Actions and coactions of finite quantum groupoids on von Neumann algebras, extensions of the matched pair procedure

Jean-Michel Vallin

UMR CNRS 6628 Université d’Orléans, Institut de Mathématiques de Jussieu, Plateau 7D, 175 rue du Chevaleret, 75013 Paris, France

Received 24 February 2006
Available online 24 April 2007
Communicated by Eva Bayer-Fluckiger

Abstract

In this work, actions and coactions of finite C*-quantum groupoids are studied in an operator algebras context. In particular we prove a double crossed product theorem, and the existence of an universal von Neumann algebra on which any finite groupoid acts outerly. We give two actually different extensions of the matched pairs procedure. In previous works, N. Andruskiewitsch and S. Natale define, for any matched pair of groupoids, two C*-quantum groupoids in duality, we give here an interpretation of them in terms of crossed products of groupoids using a single multiplicative partial isometry which gives a complete description of these structures. The second extension deals only with groups to define an other type of finite C*-quantum groupoids.

© 2007 Elsevier Inc. All rights reserved.

Keywords: Multiplicative partial isometries; Groupoids; Subfactors

1. Introduction

Multiplicative partial isometries (mpi) generalize Baaj and Skandalis multiplicative unitaries in finite dimension [BS,BBS]. They are the finite-dimensional version of so-called pseudo-multiplicative unitaries. These unitaries appeared first in a commutative context dealing with locally compact groupoids [Val0], and then in the general case for a very large class of depth two...
inclusions of von Neumann algebras in a common work with M. Enock [EV], who has developed the theory in the infinite-dimensional framework leading to F. Lesieur’s measured quantum groupoids [L].

When it is regular, any mpi generates two involutive subalgebras of the algebra of all bounded linear operators on the corresponding Hilbert space. Using a canonical pairing, these two algebras have structures generalizing involutive Hopf algebras, which are called (finite) $C^*$-quantum groupoids. The first examples of these new structures were discovered by the theoretical physicists Böhm, Szlachányi and Nill [BoSz,BoSzNi], they called them weak Hopf $C^*$-algebras.

Using general depth two and finite index inclusions $M_0 \subset M_1$ of type $II_1$ subfactors, and a special pairing between relative commutants $M_0' \cap M_2$ and $M_1' \cap M_3$, D. Nikshych and L. Vainerman, gave explicit formulas for weak Hopf $C^*$-algebra structures in duality for these two last involutive algebras [NV2]. They also found a Galois correspondence between intermediate subfactors and involutive coideals for $M_1' \cap M_3$ [NV4].

One of this article’s aims is to give examples of finite $C^*$-quantum groupoids, with an operator algebraic point of view, using several generalizations of the matched pair procedure (see [IK]). In a purely algebraic setting, N. Andruskiewitsch and S. Natale in [AN], give a construction of $C^*$-quantum groupoids dealing with matched pairs of groupoids in a sense generalizing directly the group case. We use here a special multiplicative partial isometry to give an interpretation of these examples in terms of groupoids crossed products, generalizing in finite dimension previous works in the quantum groups context. Removing the condition on the intersection of any matched pair of finite groups, we also obtain (actually different) examples of quantum groupoids.

In the second section we give definitions and properties of multiplicative partial isometries in close relation with quantum groupoids.

The third section is an approach of the actions of quantum groupoids, a notion of outerness and its translation for groupoid actions, then we obtain a double crossed product theorem.

The fourth section deals with examples. The first examples are generated by special multiplicative partial isometries associated with matched pairs of groupoids, generalizing the classical case due to S. Baaj and S. Skandalis in [BS]; we give an interpretation of them in terms of double groupoids in relation with the works of N. Andruskiewitsch and S. Natale. Then we prove the existence of quantum groupoids associated with “relatively matched pairs” of finite groups which are never of the previous type when their intersection is not abelian.

Natural questions come from these two examples of Section 4 four and will be treated later. The first problem is to find depth two inclusions of von Neumann algebras associated with double groupoids, and the second problem is to precise completely the quantum groupoids associated with “relatively matched pairs” of groupoids.

An other extension of this article will be the generalization of these constructions in direction of Lesieur’s locally compact groupoids and a characterization of these objects in terms of cleft extensions in the spirit of S. Vaes and L. Vainerman [VV].

2. Multiplicative partial isometries and $C^*$-quantum groupoids

2.1. Multiplicative partial isometries

2.1.1. Notations. Let $N$ be a finite-dimensional von Neumann algebra, so $N$ is isomorphic to a sum of matrix algebras $\bigoplus_\gamma M_{n_\gamma}$, let $\{p_\gamma\}$ be the family of minimal central projections of $N$, and let $\{e_{i,j}^\gamma/1 \leq i, j \leq n_\gamma\}$ be a given family of matrix units for $N$. Let $N^\alpha$ be the opposite von Neumann algebra of $N$, so this is $N$ with the opposite multiplication, hence a matrix unit of $N^\alpha$.
is given by the transposed of $N$'s: $\{e_{j,i}^\gamma / 1 \leq i, j \leq n_\gamma\}$. The element $f = \sum_{\gamma} \sum_{i,j} \frac{1}{n_\gamma} e_{i,j}^\gamma \otimes e_{j,i}^\gamma$ is the only projection of $N^0 \otimes N$ such that, for any $n$ in $N$: $f(n^0 \otimes 1) = f(1 \otimes n)$ and if $f(1 \otimes n) = 0$ then $n = 0$.

Let $M_1$, $M_2$ be two von Neumann algebras. Let $s$ (respectively $r$) be a faithful non-degenerate antirepresentation (respectively, representation) from $N$ to $M_1$ (respectively, $M_2$), then $s$ can be viewed as a representation $s^o$ of $N^o$. Let us define:

$$e_{s,r} = (s^o \otimes r)(f) = \sum_{\gamma} \sum_{i,j} \frac{1}{n_\gamma} s(e_{i,j}^\gamma) \otimes r(e_{j,i}^\gamma).$$

By an obvious generalization of Lemma 2.1.2 in [Val1], $e_{s,r}$ is a projection in $s(N) \otimes r(N)$, $e_{s,r}$ is the only projection $e$ in $M_1 \otimes r(N)$ which satisfies the following two conditions:

(a) For every $m_1$, $m_2$ in $M_1$ and $M_2$ respectively, the relation $e(m_1 \otimes 1) = 0$ implies $m_1 = 0$ and the relation $e(1 \otimes m_2) = 0$ implies $m_2 = 0$,
(b) for every $n$ in $N$: $e(s(n) \otimes 1) = e(1 \otimes r(n))$.

Let $H$ be a finite-dimensional Hilbert space, and let $\alpha$ (respectively, $\beta, \hat{\beta}$) be an injective non-degenerate representation (respectively, two injective non-degenerate antirepresentations) of $N$, which commute two by two pointwise. We also suppose that $\text{tr} \circ \alpha = \text{tr} \circ \beta = \text{tr} \circ \hat{\beta}$, where $\text{tr}$ is the canonical trace on $H$. One must keep in mind that $\beta$ and $\alpha$ are also a representation and an antirepresentation of $N^o$.

2.1.2. Definition. We call multiplicative partial isometry (with the base $(N, \alpha, \beta, \hat{\beta})$) every partial isometry $I$ whose initial (respectively, final) support is $e_{\hat{\beta},\alpha}$ (respectively, $e_{\alpha,\beta}$) and such that:

(1) $I$ commutes with $\beta(N) \otimes \hat{\beta}(N)$,
(2) for every $n$, $n'$ in $N$, one has: $I(\alpha(n) \otimes \beta(n')) = (\hat{\beta}(n') \otimes \alpha(n))I$,
(3) $I$ satisfies the pentagonal relation: $I_{12}I_{13}I_{23} = I_{23}I_{12}$.

