{n} H_0^I\big(\Gamma_{\mathfrak{m}_j}(M)\big) \longrightarrow H_0^I(M) \longrightarrow H_0^I\left(M \Big/ \bigoplus_{j=1}^{n} \Gamma_{\mathfrak{m}_j}(M)\right) \longrightarrow 0.$$

Proof. The short exact sequence of linearly compact R-modules

$$0 \longrightarrow L(M) \longrightarrow M \longrightarrow M/L(M) \longrightarrow 0$$

gives rise to a long exact sequence of local homology modules

$$\cdots \longrightarrow H_{i+1}^I\big(M/L(M)\big) \longrightarrow H_i^I\big(L(M)\big) \longrightarrow H_i^I(M) \longrightarrow H_i^I\big(M/L(M)\big) \longrightarrow \cdots.$$

By 4.2, $H_i^I(M/L(M)) = 0$ for all $i > 0$, as $\mathrm{Soc}(M/L(M)) = 0$. Then we get $H_i^I(M) \cong H_i^I(L(M))$ for all $i > 0$ and the short exact sequence

$$0 \longrightarrow H_0^I\big(L(M)\big) \longrightarrow H_0^I(M) \longrightarrow H_0^I\big(M/L(M)\big) \longrightarrow 0.$$

Now the conclusion follows from 4.3. □

We have an immediate consequence of 4.3 and 4.4 for the local case.

Corollary 4.5. *Let $(R, \mathfrak{m})$ be a local noetherian ring and M a semi-discrete linearly compact R-module. Then*

$$L(M) = \Gamma_{\mathfrak{m}}(M), \qquad H_i^I(M) \cong H_i^I\big(\Gamma_{\mathfrak{m}}(M)\big)$$

for all $i > 0$, and the following sequence is exact

$$0 \longrightarrow H_0^I\big(\Gamma_{\mathfrak{m}}(M)\big) \longrightarrow H_0^I(M) \longrightarrow H_0^I\big(M/\Gamma_{\mathfrak{m}}(M)\big) \longrightarrow 0.$$

We now recall the concept of *noetherian dimension* of an R-module M denoted by $\mathrm{Ndim}\, M$. Note that the notion of noetherian dimension was introduced first by R.N. Roberts [21] by the name Krull dimension. Later, D. Kirby [13] changed this terminology of Roberts and refereed to *noetherian dimension* to avoid confusion with well-known Krull dimension of finitely generated modules. Let M be an R-module. When $M = 0$ we put $\mathrm{Ndim}\, M = -1$. Then by induction, for any ordinal α, we put $\mathrm{Ndim}\, M = \alpha$ when (i) $\mathrm{Ndim}\, M < \alpha$ is false, and (ii) for every ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M, there exists a positive integer m_0 such that $\mathrm{Ndim}(M_{m+1}/M_m) < \alpha$ for all $m \geqslant m_0$. Thus M is non-zero and finitely generated if and only if $\mathrm{Ndim}\, M = 0$. If $0 \to M'' \to M \to M' \to 0$ is a short exact sequence of R-modules, then $\mathrm{Ndim}\, M = \max\{\mathrm{Ndim}\, M'', \mathrm{Ndim}\, M'\}$.

Remark 4.6.

(i) In case M is an artinian module, $\mathrm{Ndim}\, M < \infty$ (see [21]). More general, if M is a semi-discrete linearly compact module, there is a short exact sequence $0 \to N \to M \to A \to 0$ where N is finitely generated and A is artinian (see [31, Theorem]). Hence $\mathrm{Ndim}\, M = \max\{\mathrm{Ndim}\, N, \mathrm{Ndim}\, A\} < \infty$.

(ii) If M is an artinian R-module or more general, a semi-discrete linearly compact R-module, then $\mathrm{Ndim}\, M \leqslant \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Coass}(M)\}$. Especially, if M is an artinian module over a complete local noetherian ring $(R, \mathfrak{m})$, $\mathrm{Ndim}\, M = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Coass}(M)\}$ (see [28, 2.10]).

Lemma 4.7. *Let M be an R-module with* $\operatorname{Ndim} M = d > 0$ *and $x \in R$ such that $xM = M$. Then*

$$\operatorname{Ndim} 0 :_M x \leqslant d - 1.$$

Proof. Consider the ascending chain

$$0 \subseteq 0 :_M x \subseteq 0 :_M x^2 \subseteq \cdots.$$

As $\operatorname{Ndim} M = d$, there exists a positive integer n such that $\operatorname{Ndim}(0 :_M x^{n+1}/0 :_M x^n) \leqslant d - 1$. Since $xM = M$, the homomorphism $0 :_M x^{n+1}/0 :_M x^n \xrightarrow{x^n} 0 :_M x$ is an isomorphism. Therefore $\operatorname{Ndim} 0 :_M x \leqslant d - 1$. □

Theorem 4.8. *Let M be a linearly compact R-module with* $\operatorname{Ndim} M = d$. *Then*

$$H_i^I(M) = 0$$

for all $i > d$.

Proof. Let $\mathcal{M}$ be a nuclear base of M. Then, by [16, 3.11], $M = \varprojlim_{U \in \mathcal{M}} M/U$. It follows from 3.4 that

$$H_i^I(M) \cong \varprojlim_{U \in \mathcal{M}} H_i^I(M/U).$$

Note that M/U is a discrete linearly compact R-module with $\operatorname{Ndim} M/U \leqslant \operatorname{Ndim} M$. Thus we only need to prove the theorem for the case M is a discrete linearly compact R-module. Let $L(M)$ be the sum of all artinian R-submodules of M. Then $L(M)$ is artinian by 2.8. From the proof of 4.4, we have the isomorphisms

$$H_i^I(M) \cong H_i^I\big(L(M)\big)$$

for all $i > 0$. As $\operatorname{Ndim} L(M) \leqslant \operatorname{Ndim} M = d$, $H_i^I(L(M)) = 0$ for all $i > d$ by [6, 4.8] and then the proof is complete. □

Remark 4.9. In [6, 4.8, 4.10] we proved that if M is an artinian module on a local noetherian ring $(R, \mathfrak{m})$, then

$$\operatorname{Ndim} M = \max\big\{i \bigm| H_i^{\mathfrak{m}}(M) \neq 0\big\},$$

where we use the convention that $\max(\emptyset) = -1$. Therefore it raises to the following natural question that whether the above equality holds true when M is a semi-discrete linearly compact module? Unfortunately, the answer is negative in general. The following counterexample is due to H. Zöschinger. Let $(R, \mathfrak{m})$ be a complete local noetherian domain of dimension 1 and K the field of fractions of R. Consider K as an R-module. Then $\operatorname{Soc}(K) = 0$ and $\operatorname{Coass}(K) = \{0\}$, therefore $\operatorname{Ndim} K = 1$ by [31, 1.6(a)]. Since K/R is artinian, it follows by [31, Theorem] that K is a semi-discrete linearly compact R-module. As $xK = K$ for any non-zero element $x \in \mathfrak{m}$, $H_0^{\mathfrak{m}}(K) = 0$ by 4.1. Moreover, we obtain by 4.2 that $H_i^{\mathfrak{m}}(K) = 0$ for all $i > 0$. Thus

$$\operatorname{Ndim} K = 1 \neq -1 = \max\big\{i \bigm| H_i^{\mathfrak{m}}(K) \neq 0\big\}.$$

However, the following theorem gives an affirmative answer for the question when $\operatorname{Ndim} M \neq 1$.

Theorem 4.10. *Let* $(R, \mathfrak{m})$ *be a local noetherian ring and* M *a non-zero semi-discrete linearly compact* R*-module. Then*

(i) $\operatorname{Ndim} \Gamma_{\mathfrak{m}}(M) = \max\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\}$ *if* $\Gamma_{\mathfrak{m}}(M) \neq 0$;
(ii) $\operatorname{Ndim} M = \max\{i \mid H_i^{\mathfrak{m}}(M) \neq 0\}$ *if* $\operatorname{Ndim} M \neq 1$.

Proof. (i) Since $\Gamma_{\mathfrak{m}}(M)$ is the artinian R-module, we obtain from [6, 4.8, 4.10] that

$$\operatorname{Ndim} \Gamma_{\mathfrak{m}}(M) = \max\bigl\{i \bigm| H_i^{\mathfrak{m}}\bigl(\Gamma_{\mathfrak{m}}(M)\bigr) \neq 0\bigr\}.$$

Thus (i) follows from 4.5.

(ii) First, note by virtue of [31, 1.6(a)] and 4.6(ii) that if $\operatorname{Soc}(M) = 0$ then $\operatorname{Ndim} M \leqslant 1$. If $\Gamma_{\mathfrak{m}}(M) = 0$ then $\operatorname{Soc}(M) = 0$ by 4.3. So we get from the hypothesis that $\operatorname{Ndim} M = 0$. It follows that M is a finitely generated R-module and $H_0^{\mathfrak{m}}(M) \cong \widehat{M} \neq 0$, where $\widehat{M}$ is the $\mathfrak{m}$-adic completion of M. Thus (ii) is proved in this case. Assume now that $\Gamma_{\mathfrak{m}}(M) \neq 0$. By (i) we have only to show that $\operatorname{Ndim} M = \operatorname{Ndim} \Gamma_{\mathfrak{m}}(M)$. Indeed, it is trivial for the case $\operatorname{Ndim} M = 0$. Let $\operatorname{Ndim} M > 1$. From the short exact sequence $0 \to \Gamma_{\mathfrak{m}}(M) \to M \to M/\Gamma_{\mathfrak{m}}(M) \to 0$ we get

$$\operatorname{Ndim} M = \max\bigl\{\operatorname{Ndim} \Gamma_{\mathfrak{m}}(M), \operatorname{Ndim} M/\Gamma_{\mathfrak{m}}(M)\bigr\}.$$

Since $\operatorname{Soc}(M/\Gamma_{\mathfrak{m}}(M)) = 0$, $\operatorname{Ndim}(M/\Gamma_{\mathfrak{m}}(M)) \leqslant 1$. Thus $\operatorname{Ndim} M = \operatorname{Ndim} \Gamma_{\mathfrak{m}}(M)$ as required. □

A sequence of elements $x_1, \dots, x_r$ in R is said to be an *M-coregular* sequence (see [20, 3.1]) if $0 :_M (x_1, \dots, x_r) \neq 0$ and $0 :_M (x_1, \dots, x_{i-1}) \xrightarrow{x_i} 0 :_M (x_1, \dots, x_{i-1})$ is surjective for $i = 1, \dots, r$. We denote by $\operatorname{width}_I(M)$ the supremum of the lengths of all maximal M-coregular sequences in the ideal I. Note by 4.6(i) and 4.7 that

$$\operatorname{width}_I(M) \leqslant \operatorname{Ndim} M < \infty$$

when M is a semi-discrete linearly compact R-module.

Theorem 4.11. *Let* M *be a semi-discrete linearly compact* R*-module and* I *an ideal of* R *such that* $0 :_M I \neq 0$. *Then all maximal* M*-coregular sequences in* I *have the same length. Moreover*

$$\operatorname{width}_I(M) = \inf\bigl\{i / H_i^I(M) \neq 0\bigr\}.$$

Proof. It is sufficient to prove that if $\{x_1, x_2, \dots, x_n\}$ is a maximal M-coregular sequence in I, then $H_i^I(M) = 0$ for all $i < n$, and $H_n^I(M) \neq 0$. We argue by the induction on n. When $n = 0$, there does not exists an element x in I such that $xM = M$. Then $H_0^I(M) \neq 0$ by 4.1.