By Lemmas 2.3.1, 2.4.2, 2.4.6 in [Val1], one has:

2.1.3. Notations and lemma. Let $I$ be a multiplicative partial isometry with the base $(N, \alpha, \beta, \hat{\beta})$, let us denote by $S$ the set $\{(\omega \otimes I) / \omega \text{ linear form on } L(H)\}$, and let us denote by $\hat{S}$ the set $\{(i \otimes \omega)(I) / \omega \text{ linear form on } L(H)\}$. Then $S$ and $\hat{S}$ are non-degenerate subalgebras of $L(H)$.

2.1.4. Lemma. (Cf. Lemme 2.6.2 of [Val1].) $S$ and $\hat{S}$ are von Neumann subalgebras of $L(H)$ if and only if $\{(i \otimes \omega)(\Sigma I) / \omega \text{ is a linear form on } L(H)\} = \alpha(N)'$; in this case one says that $I$ is regular.

2.2. $C^*$-quantum groupoids

Let us recall the definition of a $C^*$-quantum groupoid (or a weak Hopf $C^*$-algebra):
2.2.1. Definition. (See G. Böhm, K. Szlachányi, F. Nill [BoSzNi].) A weak Hopf $C^*$-algebra is any collection $(A, \Gamma, \kappa, \epsilon)$ such that:

- $A$ is a finite-dimensional $C^*$-algebra (or von Neumann algebra),
- $\Gamma : A \to A \otimes A$ is a generalized coproduct, which means that: $(\Gamma \otimes i) \Gamma = (i \otimes \Gamma) \Gamma$,
- $\kappa$ is an antipode on $A$, i.e., a linear map from $A$ to $A$ such that $(\kappa \circ \ast)^2 = i$ (where $\ast$ is the involution on $A), \kappa(xy) = \kappa(y)\kappa(x)$ for every $x, y$ in $A$ with $(\kappa \otimes \kappa) \Gamma = \zeta \Gamma \kappa$ (where $\zeta$ is the usual flip on $A \otimes A$). We suppose also that $(m(\kappa \otimes i) \otimes i)(\Gamma \otimes i) \Gamma(x) = (1 \otimes x) \Gamma(1)$ (where $m$ is the multiplication of tensors, i.e., $m(ab) = ab$),
- $\epsilon$ is a counit, i.e. a positive linear form on $A$ such that $(\epsilon \otimes \epsilon) \Gamma = (i \otimes \epsilon) \Gamma = i$, and for every $x, y$ in $A$: $(\epsilon \otimes \epsilon)((x \otimes 1) \Gamma(1)(1 \otimes y)) = \epsilon(xy)$.

2.2.2. Results. (Cf. [NV1,NV3,BoSzNi].) If $(A, \Gamma, \kappa, \epsilon)$ is a weak Hopf $C^*$-algebra, then the following assertions are true:

1. The sets
   
   $$A_t = \{ x \in A/\Gamma(x) = \Gamma(1)(x \otimes 1) = (x \otimes 1)\Gamma(1) \},$$
   $$A_s = \{ x \in A/\Gamma(x) = \Gamma(1)(1 \otimes x) = (1 \otimes x)\Gamma(1) \}$$

   are commuting sub $C^*$-algebras of $A$ and $\kappa(A_t) = A_s$; one calls them respectively target and source Cartan subalgebra of $(A, \Gamma, \kappa, \epsilon)$.

2. The application $\epsilon_t = m(\kappa \otimes i) \Gamma$ takes values in $A_t$ and $\epsilon_s = m(\kappa \otimes i) \Gamma$ takes values in $A_s$. We call target counit the application $\epsilon_t$ and we call source counit $\epsilon_s$.

3. The $C^*$-algebra $A$ has a unique projection $p$, called the Haar projection, characterized by the relations: $\kappa(p) = p, \epsilon_t(p) = 1$ and for every $a$ in $A$, $ap = \epsilon_t(a)p$.

4. There exists a unique faithful positive linear form $\phi$, called the normalized Haar measure of $(A, \Gamma, \kappa, \epsilon)$, satisfying the following three properties:

   $$(i \otimes \phi)((1 \otimes y) \Gamma(x)) = \kappa((i \otimes \phi)(\Gamma(y)(1 \otimes x))).$$

4. The map $E^s_\phi = (\phi \otimes i) \Gamma$ (respectively, $E^t_\phi = (i \otimes \phi) \Gamma$) is the conditional expectation with values in the source (respectively, target) Cartan subalgebra, such that: $\phi \circ E^s_\phi = \phi$ (respectively, $\phi \circ E^t_\phi = \phi$), it is called a source (respectively, target) Haar conditional expectation. If $g_s = E^s_\phi(p)^\frac{1}{2}$ and $g_t = E^t_\phi(p)^\frac{1}{2}$, then one has $g_t = \kappa(g_s)$. For every $a$ in $A$, $\kappa^2(a) = g_t g_s^{-1} a g_t^{-1} g_s$, and the modular group $\sigma^\phi_{-i}$ is given by $\sigma^\phi_{-i}(a) = g_t g_s a g_t^{-1} g_s^{-1}$; this leads to a polar decomposition $\kappa = j \circ Adg_t$, where $j$ is the involutive anti-homomorphism of $A$ (coinvolution) defined by $j(x) = g_s \kappa(x) g_s^{-1}$ for any $x$ in $A$.

It is shown in [Val1], that $I$ is regular, if and only if it generates two $C^*$-quantum groupoids in duality:

2.2.3. Proposition. If $I$ is regular, then one can define two $C^*$-quantum groupoids in duality $(\hat{S}, \hat{\Gamma}, \hat{\kappa}, \hat{\epsilon})$ and $(\hat{S}, \hat{\Gamma}, \hat{\kappa}, \hat{\epsilon})$, by the formulas:
for any $s \in S$: \[ \Gamma(s) = I(s \otimes 1)I^*, \]
for any $\hat{s} \in \hat{S}$: \[ \hat{\Gamma}(\hat{s}) = I^*(1 \otimes \hat{s})I, \]
for any $\omega \in \mathcal{L}(H)_o$: \[ \kappa((\omega \otimes i)(I)) = (\omega \otimes i)(I^*), \quad \hat{\kappa}((i \otimes \omega)(I)) = (i \otimes \omega)(I^*). \]
\[ \epsilon((\omega \otimes i)(I)) = \omega(1) \quad \text{and} \quad \hat{\epsilon}((i \otimes \omega)(I)) = \omega(1). \]

These two quantum groupoids are in duality, using the following brackets:
\[ \{(\omega \otimes i)(I), (i \otimes \omega')(I)\} = (\omega \otimes \omega')(I). \]

This proposition has a converse (see [Val1, 3.2.3] and [Val3]):

2.2.4. Proposition. Let $(A, \Delta, \kappa, \epsilon)$ be any C*-quantum groupoid such that $\kappa$ is involutive on Cartan subalgebras, and let $\phi$ be its normalized Haar measure, then with GNS notations, the map $I : A_{\phi} \mapsto A_{\phi}(\Delta(x)(1 \otimes y))$ is a regular mpi on $H_\phi \otimes H_\phi$, and the GNS representation $\pi_\phi$ is an isomorphism between $(A, \Delta, \kappa, \epsilon)$ and the C*-quantum groupoid $(S, \Gamma, \kappa, \epsilon)$ given by Proposition 2.2.3.

In fact, in [Val2, Theorem 2.6.5] we have proven that any regular mpi $I$ can be decomposed to obtain an irreducible one, for which there exist separating and cyclic vectors $e$ and $\hat{e}$ for $S$ and $\hat{S}$ respectively, which are in standard form on $H$. With Tomita’s theory notations, the unitary $U = J\hat{J} = \hat{J}J$ leads to define a fourth representation $\hat{\alpha} = \text{Ad}U \circ \alpha : N \to \mathcal{L}(H)$, and two new mpi $\hat{I} = \Sigma(U \otimes 1)I(U \otimes 1)\Sigma$, $\tilde{I} = \Sigma(1 \otimes U)I(1 \otimes U)\Sigma$ over the base $(N^o, \beta, \hat{\alpha}, \alpha)$ and $(N^o, \beta, \alpha, \hat{\alpha})$. Applying Proposition 2.2.3 to $\tilde{I}$ (respectively $\hat{I}$) leads to define another C*-quantum groupoid on $\hat{S}'$ (respectively $S'$), the commutant in $\mathcal{L}(H)$ of $\hat{S}$ (respectively $S$). Let $(S', \Gamma', \kappa', \epsilon')$ and $(\hat{S}', \hat{\Gamma}', \hat{\kappa}', \hat{\epsilon}')$ be these new C*-quantum groupoids.

2.2.5. Proposition. The commutant $\hat{\alpha}(N)'$ of $\hat{\alpha}(N)$ in $\mathcal{L}(H)$, is the vector space $S\hat{S}$ (called the Weyl algebra) generated by $\{s\hat{s} / s \in S, \hat{s} \in \hat{S}\}$ in $\mathcal{L}(H)$. One has: $S \cap \hat{S} = \alpha(N), S \cap S' = \beta(N), S' \cap \hat{S} = \hat{\beta}(N)$ and $S' \cap S' = \hat{\alpha}(N)$.

2.2.6. Notation. We shall denote by $\mathfrak{G}$, the Weyl algebra $S\hat{S}$.

2.3. The commutative example

Let us recall that a groupoid $\mathcal{G}$ is a small category the morphisms of which are all invertible. In all what follows, $\mathcal{G}$ is finite. One can identify the set of object, noted $\mathcal{G}^0$, to a subset of the morphisms. So a (finite) groupoid can also be viewed as a set $\mathcal{G}$ together with a, not everywhere, defined multiplication for which there is a set of unities $\mathcal{G}^0$, two maps, source denoted by $s$ and range by $r$, from $\mathcal{G}$ to $\mathcal{G}^0$ so that the product $xy$ of two elements $x, y \in \mathcal{G}$ exists if and only if $s(x) = r(y)$; every element $x \in \mathcal{G}$ has a unique inverse $x^{-1}$, and one has $x(yz) = (xy)z$ whenever both members make sense. We refer to [R] for the fundamental structures and notations for groupoids.
Let us denote $H = l^2(\mathcal{G})$, with the usual notations. Actually there exist four natural irreducible mpi associated to $\mathcal{G}$ (for an other example see [Val1, 4.1]). Let $I_\mathcal{G}$ be the mpi defined for any $x, y \in \mathcal{G}, \xi \in l^2(\mathcal{G})$, by:

$$I_\mathcal{G}(\xi)(x, y) = \xi(xy, y) \quad \text{if} \quad s(x) = r(y) \quad \text{and} \quad I_\mathcal{G}(\xi)(x, y) = 0 \quad \text{otherwise.}$$

In this case, $N$ is equal to $C(\mathcal{G}^0)$, the commutative involutive algebras of complex valued functions on $\mathcal{G}^0$, one has $\alpha = \hat{\beta}$ and $\hat{\alpha} = \beta$, which are given by the source and target functions $s$ (respectively $r$), so for every $n \in N$: $\alpha(n) = \hat{\beta}(n) = s \circ n$ and $\hat{\alpha}(n) = \beta(n) = r \circ n$. One has $S = C(\mathcal{G})$, which is the commutative involutive algebra of complex valued functions on $\mathcal{G}$, and $\hat{S} = R(\mathcal{G}) = \{ \sum_{x \in \mathcal{G}} a_x \rho(x) \}$ (the right regular algebra of $\mathcal{G}$) where $\rho(x)$ is the partial isometry given by the formula $(\rho(x)\xi)(t) = \xi(tx)$ if $x \in \mathcal{G}^{s(t)}$ and $= 0$ otherwise, $\hat{S}' = L(\mathcal{G}) = \{ \sum_{s \in \mathcal{G}} a_s \lambda(s) \}$ (the left regular algebra of $\mathcal{G}$), where $\lambda(s)$ is the partial isometry given by the formula $(\lambda(s)\xi)(t) = \xi(s^{-1}t)$ if $t \in \mathcal{G}^{r(s)}$ and $= 0$ otherwise, $S' = S$. The two $C^*$-quantum groupoids structures on $S$ and $\hat{S}$ are given by:

- **Coproducts:**

  $$\Gamma_\mathcal{G}(f)(x, y) = \begin{cases} f(xy) & \text{if } x, y \text{ are composable and } f \in C(\mathcal{G}), \\ 0 & \text{otherwise,} \end{cases} \quad \hat{\Gamma}_\mathcal{G}(\rho(s)) = \rho(s) \otimes \rho(s).$$

- **Antipodes:**

  $$\kappa_\mathcal{G}(f)(x) = f(x^{-1}), \quad \hat{\kappa}_\mathcal{G}(\rho(s)) = \rho(s^{-1}) = \rho(s)^*.$$

- **Counities:**

  $$\epsilon_\mathcal{G}(f) = \sum_{u \in \mathcal{G}^0} f(u), \quad \hat{\epsilon}_\mathcal{G}(\rho(s)) = 1.$$

### 3. Actions of $C^*$-quantum groupoids on von Neumann modules

As our definitions have direct generalizations to infinite dimension (see [EV, Definition 7.1]) in the von Neumann algebra context, the aim of this section is to give a framework, in operator algebras, for actions of $C^*$-quantum groupoids, and to find double crossed product properties extrapolating [N] in the von Neumann algebra context and [Y] in the quantum groupoids one.

In all what follows, $I$ will be an irreducible regular mpi with the base $(N, \alpha, \beta, \hat{\beta})$, we shall use the notations of Section 2.2, in particular one has: $S_\mathcal{I} = \hat{S}_\mathcal{I} = \alpha(N), S_I = \beta(N)$ and $\hat{S}_\mathcal{I} = \hat{\beta}(N)$, and by [Val2, 3.1], $I$ leads to define two other mpi $\hat{I}$ and $\check{I}$, so one has two other quantum groupoids for the commutants $S'$ and $\hat{S}'$.

#### 3.1. Actions of $C^*$-quantum groupoids

**3.1.1. Notations.** Let $A$ be a von Neumann acting on an Hilbert space $H$, let $b$ be any unital faithful anti-representation $S_i \to A$, let $i$ be the canonical inclusion $S_i \hookrightarrow S$, let $e_{b,i}$ be the
projection given by 2.1.1 in these conditions. Let \( b' \) be any unital faithful representation \( S_t \to A \) and \( \kappa \) viewed as a restriction \( S_t \to S''_t \), we shall denote by \( e_{b',\kappa} \) the corresponding projection given by 2.1.1 (applied to \( S''_t \)).

3.1.2. Definition. With notations above, let \( b : S_t \to A \) be a unital faithful anti-representation (respectively representation), one calls a right (respectively left) action of \( (S, \Gamma, \kappa) \) on \( (A, b) \) any map \( \delta \) (respectively \( \gamma \)) such that:

1. \( \delta \) (respectively \( \gamma \)) is an injective normal homomorphism \( A \to A \otimes S \) (not unital in general),
2. \( (\delta \otimes i)\delta = (i \otimes \Gamma)\delta \) (respectively \( (\gamma \otimes i)\gamma = (i \otimes \varsigma \Gamma)\gamma \)),
3. for any \( x \in S_t : \delta(b(x)) = e_{b,i}(1 \otimes \kappa(x)) \) (respectively for any \( x \in S_t : \gamma(b'(x)) = e_{b',\kappa}(1 \otimes x) \)).

3.1.3. Definition. We shall call right (respectively left) coaction of \( (S, \Gamma, \kappa) \) on \( (A, b) \) any right (respectively left) action of \( (\hat{S}, \hat{\Gamma}, \hat{\kappa}) \).