Let $n > 0$. The short exact sequence

$$0 \longrightarrow 0 :_M x_1 \longrightarrow M \xrightarrow{x_1} M \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \longrightarrow H_i^I(M) \xrightarrow{x_1} H_i^I(M) \longrightarrow H_{i-1}^I(0:_M x_1) \longrightarrow \cdots.$$

By the inductive hypothesis, $H_i^I(0:_M x_1)=0$ for all $i<n-1$ and $H_{n-1}^I(0:_M x_1)\neq 0$. Therefore by virtue of 3.2(ii), $H_i^I(M)=x_1H_i^I(M)=\bigcap_{t>0}x_1^tH_i^I(M)=0$ for all $i<n$. Now, it follows from the exact sequence

$$\cdots \longrightarrow H_n^I(M) \xrightarrow{x_1} H_n^I(M) \longrightarrow H_{n-1}^I(0:_M x_1) \longrightarrow 0$$

and $H_{n-1}^I(0:_M x_1)\neq 0$ that $H_n^I(M)\neq 0$ as required. □

Remark 4.12. We give here an example which shows that the condition $\Gamma_{\mathfrak{m}}(M)\neq 0$ in Theorem 4.10(i) is needful. Let R be the ring and K the R-module as in 4.9. Set $M=N\oplus K$, where N is a finitely generated R-module satisfying $\operatorname{depth}_{\mathfrak{m}} N\geqslant 1$. Then, it is easy to check that $\Gamma_{\mathfrak{m}}(M)=0$ and $H_i^{\mathfrak{m}}(M)\cong H_i^{\mathfrak{m}}(K)=0$ for all $i\geqslant 1$ and $H_0^{\mathfrak{m}}(M)\cong H_0^{\mathfrak{m}}(N)\cong \widehat{N}\neq 0$. Therefore

$$\operatorname{Ndim}\Gamma_{\mathfrak{m}}(M)=-1\neq 0=\max\{i \mid H_i^{\mathfrak{m}}(M)\neq 0\}.$$

We have seen in Remark 4.9 the existence of a non-zero semi-discrete linearly compact module K such that $H_i^{\mathfrak{m}}(K)=0$ for all $i\geqslant 0$. Below, we give a characterization for this class of semi-discrete linearly compact modules. This corollary also shows that we cannot drop the condition $0:_M I\neq 0$ in the assumption of Theorem 4.11.

Corollary 4.13. *Let $(R,\mathfrak{m})$ be a local noetherian ring and M a non-zero semi-discrete linearly compact module. Then $H_i^{\mathfrak{m}}(M)=0$ for all $i\geqslant 0$ if and only if there exists an element $x\in\mathfrak{m}$ such that $xM=M$ and $0:_M x=0$.*

Proof. Let $H_i^{\mathfrak{m}}(M)=0$ for all $i\geqslant 0$. We obtain by 4.1 that $xM=M$ for some $x\in\mathfrak{m}$. On the other hand, it follows from the short exact sequence $0\to 0:_M x\to M\xrightarrow{x} M\to 0$ that $H_i^{\mathfrak{m}}(0:_M x)=0$ for all $i\geqslant 0$. Since $0:_M x$ is artinian by [31, Corollary 1], $0:_M x=0$ by [6, 4.10]. Conversely, suppose that $xM=M$ and $0:_M x=0$, then for all $i\geqslant 0$

$$H_i^{\mathfrak{m}}(M)=xH_i^{\mathfrak{m}}(M)=\bigcap_{t>0}x^tH_i^{\mathfrak{m}}(M)=0$$

by 3.2(ii). □

5. Noetherian local homology modules

First, the following criterion for a module to be noetherian is useful for the investigation of the noetherian property of local homology modules.

Lemma 5.1. *Let J be a finitely generated ideal of a commutative ring R such that R is complete with respect to the J-adic topology and M an R-module. If M/JM is a noetherian R-module and M is J-separated (i.e., $\bigcap_{t>0}J^tM=0$), then M is a noetherian R-module.*

Proof. Set

$$K = \bigoplus_{t \geqslant 0} J^t M / J^{t+1} M$$

the associated graded module over the graded ring

$$\mathrm{Gr}_J(R) = \bigoplus_{t \geqslant 0} J^t / J^{t+1}.$$

Let $x_1, x_2, \dots, x_s$ be a system of generators of J and $(R/J)[T_1, \dots, T_s]$ the polynomial ring of variables $T_1, T_2, \dots, T_s$. The natural epimorphism

$$g : (R/J)[T_1, \dots, T_s] \longrightarrow \mathrm{Gr}_J(R)$$

leads K to be an $(R/J)[T_1, \dots, T_s]$-module. We write $K_t = J^t M / J^{t+1} M$ for all $t \geqslant 0$, then $K_0 = M/JM$ is a noetherian R/J-module by the hypothesis. On the other hand, it is easy to check that

$$K_{t+1} = \sum_{i=1}^{s} T_i K_t$$

for all $t \geqslant 0$. Thus K satisfies the conditions of [12, 1(i)]. Then K is a noetherian $(R/J)[T_1, \dots, T_s]$-module and so K is a noetherian $\mathrm{Gr}_J(R)$-module. Since M is J-separated by the hypothesis, M is a noetherian R-module by [2, 10.25]. $\square$

Theorem 5.2. *Let $(R, \mathfrak{m})$ be a local noetherian ring and M a semi-discrete linearly compact R-module. Then $H_i^{\mathfrak{m}}(M)$ is a noetherian $\widehat{R}$-module for all $i \geqslant 0$.*

Proof. We prove the theorem by induction on i. If $i = 0$, we have $H_0^{\mathfrak{m}}(M) \cong \Lambda_{\mathfrak{m}}(M)$. As M is a semi-discrete linearly compact R-module, $M/\mathfrak{m}M$ is also a semi-discrete linearly compact $R/\mathfrak{m}$-module. By virtue of [16, 5.2], $M/\mathfrak{m}M$ is a finite dimensional vector $R/\mathfrak{m}$-space. Then $\Lambda_{\mathfrak{m}}(M)$ is a noetherian $\widehat{R}$-module by [7, 7.2.9]. Let $i > 0$. Combining 3.10 with 4.1 we may assume, by replacing M with $\bigcap_{t>0} \mathfrak{m}^t M$, that there is an element $x \in \mathfrak{m}$ such that $xM = M$. Then the short exact sequence of linearly compact modules

$$0 \longrightarrow 0 :_M x \longrightarrow M \xrightarrow{x} M \longrightarrow 0$$

gives rise to a long exact sequence of local homology modules

$$\cdots \longrightarrow H_i^{\mathfrak{m}}(M) \xrightarrow{x} H_i^{\mathfrak{m}}(M) \xrightarrow{\delta} H_{i-1}^{\mathfrak{m}}(0 :_M x) \longrightarrow \cdots.$$