3.1.4. Remarks.

1. \( A \) becomes a right (respectively left) module over \( S_t \).
2. If \( S_t \subset Z(S) \) (the center of \( S \)), for every \( a \) in \( A \) and every \( x \) in \( S_t \), one has: \( \delta(ab(x)) = \delta(b(x)a) \), hence \( b(S_t) \subset Z(A) \). In particular, when it is not a quantum group, such a quantum groupoid cannot act on a factor.
3. If \( \alpha \) is a left action of \( (S, \Gamma, \kappa) \) on \( (A, b) \), it is a right action of \( (S, \varsigma \Gamma, \kappa) \), associated with \( \Sigma \hat{I}^* \Sigma \) on \( (A, b \circ \kappa) \).
4. \( \Gamma \) is a right action of \( (S, \Gamma, \kappa) \) on \( (S, \kappa|\beta(N)) \).

Hence till the end we shall only deal with right actions.

3.1.5. Lemma and definition. Let \( \delta \) be a right action of \( (S, \Gamma, \kappa) \) on \( (A, b) \), then one has:

\[
\{a \in A/\delta(a) = e_{b,i}(a \otimes 1)\} = \{a \in A \cap b(S_t)'/\delta(a) = e_{b,i}(a \otimes 1)\},
\]

this is a von Neumann subalgebra of \( A \cap b(S_t)' \), we shall call it the fixed point subalgebra of \( \delta \) and note it \( A^\delta \).

Proof. As \( S_t \) and \( \kappa(S_t) \) commute, hence for every \( y \) in \( S_t \), one has:

\[
\delta(b(y)) = e_{b,i}(1 \otimes \kappa(y)) = (1 \otimes \kappa(y))e_{b,i} = e_{b,i}(1 \otimes \kappa(y))e_{b,i},
\]

hence for any \( a \) in \( A \), such that \( \delta(a) = e_{b,i}(a \otimes 1) \), one has: \( \delta(ab(y)) = e_{b,i}(a \otimes \kappa(y))e_{b,i} = \delta(b(y)a) \), the lemma follows.

3.1.6. Proposition. Let \( \delta \) be a right action of \( (S, \Gamma, \kappa) \) on \( (A, b) \), then the map:

\[
T_\delta = (i \otimes \phi)\delta
\]

is a faithful conditional expectation \( A \to A^\delta \).

Proof. If \( (n_j) \) is a matrix unit for \( N \), then \( (\beta(n_j)) \) (respectively \( \alpha(n_j) \)) is also a matrix unit for \( S_t \) (respectively \( \hat{S}_t \)), so, for any \( a \in A \), by 2.1.1 one has:
\[ T_\delta(a) = (i \otimes \phi)(e_{b,i}(a \otimes 1)) = (i \otimes \phi) \left( \sum_j b(\beta(n_j)) \otimes \beta_{n_j^*} \right) a \]
\[ = (b \circ \kappa)(i \otimes \phi)(\Gamma(1))a = (b \circ \kappa)(1)a = a. \]

For every \( a \in A \), let us use the notations \( \delta(a) = a_1 \otimes a_2 \), one has:
\[ \delta(T_\delta(a)) = \delta((i \otimes \phi)\delta(a)) = \delta((i \otimes i \otimes \phi)(i \otimes \Gamma)\delta(a)) = \delta((i \otimes (i \otimes \phi)\Gamma)\delta(a)) = (i \otimes E_i)\delta(a) = e_{b,i}(i \otimes E_i)\delta(a) = e_{b,i}(a_1 \otimes E_i(a_2)) = e_{b,i}(a_1 b(E_i(a_2)) \otimes 1). \]

So one deduces that:
\[ (i \otimes \phi)\delta(T_\delta(a)) = (i \otimes \phi)(e_{b,i}(a_1 b(E_i(a_2)) \otimes 1)) = (i \otimes \phi)(e_{b,i})a_1 b(E_i(a_2)) = a_1 b(E_i(a_2)) \]
but on the other hand:
\[ (i \otimes \phi)\delta(T_\delta(a)) = (i \otimes \phi)(e_{b,i}(a_1 \otimes E_i(a_2))) = (i \otimes \phi)(\delta(1)\delta(a)) = (i \otimes \phi)\delta(a) = T_\delta(a). \]

Hence one has: \( T_\delta(a) = a_1 b(E_i(a_2)) \), replacing this in the expression of \( \delta(T \delta(a)) \), one has:
\[ \delta(T_\delta(a)) = e_{b,i}(a_1 b(E_i(a_2)) \otimes 1) = e_{b,i}(T_\delta(a) \otimes 1). \]
this implies that \( T_\delta(A) = A^\delta \).

Using the fact that any element of \( A^\delta \) commutes with \( b(S_t) \), for any \( b, c \) in \( A^\delta \) and any \( a \) in \( A \), one has:
\[ T_\delta(cab) = (i \otimes \phi)cab = (i \otimes \phi)(\delta(c)\delta(a)\delta(b)) = (i \otimes \phi)(e_{b,i}(c \otimes 1)\delta(a)e_{b,i}(b \otimes 1)) \]
\[ = (i \otimes \phi)((c \otimes 1)e_{b,i}\delta(a)e_{b,i}(b \otimes 1)) = c(i \otimes \phi)(\delta(1)\delta(a)\delta(1))b \]
\[ = cT_\delta(a)b. \]

The proposition follows. \( \Box \)

3.1.7. Definition. Let \( \delta \) (respectively \( \hat{\delta} \)) be a right action (respectively coaction) of \((S, \Gamma, \kappa)\) on a von Neumann module \((A, b)\) (respectively \((A, \hat{b})\)), the crossed product \( A \rtimes_\delta S \) (respectively \( A \hat{\rtimes}_{\hat{\delta}} \hat{S} \)) is the von Neumann subalgebra of \( eb,i(A \otimes L(H))eb,i \) (respectively \( e_{\hat{b},i}(A \otimes \hat{L}(H))e_{\hat{b},i} \)) generated by \( \delta(A) \) and \( e_{b,i}(1 \otimes \hat{S}) \) (respectively \( \hat{\delta}(A) \) and \( e_{\hat{b},i}^* (1 \otimes S') \)).

3.1.8. Remarks.

(1) One must keep in mind that the crossed product is degenerated in \( A \otimes \hat{L}(H) \), and its unit element is \( e_{b,i} \).
(2) If \( \alpha \) is a left action, as it is a right action one can define also a crossed product.
(3) As a matter of facts, in this Baaj and Skandalis formalism, \( \hat{S} \) is not equal to the one given by Vaes’ theory, but our crossed product do generalize the quantum groups one.
3.1.9. Lemma. The crossed product \( A \rtimes_S \) is the vector subspace of \( e_{b,i}(A \otimes L(H))e_{b,i} \) (respectively \( e_{b,k}(A \otimes L(H))e_{b,k} \)) generated by the products \( \delta(a)(1 \otimes \hat{b}) \ a \in A, \hat{b} \in \hat{S} \).

**Proof.** The equality \( (\delta \otimes i)\delta = (i \otimes \Gamma)\delta \) makes it possible to show that for any \( a \in A \), one has:

\[
I_{23}(\delta(a) \otimes 1) I^*_2 = \sum_i (\delta(a_i) \otimes c_i) I_{23}
\]

Hence for any linear form \( \omega \) on \( L(H) \), one has:

\[
(1 \otimes (i \otimes \omega)(I)) \delta(a) = (i \otimes i \otimes \omega)(I_{23}(\delta(a) \otimes 1)) = (i \otimes i \otimes \omega)(\sum_i (\delta(a_i) \otimes c_i) I_{23}) = \sum_i \delta(a_i)(1 \otimes (i \otimes (\omega.c_i))(I)).
\]

This leads to the lemma \( \square \)

3.1.10. Notation. Let us denote \( I' = \Sigma(\hat{\Gamma})' \Sigma = \Sigma(U \otimes U) I^*(U \otimes U) \Sigma \), hence \( I' \) is an mpi belonging to \( S' \otimes \hat{S}' \) over the base \( (N, \hat{\alpha}, \hat{\beta}, \hat{\beta}) \), which gives on \( \hat{S}' \) the opposite coproduct \( \hat{\Gamma}'{}^{opp} = \varsigma \hat{\Gamma}' \).

3.1.11. Proposition. Let \( \delta \) be a right action of \( (S, \Gamma, \kappa) \) on \( (A, b) \), let \( \hat{b} \) be the map \( \hat{S}_i = \alpha(N) \rightarrow A \rtimes_S \) defined for every \( n \in N \) by:

\[
\hat{b}(a(n)) = e_{b,i}(1 \otimes \hat{b}(n)).
\]

let \( \hat{\delta} \) be the map defined for every \( x \in A \rtimes_S \) by:

\[
\hat{\delta}(x) = \tilde{I}_{23}(x \otimes 1) \tilde{I}^*_2
\]

then \( (\hat{\delta}, \hat{b}) \) is a (right) coaction of \( (S, \Gamma, \kappa) \) on \( (A \rtimes_S, \hat{b}) \) and \( (A \rtimes_S) \rtimes_{\hat{\delta}} \hat{S} \) is isomorphic to the von Neumann subalgebra of \( \delta(1)(A \otimes L(H))\delta(1) \) generated by \( \delta(A) \) and \( \delta(1)(1 \otimes \hat{S}S') (= \delta(1)(1 \otimes \beta(N)')) \).

**Proof.** By [Val3, 3.1], the initial support of \( \tilde{I} \) is \( e_{\alpha,\hat{\beta}} \), but \( \hat{\alpha}(N) \) commutes with \( S \) and \( \hat{S} \), this implies that \( \hat{\delta} \) is a normal homomorphism on \( A \rtimes_S \); as \( \tilde{I} \in S' \otimes \hat{S} \) and his final support is \( e_{\hat{\beta},\alpha} \), for any \( a \) in \( A \) one has:

\[
(\hat{\delta}(\delta(a))(1 \otimes e_{\hat{\beta},\alpha})(\delta(a) \otimes 1))
\]
and due to [Val2, Proposition 3.1.4], for every \( \hat{b} \) in \( \hat{S} \), one has:

\[
\hat{\delta}(e_{b,i}(1 \otimes \hat{b})) = (e_{b,i} \otimes 1)(1 \otimes \hat{\Gamma}(\hat{b})).
\]

So \( \hat{\delta} \) takes its values in \( \mathcal{A} \rtimes_{\hat{\delta}} S \otimes \hat{\mathcal{S}} \), and:

\[
\hat{\delta}(\delta(a)(1 \otimes \hat{b})) = (\delta(a) \otimes 1)(1 \otimes \hat{\Gamma}(\hat{b}))
\]

also for every \( x \in \hat{S}_r \), one has:

\[
\hat{\delta}(\hat{b}(x)) = \hat{\delta}(\delta(1)(1 \otimes \hat{k}(x))) = (\delta(1) \otimes 1)(1 \otimes \hat{\Gamma}(\hat{k}(x))) = \hat{\delta}(\hat{b}(1))(1 \otimes 1 \otimes \hat{k}(x)).
\]

But, if \((n_j)\) is a matrix unit for \( N \), then \((\beta(n_j))\) (respectively \(\alpha(n_j)\)) is also a matrix unit for \( S_r \) (respectively \(\hat{S}_r \)), so by 2.1.1 one has:

\[
e_{b,i} = \sum_j \hat{b}(\alpha(n_j)) \otimes \alpha(n_j) = \sum_j e_{b,i}(1 \otimes \hat{\beta}(n_j) \otimes \alpha(n_j)) = (\delta(1) \otimes 1)(1 \otimes \hat{\Gamma}(1)) = \hat{\delta}(\hat{b}(1))
\]

by replacing this in the previous computation: \( \hat{\delta}(\hat{b}(x)) = e_{b,i}(1 \rtimes_{\hat{\delta}} S \otimes \hat{k}(x)) \).

Now let us verify condition (2) of Definition 3.1.2, using Sweedler notations, for any \( a \) in \( \mathcal{A} \) and any \( \hat{b} \) in \( \hat{S} \), one has:

\[
(\hat{\delta} \otimes i)(\delta(a)(1 \otimes \hat{b})) = (\delta \otimes i)((\delta(a) \otimes 1)(1 \otimes \hat{\Gamma}(\hat{b}))) = (\delta \otimes i)(\delta(a)(1 \otimes \hat{b}_1) \otimes \hat{b}_2)
\]

\[
= \hat{\delta}(\delta(a)(1 \otimes \hat{b}_1)) \otimes \hat{b}_2
= (\delta(a) \otimes 1)(1 \otimes \hat{\Gamma}(\hat{b}_1)) \otimes \hat{b}_2 = (\delta(a) \otimes 1 \otimes 1)(1 \otimes (\hat{\Gamma} \otimes i) \hat{\Gamma}(\hat{b}))
= (\delta(a) \otimes \hat{\Gamma}(1))(1 \otimes (i \otimes \hat{\Gamma}) \hat{\Gamma}(\hat{b})) = (i \otimes i \otimes \hat{\Gamma})(\delta(a) \otimes 1)(1 \otimes \hat{\Gamma}(\hat{b}))
= (i \otimes i \otimes \hat{\Gamma}) \hat{\delta}(\delta(a)(1 \otimes \hat{b})).
\]

One can deduce that: \( (\hat{\delta} \otimes i) \hat{\delta} = (i \otimes i \otimes \hat{\Gamma}) \hat{\delta} \).

Now let us define the one to one morphism \( \gamma \) on \( \mathcal{A} \otimes \mathcal{L}(\mathcal{H}) \) by:

\[
\gamma(x) = I_{23}^\ast(\delta \otimes i)(x)I_{23}, \quad \forall x \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}).
\]

Obvious computations give that, for any \( a \) in \( \mathcal{A} \), \( \hat{s} \) in \( \hat{S} \), \( s' \) in \( S' \), one has:

\[
\gamma(\delta(a)) = \hat{\delta}(1)(\delta(a) \otimes 1), \quad \gamma(\delta(1)(1 \otimes \hat{s})) = \hat{\delta}(1)(1 \otimes \hat{\Gamma}(\hat{s})), \quad \gamma(\delta(1)(1 \otimes s')) = \hat{\delta}(1)(1 \otimes 1 \otimes s').
\]

Hence, \( \gamma \) is an isomorphism between the von Neumann subalgebra of \( \delta(1)(A \otimes \mathcal{L}(\mathcal{H}))(1 \otimes \hat{\delta}(1)) \)

generated by \( \delta(A) \) and \( \delta(1)(1 \otimes \hat{SS}') \) (= \( \delta(1)(1 \otimes \beta(N)'(1)) \)), and \( (A \rtimes_{\hat{\delta}} S) \rtimes_{\hat{\delta}} \hat{S} \). \( \square \)
3.2. Actions of groupoids

Let us explain what is an action $\alpha$ of $(C(\mathcal{G}), \Gamma_\mathcal{G}, \kappa_\mathcal{G})$, where $\mathcal{G}$ is any finite groupoid, on a von Neumann module $(A, b)$. In fact, the map $b$ is clearly equivalent to the data of a decomposition $A = \bigoplus_{u \in \mathcal{G}^0} A_u$, where each $A_u$ is a von Neumann algebra, the relation is given, for every $u \in \mathcal{G}^0$, by $b(\delta_u) = 1_u$, where $\delta_u$ is the Dirac function for $u$ and $1_u$ the identity element of $A_u$ (a projection in $Z(A)$). Hence $A$ is a module over $C(\mathcal{G}^0)$.