If $0 :_M x = 0$, then $H_i^{\mathfrak{m}}(M) = x H_i^{\mathfrak{m}}(M) = \bigcap_{t>0} x^t H_i^{\mathfrak{m}}(M) = 0$ for all $i \geqslant 0$ by 3.2(ii). We now assume that $0 :_M x \neq 0$. By the inductive hypothesis, $H_{i-1}^{\mathfrak{m}}(0 :_M x)$ is a noetherian $\widehat{R}$-module. Set $H = H_i^{\mathfrak{m}}(M)$, we have $H/xH \cong \operatorname{Im} \delta \subseteq H_{i-1}^{\mathfrak{m}}(0 :_M x)$. It follows that H/xH is a noetherian $\widehat{R}$-module. Thus $H/\hat{\mathfrak{m}}H$ is also a noetherian $\widehat{R}$-module. Moreover, $\bigcap_{t>0} \hat{\mathfrak{m}}^t H = \bigcap_{t>0} \mathfrak{m}^t H_i^{\mathfrak{m}}(M) = 0$. Therefore H is a noetherian $\widehat{R}$-module by 5.1. $\square$

Theorem 5.3. *Let $(R,\mathfrak{m})$ be a local noetherian ring and M a semi-discrete linearly compact R-module with* $\operatorname{Ndim} M = d$. *Then $H^I_d(M)$ is a noetherian module on $\Lambda_I(R)$.*

Proof. We argue by induction on d. If $d = 0$, M is a finitely generated R-module, and so is I-separated. By 3.8, $H^I_0(M) \cong \Lambda_I(M) \cong M$, therefore $H^I_0(M)$ is a noetherian $\Lambda_I(R)$-module. Let $d > 0$. From 3.10 we have $H^I_d(M) \cong H^I_d(\bigcap_{t>0} I^t M)$. If $\operatorname{Ndim}(\bigcap_{t>0} I^t M) < d$, then $H^I_d(M) = 0$ by 4.8 and then there is nothing to prove. If $\operatorname{Ndim}(\bigcap_{t>0} I^t M) = d$, by 4.1 we may assume, by replacing M with $\bigcap_{t>0} I^t M$, that there is an element $x \in I$ such that $xM = M$. Then, from the short exact sequence of linearly compact modules

$$0 \longrightarrow 0 :_M x \longrightarrow M \xrightarrow{x} M \longrightarrow 0$$

we get an exact sequence of local homology modules

$$H^I_d(M) \xrightarrow{x} H^I_d(M) \xrightarrow{\delta} H^I_{d-1}(0 :_M x).$$

Note by 4.7 that $\operatorname{Ndim}(0 :_M x) \leqslant d-1$. If $\operatorname{Ndim}(0 :_M x) < d-1$, then $H^I_{d-1}(0 :_M x) = 0$ by 4.8 and therefore

$$H^I_d(M) = xH^I_d(M) = \bigcap_{t>0} x^t H^I_d(M) = 0$$

by 3.2(ii). Assume that $\operatorname{Ndim}(0 :_M x) = d-1$. It follows by the inductive hypothesis that $H^I_{d-1}(0 :_M x)$ is a noetherian $\Lambda_I(R)$-module. On the other hand, we have $H^I_d(M)/xH^I_d(M) \cong \operatorname{Im}\delta \subseteq H^I_{d-1}(0 :_M x)$. Thus $H^I_d(M)/xH^I_d(M)$ is a noetherian $\Lambda_I(R)$-module. Therefore $H^I_d(M)/JH^I_d(M)$ is a noetherian $\Lambda_I(R)$-module, where $J = I\Lambda_I(R)$. Moreover, since $\bigcap_{t>0} J^t H^I_d(M) = \bigcap_{t>0} I^t H^I_d(M) = 0$ and $\Lambda_I(R)$ is complete in J-adic topology, $H^I_d(M)$ is a noetherian $\Lambda_I(R)$-module by 5.1 as required. □

6. Macdonald duality

Henceforth $(R,\mathfrak{m})$ will be a local noetherian ring with the maximal ideal $\mathfrak{m}$. Suppose now that the topology on R is the $\mathfrak{m}$-adic topology.

Let M be an R-module and $E(R/\mathfrak{m})$ the injective envelope of $R/\mathfrak{m}$. The module $D(M) = \operatorname{Hom}(M, E(R/\mathfrak{m}))$ is called Matlis dual of M. If M is a Hausdorff linearly topologized R-module, then *Macdonald dual* of M is defined by $M^* = \operatorname{Hom}(M, E(R/\mathfrak{m}))$ the set of continuous homomorphisms of R-modules (see [16, §9]). In case $(R,\mathfrak{m})$ is local complete, the topology on M^* is defined as in [16, 8.1]. Moreover, if M is semi-discrete, then the topology of M^* coincides with that induced on it as a submodule of $E(R/\mathfrak{m})^M$, where $E(R/\mathfrak{m})^M = \prod_{x\in M}(E(R/\mathfrak{m}))_x$, $(E(R/\mathfrak{m}))_x = E(R/\mathfrak{m})$ for all $x \in M$ (see [16, 8.6]).

Lemma 6.1. *(See [16, 5.8].) A Hausdorff linearly topologized R-module M is semi-discrete if and only if $D(M) = M^*$.*

Lemma 6.2. *(See [16, 5.7].) Let M be a Hausdorff linearly topologized R-module and $u : M \to A^*$ a homomorphism. Then the following statements are equivalent*:

(a) *u is continuous,*
(b) $\ker u$ *is open,*
(c) $\ker u$ *is closed.*

A Hausdorff linearly topologized R-module is $\mathfrak{m}$-*primary* if each element of M is annihilated by a power of $\mathfrak{m}$. A Hausdorff linearly topologized R-module M is *linearly discrete* if every $\mathfrak{m}$-primary quotient of M is discrete. It should be noted that if M is linearly discrete, then M is semi-discrete (see [16, 6.2]). The direct limit of a direct system of linearly discrete R-modules is linearly discrete. If $f : M \to N$ is an epimorphism of Hausdorff linearly topologized R-modules in which M is linearly discrete, then f is continuous (see [16, 6.2, 6.7, 6.8]). Then I.G. Macdonald [16] established the duality between linearly discrete and linearly compact modules as follows.