3.2.1. Definition. An action of $\mathcal{G}$ on $A$ is any covariant functor $F$ from the category $\mathcal{G}$ to the category whose objects are the element of the set $\{A_u, u \in \mathcal{G}^0\}$ and the morphisms the von Neumann algebras isomorphisms, and such that, for any $u \in \mathcal{G}^0$, $F(u) = A_u$.

Hence, for any $g \in \mathcal{G}$, it exists a morphism $\alpha_g (\equiv F(g)) : A_{s(g)} \mapsto A_{r(g)}$, in order that for any pair $(g, g')$ of composable elements, one has: $\alpha_{gg'} = \alpha_g \alpha_{g'}$.

As it can be decomposed in its connection classes, we can suppose that $\mathcal{G} = \bigsqcup_i X_i \times X_i \times G_i$, where $X_i$ is a finite set and $G_i$ is a finite group. In fact one has $\mathcal{G}^0 = \bigsqcup_i X_i$ and $G_i$ is isomorphic to the isotropy group $G^0_u$ for any $u \in X_i$.

3.2.2. Proposition. Any finite groupoid $\mathcal{G}$ acts on $(R^{G^0}, b_{G^0})$, where $R$ is the hyperfinite type $II_1$ factor and $b_{G^0} : f \mapsto (f(u)1_u)_{u \in G^0}$.

Proof. Due to the previous remark, one can suppose that $\mathcal{G} = X \times X \times G$ where $X$ is a finite set and $G$ is a group. As it is well known, there exists an action $\beta$ (even outer) of $G$ on $R$. Of course $R^{G^0}$ can be decomposed in its cartesian components, each of them is in fact $R$ itself; up to this identification, one can define for any $(x, y, g) \in \mathcal{G}$ $\alpha_{(x,y,g)} = \beta_g$, one easily sees that this is an action. □

3.2.3. Remark. Let $\alpha : A \rightarrow A \otimes C(\mathcal{G})$ be an action of $(C(\mathcal{G}), \Gamma_\mathcal{G}, \kappa_\mathcal{G})$ on $(A, b)$. The action $\alpha$ can also be viewed as a normal homomorphism from $A$ to the von Neumann algebra $A \otimes L(l^2(\mathcal{G}))$, in this sense the following lemma is true:

3.2.4. Lemma. Let $\alpha$ be an action of $(C(\mathcal{G}), \Gamma_\mathcal{G}, \kappa_\mathcal{G})$ on $(A, b)$, using the notations of 2.3, for any $g \in \mathcal{G}$ one has: $(1 \otimes \lambda(g)^*)\alpha(A)(1 \otimes \lambda(g)) \subset \alpha(A)$.

Proof. This is just a generalization of the demonstration of Proposition 1.3(ii) in [E1], replacing the unitary $W_G$ by the adjoint of the regular mpi defined in [Val1, 4.1]. □

In fact the two notions of action are equivalent:

3.2.5. Proposition.

(i) For any action of $\mathcal{G}$ on $(A, b)$, the map $\delta_\alpha$ (respectively $\gamma_\alpha$) : $A \mapsto A \otimes C(G) (= C(G, A))$ defined for every $a \in A$ by $\delta_\alpha(a) : g \mapsto \alpha_g(a_{s(g)})$ (respectively $\gamma_\alpha(a) = \alpha^{-1}g(a_{r(g)})$) is a right (respectively left) action of $(C(\mathcal{G}), \Gamma_\mathcal{G}, \kappa_\mathcal{G})$ on $(A, b)$.

(ii) For any left (respectively right) action $\gamma$ (respectively $\delta$) of $(C(\mathcal{G}), \Gamma_\mathcal{G}, \kappa_\mathcal{G})$ on $(A, b)$ there exists a unique action of $\mathcal{G}$ on $(A, b)$, such that $\gamma = \gamma_\alpha$ (respectively $\delta = \delta_\alpha$).
Proof. Using Lemma 3.2.4, one can exactly use the arguments in [E1, Proposition 1.3].

3.2.6. Remarks.

(1) Our definition of an action agrees with the algebraic definition due to Vainerman and Nikshych in [NV2]: let $\gamma$ (respectively $\delta$) be a left (respectively right) action of $(C(\mathcal{G}), \Gamma_G, \kappa_G)$ on $(A, b)$, then if for any $a$ in $A$ and $h$ in $\mathcal{G}$ we define: $\lambda(h) a = \gamma(ah)$ (respectively $a \triangleright \rho(h) = \delta(a)h$), this is a left (respectively right) action and the same formula can be used for the inverse assertion.

(2) As $\delta(1) = e_{b,r} \neq 1$, $\delta(A)$ is degenerate in $C(G, A)$, so it is better to restrict $\delta$ to $e_{b,r}(A \otimes C(\mathcal{G}))e_{b,r}$, which can be identified with $\bigoplus_{g \in \mathcal{G}} A_{r(g)}$, and $\delta(a)$ can be identified with $(\alpha_g(a_{s(g)}))_g$, in that way $\delta$ is unital.

3.2.7. Notations. Let us consider $H = L^2(A)$, the standard Hilbert space of $A$, then $H$ has an orthogonal decomposition $H = \bigoplus_{h \in \mathcal{G}} H_h$; for $j \in \{s, r\}$, $e_{b,j}(A \otimes C(\mathcal{G}))e_{b,j}$ can also be represented as a “diagonal” von Neumann algebra acting on $\bigoplus_{g \in \mathcal{G}} H_{j(g)}$. For any $h \in \mathcal{G}$, one can define the operator $(1_b \otimes_r \rho(h))$ on $\bigoplus_{g \in \mathcal{G}} H_{r(g)}$, by the formula $(1_b \otimes_r \rho(h))(\xi_g)_{g \in \mathcal{G}} = (\eta_g)_{g \in \mathcal{G}}$, where $\eta_g = 0$ if $s(g) \neq r(g)$ and $\eta_g = \xi_{gh}$ otherwise. For any $a \in A$, one also can define the operator $(a_h \otimes_s 1) = \bigoplus a_{s(g)}$ which acts on $\bigoplus H_{s(g)}$. Let us also denote by $u_g$ the canonical implementation of $\alpha_g$ for any $g \in \mathcal{G}$ (Theorem 2.18 of [H]), so the operator $U = \bigoplus u_g$ is a unitary $\bigoplus H_{s(g)} \to \bigoplus H_{r(g)}$. Hence, obviously one has:

3.2.8. Proposition. For any $a$ in $A$, one has: $\delta(a) = U(1_b \otimes_s a)U^*.$

3.3. Crossed product by groupoids actions and outer actions

In all what follows $\alpha$ is an action of $\mathcal{G}$ on $(A, b)$ and $\delta$ (= $\delta_\alpha$) is the right action of $(C(\mathcal{G}), \Gamma_G, \kappa_G)$ on $(A, b)$ associated with $\alpha$. First let us give a simple description of this crossed product.

3.3.1. Remark. The crossed product of $(A, b)$ by $\mathcal{G}$ is the sub-von Neumann algebra of $\mathcal{L}(\bigoplus_{g \in \mathcal{G}} H_{r(g)})$ generated by $\delta(A)$ and the operators $(1_b \otimes_r \rho(h))$.

3.3.2. Lemma.

(i) For any $a$ in $A$ and $h$ in $\mathcal{G}$, one has:

\[
(1_b \otimes_r \rho(h))\delta(a) = \delta(\alpha_h(a_{s(h)}))(1_b \otimes_r \rho(h)).
\]

(ii) The crossed product $A \rtimes_\delta C(\mathcal{G})$ is the vector space generated by the products $\delta(a)(1_b \otimes_r \rho(h))$ for any $(a, h)$ in $A_b \times_r \mathcal{G}$ (= $\{(a, h) \in A \times \mathcal{G}, a \in A_{r(h)}\}$).

(iii) $A \rtimes_\delta C(\mathcal{G})$ is the set of elements in $\mathcal{L}(\bigoplus_{g \in \mathcal{G}} H_{r(g)})$, which can be decomposed as a sum of the form $\sum_{h \in \mathcal{G}} \delta(x^h)(1_b \otimes_r \rho(h))$, where $x^h \in A_{r(h)}$ for all $h \in \mathcal{G}$, and this decomposition is unique.
Proof. For any \( a \) in \( A \), \( h \) in \( G \) and \((\xi_g)_{g \in G}\) in \( \bigoplus_{g \in G} H_{r(g)} \), one has:

\[
(1_b \otimes_r \rho(h))\delta(a)((\xi_g)_{g \in G}) = (1_b \otimes_r \rho(h))((\alpha_g(a_{s(g)})\xi_g)_{g \in G}) = (1_b \otimes_r \rho(h))((\alpha_g(a_{s(g)})\xi_g)_{g \in G}) = (\eta_g)_{g \in G}
\]

where one has:

\[
\eta_g = \begin{cases} 
  \alpha_g(\alpha_h(a_{s(h)}))\xi_{gh} & \text{if } s(g) = r(h), \\
  0 & \text{otherwise}.
\end{cases}
\]

On the other side for any \( b \) in \( A \), \( h \) in \( G \) and \((\xi_g)_{g \in G}\) in \( \bigoplus_{g \in G} H_{r(g)} \), one has:

\[
\delta(b)(1_b \otimes_r \rho(h))((\xi_g)_{g \in G}) = (\delta(b)(\eta'_g))_{g \in G}
\]

where one has:

\[
\eta'_g = \begin{cases} 
  \xi_{gh} & \text{if } s(g) = r(h), \\
  0 & \text{otherwise}.
\end{cases}
\]

hence \( \delta(b)(1_b \otimes_r \rho(h))((\xi_g)_{g \in G}) = (\alpha_g(b_{s(g)})\eta'_g)_{g \in G} = (\eta''_g)_{g \in G} \), so one deduces that:

\[
\eta''_g = \begin{cases} 
  \alpha_g(b_{r(h)})\xi_{gh} & \text{if } s(g) = r(h), \\
  0 & \text{otherwise}.
\end{cases}
\]

Hence we have: \((1_b \otimes_r \rho(h))\delta(a) = \delta(b)(1_b \otimes_r \rho(h))\) for any \( a, b, h \) such that \( b_{r(h)} = \alpha_h(a_{s(h)}) \), (i) and (ii) follow immediately.

If one chooses for any \( u \in G^0 \) a base \((a^k_u)\) of \( A_u \), one easily sees that the family \((\delta(a^k_u)(1_b \otimes_r \rho(h)))\) is free, hence (iii) is a consequence of (ii). \( \square \)

3.3.3. Corollary. The map: \( \sum_{h \in G} \delta(x^h)(1_b \otimes_r \rho(h)) \rightarrow x^h \otimes \rho(h) \) leads to an isomorphism between \( A \rtimes_\delta C(G) \) and the corresponding crossed product by L. Vainerman and D. Nikshych.

3.3.4. Definition. Let us call isotropic subgroupoid of \( G \), the subgroupoid of \( G \), denoted \( \text{iso}(G) \), equal to \{ \( h \in G, s(h) = r(h) \} \).

3.3.5. Remark. Obviously \( \text{iso}(G) \) is the disjoint union of the isotropic groups \( G^0_u \).

3.3.6. Lemma. Let us suppose that \( Z(A) \) is isomorphic to \( C(G^0) \) (or equivalently each \( A_u \) is a factor). An element \( x \in A \rtimes_\delta C(G) \) commutes with \( \delta(A) \) if and only if for any \( h \notin \text{iso}(G) \), one has \( x^h = 0 \), and for any \( h \in \text{iso}(G) \) with \( x^h \neq 0 \), then \( \alpha_h \) is inner and there exist a complex number \( \lambda_h \) and a unitary \( u_h \) in \( A_{r(h)} \) such that \( x^h = \lambda_h u_h \) and \( \alpha_h = A^d u_h \) (for all \( a \in A_{r(h)} \): \( \alpha_h(a) = (u^h)^{-1} a u^h \)).

Proof. Due to Lemma 3.3.2, for any \( x \in A \rtimes_\delta C(G) \), if \( \delta(a)x = x\delta(a) \) for all \( a \) in \( A \), then for any \( a \) in \( A \) and \( h \) in \( G \), one has: \( \delta(a_{r(h)})(x^h)(1_b \otimes_r \rho(h)) = \delta(x^h\alpha_h(a_{s(h)}))(1_b \otimes_r \rho(h)) \). Hence, one has: \( a_{r(h)}x^h = x^h\alpha_h(a_{s(h)}) \).
If \( h \notin \text{iso}(\mathcal{G}) \), let \( a \) be the element \( \alpha_{h^{-1}}(x^h)^* \) and so \( a_{r(h)} = 0 \), one deduces that: \( 0 = x^h \alpha_{h}(\alpha_{h}(x^h)) = x^h(x^h)^*, \) so \( x^h = 0 \).

If \( h \in \text{iso}(\mathcal{G}) \), let us suppose \( x^h \neq 0 \), then for any \( a \in A_{r(h)} \), one has: \( ax^h x^h = (a^* x^h)^* = (x^h \alpha_{h}(a^*))^* = \alpha_{h}(a)(x^h)^* \). Hence for any \( b \in A_{r(h)} \), we have: \( x^h b x^h = x^h \alpha_{h}(b)(x^h)^* = bx^h(x^h)^* \), as \( A_{r(h)} \) is a factor, this implies that there exists a strictly positive real number \( \mu \) for which \( x^h(x^h)^* = \mu 1 \), so \( u_h = 1/\sqrt{\mu} x_h \) is a unitary in \( A_{r(h)} \), one has \( x_h = 1/\mu u_h \) and for all \( a \in A_{r(h)} \), \( \alpha_{h}(a) = (u^h)^{-1} au^h \).

### 3.3.7. Remark.
The von Neumann algebra \( \delta(A) \cap A \rtimes_{\delta} C(\mathcal{G}) \), the relative commutant of \( \delta(A) \) in \( A \rtimes_{\delta} C(\mathcal{G}) \), contains \( Z(A) \rtimes_{\delta} \beta(C(\mathcal{G}^0)) \), whose elements are of the form: \( \sum_{u \in G^0} \delta(x^u)(1_b \otimes \rho(u)) \), for \( x^u \in Z(A_u) \) which, in the case when \( Z(A) = b(C(\mathcal{G}^0)) \), is just \( 1_b \otimes \beta(C(\mathcal{G}^0)) \).

### 3.3.8. Definition.

(i) A right action \( \delta \) of \( (C(\mathcal{G}), \Gamma_{\mathcal{G}}, \kappa_{\mathcal{G}}) \) on a von Neumann module \( (A, b) \) is said to be outer if and only if \( \delta(A) \cap A \rtimes_{\delta} C(\mathcal{G}) \) is equal to \( Z(A) \rtimes_{\delta} \beta(C(\mathcal{G}^0)) \).

(ii) An action \( \alpha \) of \( \mathcal{G} \) on a von Neumann module \( (A, b) \) is said to be outer if and only if for any \( h \in \text{iso}(\mathcal{G}) \) such that \( h \notin G^0 \), one has \( \alpha_h \) not trivial in \( \text{Out}(A_{r(h)}) \).