Theorem 6.3. *(See [16, 9.3, 9.12, 9.13].) Let $(R, \mathfrak{m})$ be a complete local noetherian ring.*

(i) *If M is linearly compact, then M^* is linearly discrete (hence semi-discrete). If M is semi-discrete, then M^* is linearly compact.*
(ii) *If M is linearly compact or linearly discrete, then we have a topological isomorphism $\omega : M \xrightarrow{\cong} M^{**}$.*

The following duality theorem between local homology and local cohomology modules is the main result of this section.

Theorem 6.4.

(i) *Let M be an R-module. Then for all $i \geqslant 0$,*

$$L_i^I\big(D(M)\big) \cong H_i^I\big(D(M)\big) \cong D\big(H_I^i(M)\big).$$

(ii) *If M is a linearly compact R-module, then for all $i \geqslant 0$,*

$$H_i^I(M^*) \cong \big(H_I^i(M)\big)^*.$$

Moreover, if $(R, \mathfrak{m})$ is a complete local noetherian ring, then

$$H_I^i(M^*) \cong \big(H_i^I(M)\big)^*.$$

(iii) *If $(R, \mathfrak{m})$ is a complete local noetherian ring and M a semi-discrete linearly compact R-module, then we have topological isomorphisms of R-modules for all $i \geqslant 0$,*

$$H_I^i(M^*) \cong \big(H_i^I(M)\big)^*,$$
$$H_i^I(M^*) \cong \big(H_I^i(M)\big)^*.$$

To prove Theorem 6.4 some auxiliary lemmas are necessary. First, we show that the Macdonald dual functor $(-)^*$ is exact on the category of linearly compact R-modules and continuous homomorphisms.

Lemma 6.5. *Let*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence of linearly compact R-modules, in which the homomorphisms f, g are continuous. Then the induced sequence

$$0 \longrightarrow P^* \xrightarrow{g^*} N^* \xrightarrow{f^*} M^* \longrightarrow 0$$

is exact.

Proof. By [16, 5.5] f is an open mapping, so replace M by $f(M)$ we may assume that M is a close submodule of N. Therefore, by [16, 5.9], for any continuous homomorphism $h : M \to E(R/\mathfrak{m})$ there is a continuous homomorphism $\varphi : N \to E(R/\mathfrak{m})$ which extends h. Thus f^* is surjective. It is easy to see that g^* is injective and $\operatorname{Im} g^* \subseteq \ker f^*$. So it remains to show that $\ker f^* \subseteq \operatorname{Im} g^*$. Let $\psi \in \ker f^*$, we have $\psi(\ker g) = \psi(f(M)) = 0$. Then ψ induces a homomorphism $\phi : P \to E(R/\mathfrak{m})$ such that $\phi \circ g = \psi$. It follows $\ker \phi = g(\ker \psi)$. Since ψ is continuous, $\ker \psi$ is open by 6.2. Moreover, g is open, so $\ker \phi$ is also open. Therefore ϕ is continuous by 6.2. Thus $\psi \in \operatorname{Im} g^*$. This finishes the proof. □

Note that submodules and homomorphic images of a semi-discrete module are also semi-discrete. The following consequence shows that the converse is also true in the category of linearly compact R-modules.

Corollary 6.6. *Let*

$$0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$$

be a short exact sequence of linearly compact R-modules with continuous homomorphisms f, g. If M and P are semi-discrete, then N is also semi-discrete.

Proof. It follows from 6.1 and the hypothesis that $P^* = D(P)$ and $M^* = D(M)$. We now have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^* & \xrightarrow{g^*} & N^* & \xrightarrow{f^*} & M^* & \longrightarrow & 0 \\ & & \| & & \downarrow j & & \| & & \\ 0 & \longrightarrow & D(P) & \xrightarrow{D(g)} & D(N) & \xrightarrow{D(f)} & D(M) & \longrightarrow & 0, \end{array}$$

in which j is an inclusion and rows are exact by 6.5 and [22, 3.16]. It follows that $N^* = D(N)$, thus N is semi-discrete by 6.1(i). □

Lemma 6.7. *Let N be a finitely generated R-module and M a linearly compact R-module. Then*

$$\big(\operatorname{Tor}_i^R(N,M)\big)^* \cong \operatorname{Ext}_R^i(N,M^*),$$
$$\operatorname{Tor}_i^R(N,M^*) \cong \big(\operatorname{Ext}_R^i(N,M)\big)^*$$

for all $i \geqslant 0$.

Proof. Let

$$\mathbf{F}_\bullet : \cdots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow N \longrightarrow 0$$

be a free resolution of N, in which the free R-modules F_i are finitely generated. Consider $\mathbf{F}_\bullet \otimes_R M$ as a complex of linearly compact R-modules with continuous differentials. Since the Macdonald dual functor $(-)^*$ is exact on the category of linearly compact R-modules and the continuous homomorphisms by 6.5, it follows by [19, 6.1, Theorem 1] that

$$\big(H_i(\mathbf{F}_\bullet \otimes_R M)\big)^* \cong H^i\big((\mathbf{F}_\bullet \otimes_R M)^*\big).$$

On the other hand, by virtue of [16, 2.5] we have

$$(\mathbf{F}_\bullet \otimes_R M)^* \cong \operatorname{Hom}_R(\mathbf{F}_\bullet, M^*).$$

Therefore

$$\begin{aligned}\big(\operatorname{Tor}_i^R(N,M)\big)^* &\cong \big(H_i(\mathbf{F}_\bullet \otimes_R M)\big)^*\\ &\cong H^i\big(\operatorname{Hom}_R(\mathbf{F}_\bullet, M^*)\big)\\ &\cong \operatorname{Ext}_R^i(N,M^*).\end{aligned}$$

The proof of the second isomorphism is similar. □

If M is a linearly topologized R-module, then the module $\operatorname{Ext}_R^i(R/I^t, M)$ is also a linearly topologized R-module by the topology defined as in 2.5. Since the local cohomology module $H_I^i(M) = \varinjlim_t \operatorname{Ext}_R^i(R/I^t, M)$ is a quotient module of $\bigoplus_t \operatorname{Ext}_R^i(R/I^t, M)$, it becomes a linearly topologized R-module with the quotient topology.

Lemma 6.8. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring. If M is a semi-discrete linearly compact R-module, then M is linearly discrete and therefore the local cohomology modules $H_I^i(M)$ are linearly discrete R-modules for all $i \geqslant 0$.*

Proof. We first show that if M is a semi-discrete linearly compact R-module, then M is linearly discrete. Indeed, since M is semi-discrete, M^* is linearly compact and hence M^{**} is linearly discrete by 6.3(i). On the other hand, since M is a linearly compact R-module, we have by 6.3(ii) a topological isomorphism $M \cong M^{**}$. Therefore M is linearly discrete. Now, by the same argument as in the proof of 2.5 we can prove that $\{\operatorname{Ext}_R^i(R/I^t, M)\}_t$ is a direct system of semi-discrete linearly compact R-modules with the continuous homomorphisms, and therefore it is a direct system of linearly discrete modules. Thus, by [16, 6.7] $H_I^i(M) = \varinjlim_t \operatorname{Ext}_R^i(R/I^t, M)$ are linearly discrete for all $i \geqslant 0$. □

Now we are able to prove the duality Theorem 6.4.

Proof of Theorem 6.4. (i) was proved in [24, 5.6] and [6, 3.3(ii)].

(ii) Note by [16, 2.6] that for a direct system $\{M_t\}$ of Hausdorff linearly topologized R-modules with the continuous homomorphisms we have an isomorphism $\varprojlim_t M_t^* \cong (\varinjlim_t M_t)^*$. Moreover, since $\{\operatorname{Ext}_R^i(R/I^t, M)\}_t$ forms a direct system of linearly compact R-modules with continuous homomorphisms by 2.5, we get by 6.7 that

$$\begin{aligned} H_i^I(M^*) &= \varprojlim_t \operatorname{Tor}_i^R\big(R/I^t, M^*\big) \\ &\cong \varprojlim_t \big(\operatorname{Ext}_R^i\big(R/I^t, M\big)\big)^* \\ &\cong \Big(\varinjlim_t \operatorname{Ext}_R^i\big(R/I^t, M\big)\Big)^* = \big(H_I^i(M)\big)^*. \end{aligned}$$

To prove the second isomorphism note by [16, 9.14] that for an inverse system $\{M_t\}$ of linearly compact modules over complete local noetherian ring with continuous homomorphisms we have an isomorphism $(\varprojlim_t M_t)^* \cong \varinjlim_t M_t^*$, and that $\{\operatorname{Tor}_i^R(R/I^t, M)\}_t$ forms an inverse system of linearly compact R-modules with continuous homomorphisms by 2.6. It follows by 6.7 that

$$\begin{aligned} H_I^i(M^*) &= \varinjlim_t \operatorname{Ext}_R^i\big(R/I^t, M^*\big) \\ &\cong \varinjlim_t \big(\operatorname{Tor}_i^R\big(R/I^t, M\big)\big)^* \\ &\cong \Big(\varprojlim_t \operatorname{Tor}_i^R\big(R/I^t, M\big)\Big)^* = \big(H_i^I(M)\big)^*. \end{aligned}$$

(iii) Let us prove the first isomorphism. From (ii), it is the algebraic isomorphism. Thus, by [16, 6.8], we only need to show that both $H_I^i(M^*)$ and $(H_i^I(M))^*$ are linearly discrete. Indeed, it follows from 3.3 and 6.3(i) that $(H_i^I(M))^*$ is linearly discrete. On the other hand, since M is semi-discrete linearly compact, M^* is linearly compact and linearly discrete. Therefore the local cohomology modules $H_I^i(M^*)$ are linearly discrete by 6.8, and the first topological isomorphism is proved. The second topological isomorphism follows from the first one and 6.3(ii). □

Corollary 6.9. *Let $(R, \mathfrak{m})$ be a complete local noetherian ring.*

(i) *If M is linearly compact R-module, then for all $i \geqslant 0$,*

$$\begin{aligned} H_I^i(M) &\cong \big(H_i^I(M^*)\big)^*, \\ H_i^I(M) &\cong \big(H_I^i(M^*)\big)^*. \end{aligned}$$

(ii) *If M is a semi-discrete linearly compact R-module, then we have topological isomorphisms of R-modules for all $i \geqslant 0$,*

$$H_I^i(M) \cong \big(H_i^I(M^*)\big)^*,$$
$$H_i^I(M) \cong \big(H_I^i(M^*)\big)^*.$$

Proof. (i) follows from 6.4(ii), 6.3(ii).
(ii) follows from 6.4(iii) and 6.3(ii). □

7. Local cohomology of semi-discrete linearly compact modules

In this section $(R, \mathfrak{m})$ is a local noetherian ring with the $\mathfrak{m}$-adic topology. We denote by $(\widehat{R}, \hat{\mathfrak{m}})$ the $\mathfrak{m}$-adic completion of R with the maximal ideal $\hat{\mathfrak{m}}$ and $\widehat{M}$ the $\mathfrak{m}$-adic completion of the module M. Recall that an artinian R-module A has a natural structure of module over $\widehat{R}$ as follows (see [23, 1.11]): Let $\hat{a} = (a_n) \in \widehat{R}$ and $x \in A$; since $\mathfrak{m}^k x = 0$ for some positive integer k, $a_n x$ is constant for all large n, and we define $\hat{a}x$ to be this constant value. Then we have the following generalization of this fact for linearly compact R-modules.

Lemma 7.1. *Let M be a linearly compact R-module. Then the following statements are true.*

(i) *M has a natural structure of linearly compact module over $\widehat{R}$. Moreover, a subset N of M is a linearly compact $\widehat{R}$-submodule if and only if N is a closed R-submodule.*
(ii) *Assume in addition that M is a semi-discrete R-module. Then M is also a semi-discrete linearly compact $\widehat{R}$-module.*

Proof. (i) Assume that $\{U_i\}_{i\in J}$ is a nuclear base of M consisting of submodules. Then $M \cong \varprojlim_{i\in J} M/U_i$, in which M/U_i is an artinian R-module for all $i \in J$ by [16, 3.11, 4.1, 5.5]. It should be noted by [23, 1.11] that an artinian module over a local noetherian ring $(R, \mathfrak{m})$ has a natural structure of artinian module over $\widehat{R}$ so that a subset of M is an R-submodule if and only if it is an $\widehat{R}$-submodules. Thus $\{M/U_i\}_{i\in J}$ can be regard as an inverse system of artinian $\widehat{R}$-modules with $\widehat{R}$-homomorphisms. Therefore, by passing to the inverse limits, M has a natural structure of linearly compact module over $\widehat{R}$.

It is clear that a linearly compact $\widehat{R}$-submodule of M is a closed R-submodule. Now, if N is a closed R-module of M, then $N \cong \varprojlim_{i\in J} N/(N \cap U_i)$. Since

$$N/(N \cap U_i) \cong (N + U_i)/U_i \subseteq M/U_i,$$

$N/(N \cap U_i)$ can be considered as an artinian R-submodule of M/U_i, so it is an artinian $\widehat{R}$-submodule. Moreover, the homomorphisms of the inverse system $\{N/(N \cap U_i)\}_{i\in J}$ are induced from the inverse system $\{M/U_i\}_{i\in J}$. Therefore, by 2.3(iv) N is a linearly compact $\widehat{R}$-submodule of M.

(ii) follows immediately from (i) by the fact that all submodules of a semi-discrete linearly compact module are closed. □

Remember that the (*Krull*) *dimension* $\dim_R M$ of a non-zero R-module M is the supremum of lengths of chains of primes in the support of M if this supremum exists, and ∞ otherwise. If M is finitely generated, then $\dim M = \max\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass} M\}$. For convenience, we set $\dim M = -1$ if $M = 0$.

Corollary 7.2. *Let M be a semi-discrete linearly compact R-module. Then*

(i) $\operatorname{Ndim}_R M = \operatorname{Ndim}_{\widehat{R}} M$;
(ii) $\dim_R M = \dim_{\widehat{R}} M$.

Proof. (i) follows immediately from 7.1(ii) and the definition of noetherian dimension.

(ii) In the special case M is a finitely generated R-module, from 3.8 we have $M \cong \Lambda_{\mathfrak{m}}(M) = \widehat{M}$, and therefore $\dim_R M = \dim_{\widehat{R}} M$ by [3, 6.1.3]. For any semi-discrete linearly compact module M, there is by [31, Theorem] a short exact sequence $0 \to N \to M \to A \to 0$, where N is finitely generated and A is artinian. As $\dim_R A = \dim_{\widehat{R}} A = 0$ and $\dim_R N = \dim_{\widehat{R}} N$, we get $\dim_R M = \dim_{\widehat{R}} M$. □

Remark 7.3.

(i) Denote by $\mathcal{C}$ the category of semi-discrete linearly compact R-modules. It is well known that the category $\mathcal{C}$ contains the category of artinian R-modules and also the category of finitely generated R-modules if R is complete. However, there are many semi-discrete linearly compact R-modules which are neither artinian nor finitely generated. The first example for this conclusion is the module K in Remark 4.9. More general, let R be complete ring, A an artinian R-module with $\operatorname{Ndim} A > 0$ and N a finitely generated R-module with $\dim N > 0$. Then $M = A \oplus N$ is semi-discrete linearly compact. Further, let Q be a quotient module of M, then Q is also a semi-discrete linearly compact R-module.
(ii) If $M \in \mathcal{C}$, then by 6.1 the Matlis dual $D(M)$ and the Macdonald dual M^* are the same. Moreover, M is linearly discrete by 6.8 and can be regarded by 7.1 as an $\widehat{R}$-module, therefore the Macdonald dual functor $(-)^*$ is a functor from $\mathcal{C}$ to itself and we have by 6.3 a topological isomorphism $\omega : M \xrightarrow{\cong} M^{**}$. Thus, $(-)^*$ is an equivalent functor on the category $\mathcal{C}$.

Lemma 7.4. *Let M be a semi-discrete linearly compact R-module. Then*

$$\operatorname{Ndim}_R M^* = \dim_R M \quad \textit{and} \quad \operatorname{Ndim}_R M = \dim_R M^*.$$

Proof. From 7.1(ii) and 7.2 we may assume that $(R, \mathfrak{m})$ is a complete ring. If M is finitely generated R-module, M^* is artinian. Keep in mind in our case that $M^* = D(M)$, then the equality $\operatorname{Ndim} M^* = \dim M$ follows from the well-known facts of Matlis duality. If M is artinian, then it is clear that $\operatorname{Ndim} M^* = \dim M = 0$. Suppose now that M is semi-discrete linearly compact. There is by [31, Theorem] a short exact sequence $0 \to N \to M \to A \to 0$ in which N is finitely generated and A is artinian. Thus we get by Macdonald duality an exact sequence $0 \to A^* \to M^* \to N^* \to 0$, where N^* is artinian and A^* is finitely generated. Then

$$\begin{aligned} \operatorname{Ndim} M^* &= \max\{\operatorname{Ndim} N^*, \operatorname{Ndim} A^*\} \\ &= \max\{\dim N, \dim A\} = \dim M. \end{aligned}$$

The second equality follows from 7.3(ii). □

Now we are able to extend well-known results in Grothendieck's local cohomology theory of finitely generated R-modules for semi-discrete linearly compact modules.

Theorem 7.5. *Let M be a non-zero semi-discrete linearly compact R-module. Then*

(i) $\dim_R M = \max\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$ *if* $\dim_R M \neq 1$;
(ii) $\dim_R(\widehat{M}) = \max\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}$ *if* $\widehat{M} \neq 0$.

Proof. (i) Note by 7.1(ii) and 7.2 that M is a semi-discrete linearly compact $\widehat{R}$-module with $\dim_R M = \dim_{\widehat{R}} M$. Moreover, the natural homomorphism $f: R \to \widehat{R}$ gives by [3, 4.2.1] an isomorphism $H^d_{\mathfrak{m}}(M) \cong H^d_{\hat{\mathfrak{m}}}(M)$. Thus, we may assume without any loss of generality that $(R, \mathfrak{m})$ is a complete local noetherian ring. As M is a semi-discrete linearly compact R-module, M^* is also a semi-discrete linearly compact R-module by 6.3(i). Recall by [16, 5.6] that $M^* = 0$ if only if $M = 0$. Then, since $1 \neq \dim M = \operatorname{Ndim} M^*$, it follows from 7.4, 4.10(ii) and 6.4(ii) that

$$\begin{aligned}\dim M = \operatorname{Ndim} M^* &= \max\bigl\{i \bigm| H_i^{\mathfrak{m}}(M^*) \neq 0\bigr\}\\ &= \max\bigl\{i \bigm| H^i_{\mathfrak{m}}(M)^* \neq 0\bigr\}\\ &= \max\bigl\{i \bigm| H^i_{\mathfrak{m}}(M) \neq 0\bigr\}.\end{aligned}$$

(ii) The continuous epimorphisms $M \to M/\mathfrak{m}^t M$ for all $t > 0$ induce by 2.4 a continuous epimorphism $\pi: M \to \widehat{M}$. Moreover π is the open homomorphism by [16, 5.5]. Thus $\widehat{M}$ is also a semi-discrete linearly compact R-module. It follows from 7.2 that $\dim_R \widehat{M} = \dim_{\widehat{R}} \widehat{M}$. Hence, as in the proof of (i) we may assume without any loss of generality that $(R, \mathfrak{m})$ is a complete local noetherian ring. Note that $H^I_0(M) \cong \Lambda_I(M)$ and $H^0_I(M) \cong \Gamma_I(M)$, hence we have by 6.4(ii)

$$0 \neq (\widehat{M})^* = \bigl(H^{\mathfrak{m}}_0(M)\bigr)^* \cong \Gamma_{\mathfrak{m}}(M^*).$$

Thus, by virtue of 7.4, 4.10(i) and 6.4(ii) we get

$$\begin{aligned}\dim \widehat{M} = \operatorname{Ndim}(\widehat{M})^* = \operatorname{Ndim} \Gamma_{\mathfrak{m}}(M^*) &= \max\bigl\{i \bigm| H_i^{\mathfrak{m}}(M^*) \neq 0\bigr\}\\ &= \max\bigl\{i \bigm| \bigl(H^i_{\mathfrak{m}}(M)\bigr)^* \neq 0\bigr\}\\ &= \max\bigl\{i \bigm| H^i_{\mathfrak{m}}(M) \neq 0\bigr\}.\end{aligned}$$

The proof is complete. □

Remark 7.6.

(i) The condition $\operatorname{Ndim} M \neq 1$ in Theorem 7.5(i) is necessary. Indeed, take the ring R and the semi-discrete linearly compact R-module K as in Remark 4.9 and set $L = K^*$. It follows from 6.4(iii) and 4.9 that $H^i_{\mathfrak{m}}(L) \cong H_i^{\mathfrak{m}}(K)^* = 0$ for all $i \geqslant 0$. Hence

$$\dim L = \operatorname{Ndim} K = 1 \neq -1 = \max\bigl\{i \bigm| H^i_{\mathfrak{m}}(L) \neq 0\bigr\}.$$

(ii) The condition $\widehat{M} \neq 0$ in Theorem 7.5(ii) can also not be dropped as the following example shows. Set $M = L \oplus A$, where A is an artinian R-module satisfying $\operatorname{width}_{\mathfrak{m}} A \geqslant 1$. Then there is an element $x \in \mathfrak{m}$ such that $xM = M$, and therefore $\widehat{M} = 0$. It is easy to see that $H^i_{\mathfrak{m}}(M) = H^i_{\mathfrak{m}}(L) = 0$ for all $i \geqslant 1$ and $H^0_{\mathfrak{m}}(M) \cong H^0_{\mathfrak{m}}(A) = A$. Thus

$$\dim \widehat{M} = -1 \neq 0 = \max\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

To complete the unusual behavior on the vanishing theorem of local cohomology for semi-discrete linearly compact modules we give a characterization of semi-discrete linearly compact modules, whose all local cohomology modules are vanished.

Corollary 7.7. *Let M be a semi-discrete linearly compact R-module. Then $H^i_{\mathfrak{m}}(M) = 0$ for all $i \geqslant 0$ if and only if there exists an element $x \in \mathfrak{m}$ such that $xM = M$ and $0 :_M x = 0$.*

Proof. The conclusion follows from 4.13 by using 6.4(ii) and 7.3(ii). □

Recall that a sequence of elements $x_1, \ldots, x_r$ in R is said to be an *M-regular* sequence if $M/(x_1, \ldots, x_r)M \neq 0$ and $M/(x_1, \ldots, x_{i-1})M \xrightarrow{x_i} M/(x_1, \ldots, x_{i-1})M$ is injective for $i = 1, \ldots, r$. Denote by $\operatorname{depth}_I(M)$ the supremum of the lengths of all maximal M-regular sequences in I. Then we have the following result.

Theorem 7.8. *Let M be a semi-discrete linearly compact R-module such that $M/IM \neq 0$. Then*

$$\operatorname{depth}_I(M) = \inf\{i / H^i_I(M) \neq 0\}.$$

Proof. Note by [20, §3] that $x_1, \ldots, x_r$ is an M-regular sequence if and only if it is a $D(M)$-coregular sequence. Since M is semi-discrete linearly compact, $D(M) = M^*$ and therefore $\operatorname{depth}_I(M) = \operatorname{width}_I(M^*)$. On the other hand, M^* is a semi-discrete linearly compact R-module by 6.3(i) and $0 :_{M^*} I \cong (M/IM)^* \neq 0$. Thus the conclusion follows by virtue of 4.11 and 6.4(ii). □

Following is the artinianness of local cohomology modules.

Theorem 7.9. *Let M be a semi-discrete linearly compact R-module with $\dim_R M = d$. Then the following statements are true.*

(i) *The local cohomology modules $H^i_{\mathfrak{m}}(M)$ are artinian R-modules for all $i \geqslant 0$.*
(ii) *The local cohomology module $H^d_I(M)$ is artinian.*

Proof. Note first that if A is an artinian $\widehat{R}$-module, then A is an artinian R-module. Therefore, from the independent of the base ring of local cohomology and 7.1 we may assume without loss of generality that R is complete. Then, by applying the duality between local homology and local cohomology 6.4, the statement (i) follows from 5.2 and the statement (ii) from 5.3. □

Finally, as an immediate consequence of Theorem 7.9 we get the following well-known result.

Corollary 7.10. *(See [3, 7.1.3, 7.1.6].) Let M be a finitely generated R-module with $\dim_R M = d$. Then the local cohomology modules $H^i_{\mathfrak{m}}(M)$ and $H^d_I(M)$ are artinian R-modules for all $i \geqslant 0$.*

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