### 3.3.9. Remark.
The transitive groupoid \( \mathcal{G} = X \times X \times \text{Out} R \) acts outerly on \( R^X \).

As a consequence of Lemma 3.3.6, one has:

### 3.3.10. Proposition.
*In the case when \( Z(A) = b(C(\mathcal{G}^0)) \), any right action \( \delta \) of the quantum groupoid \( (C(\mathcal{G}), \Gamma_{\mathcal{G}}, \kappa_{\mathcal{G}}) \) on a von Neumann module \( (A, b) \) is outer if and only if the action of \( \mathcal{G} \) on \( (A, b) \), canonically associated with \( \delta \), is outer.*

And finally:

### 3.3.11. Proposition.
*Any finite groupoid \( \mathcal{G} \) acts outerly on the von Neumann module \( (R^{G^0}, b_{G^0}) \).*

**Proof.** As in Proposition 3.2.2, one can suppose \( \mathcal{G} = X \times X \times G \) with the same notations, then \( \text{iso}(\mathcal{G}) = \{(x, x, g), x \in X, g \in G\} \), so for any \( h \in \text{iso}(\mathcal{G}) \) not in \( G^0 \), there is \( x \in X \) and \( g \in G \), \( g \) different from the unit element, such that \( h = (x, x, g) \), then \( \alpha_h = \beta_g \) which can be taken in \( \text{Out}(R) \) and obviously \( Z(A) = b(C(\mathcal{G}^0)) \), the proposition follows from Lemma 3.3.6.

### 3.4. Double crossed products

Now, let us give a refinement of Proposition 3.1.11 in the commutative case, that is \( S = C(\mathcal{G}) \) (see also [Y, Theorem 6.4] for more general groupoids).

### 3.4.1. Lemma.
*Let \( \delta \) be a right action of \( (C(\mathcal{G}), \Gamma_{\mathcal{G}}, \kappa_{\mathcal{G}}) \) on a von Neumann module \( (A, b) \), then \( \delta(A)(1 \otimes C(\mathcal{G})) = \delta(1)(A \otimes C(\mathcal{G})) \).*

**Proof.** Clearly, one has: \( \delta(A)(1 \otimes C(\mathcal{G})) \subset \delta(1)(A \otimes C(\mathcal{G})) \). On the other hand, using the identification of \( \delta(1)(A \otimes C(\mathcal{G})) \) with the set of functions \( \phi : \mathcal{G} \rightarrow A \) such that for any \( g \in \mathcal{G} \), one has: \( \phi(g) \in A_{r(g)} \), one easily sees that \( \phi = \sum_{g} \delta(\alpha_{g^{-1}}(\phi(g)))(1 \otimes \delta_{g}) \), the lemma follows.
3.4.2. Theorem. Let $\delta$ be a right action of $(C(\mathcal{G}), \Gamma_\delta, \kappa_\delta)$ on a von Neumann module $(A, b)$, then the double crossed product $(A \rtimes_\delta C(\mathcal{G})) \rtimes_\delta \mathcal{R}(\mathcal{G})$ is isomorphic to $\delta(1)(A \otimes \mathcal{W}(\mathcal{G}))\delta(1)$, where $\mathcal{W}(\mathcal{G})$ (the Weyl algebra of $C(\mathcal{G})$) is the commutant in $\mathcal{L}(l^2(\mathcal{G}))$ of $r(l^\infty(\mathcal{G}^0)) = \hat{\alpha}(N)'$ which is also the von Neumann subalgebra generated by $C(\mathcal{G})$ and $\mathcal{R}(\mathcal{G}) = SS$.

Proof. Using Proposition 3.1.11, and the fact that $S = S'$, one deduces that: $(A \rtimes_\delta C(\mathcal{G})) \rtimes_\delta \mathcal{R}(\mathcal{G})$ is isomorphic to $\delta(1)(1 \otimes S\hat{S})\delta(1)$. Now thanks to Lemma 3.4.1 and the fact that $S\hat{S}$ is the vector space generated by $\{s\hat{s} / s \in S, \hat{s} \in S\}$ (and also equal to $\beta(N)'$), the double crossed product is also isomorphic to $\delta(1)(A \otimes S\hat{S})\delta(1)$, the theorem follows. □

3.5. Action of a groupoid on a fibered space over its base

Let us suppose now that $A$ is commutative and finite-dimensional, hence there is a finite set $X$ such that $A = C(X)$, the von Neumann algebra of functions on $X$, the existence of $b$ leads to a partition $X = \bigsqcup_{g \in G^0} X_u$, and for each $u \in G^0$, one has: $A_u = C(X_u)$.

A left (respectively right) action of the groupoid $\mathcal{G}$ on $(A, b)$ is given by a covariant (respectively contravariant) functor between the small category $\mathcal{G}$ and by denoted bijection between the two notions is given by the following formulae:

Let $\gamma(\cdot) \in \alpha(N)$ such that for any $g, g' \in G$ which are composable, one has for any $x$ in $X^{s(g')}$ (respectively $X^{r(g)}$): $g \triangleright (g' \triangleright x) = g g' \triangleright x$ (respectively $(x \triangleright g) \triangleright g' = x \triangleright g' g$). The bijection between the two notions is given by the following formulae:

for any $a$ in $A$ and $g$ in $\mathcal{G}$: $\gamma(a)_g = g^{-1} \triangleright a_r(g)$ (respectively $\delta(a)_g = a_s(g) \triangleright g$).

The crossed product $C(\mathcal{G}) \rtimes_\gamma A$ (respectively $A \rtimes_\delta C(\mathcal{G})$) can also be interpreted as the image of a certain $*$-algebra representation.

Let us denote by $X_b \times_r \mathcal{G} = \{(x, g) \in X \times \mathcal{G} / b(x) = r(g)\}$, the fiber product of $X$ and $\mathcal{G}$, and by $L^1(X_b \times_r \mathcal{G})$ the vector space of functions on this set. One can give to $L^1(X_b \times_r \mathcal{G})$ a $*$-algebra structure denoted by $(L^1(X_b \times_r \mathcal{G}), \star_\gamma, \#_\gamma)$ (respectively $(L^1(X_b \times_r \mathcal{G}), \star_\delta, \#_\delta)$).

For any functions $F, F'$ in $L^1(X_b \times_r \mathcal{G})$ and any $(x, g)$ in $X_b \times_r \mathcal{G}$, one has:

$$F \star_\gamma F'(x, g) = \sum_{r(h) = r(g)} F(x, h) F'(h^{-1} \triangleright x, h^{-1} g),$$

$$F \#_\gamma(x, g) = F(g^{-1} \triangleright x, g^{-1})$$

(respectively $$F \star_\delta F'(x, g) = \sum_{r(h) = r(g)} F(x, h) F'(x \triangleright h, h^{-1} g) F \#_\delta(x, g) = F(x \triangleright g, g^{-1}).$$

One must keep in mind that these functions have the good support. One can define a left (respectively right) regular representation of $L^1(X_b \times_r \mathcal{G})$ in $l^2(X_b \times_s \mathcal{G})$ (respectively $l^2(X_b \times_r \mathcal{G})$) denoted $L_\gamma$ (respectively $R_\delta$); for any $\xi$ in $l^2(X_b \times_s \mathcal{G})$ (respectively $l^2(X_b \times_r \mathcal{G})$) any $F$ in $L^1(X_b \times_r \mathcal{G})$ and any $(x, g)$ in $X_b \times_s \mathcal{G}$ (respectively $X_b \times_r \mathcal{G}$):

$$L_\gamma(F)\xi(x, g) = \sum_{r(h) = r(g)} F(g \triangleright x, h)\xi(x, h^{-1} g)$$

(respectively $$R_\delta(F)\xi(x, g) = \sum_{r(h) = s(g)} F(x \triangleright g, h)\xi(x, gh).$$


4.1.1. Notations.

With these definitions one can also formulate an alternative definition of the crossed products:
\[ L^Y(L^1(X_b \times_r G)) = C(G) \ltimes_Y A \quad \text{and} \quad R^\delta(L^1(X_b \times_r G)) = A \rtimes_\delta C(G). \]
In fact, if, for any \((a, g)\) in \(X \times G\), one denotes \(\chi_a\) (respectively, \(\chi_{(a,g)}\)), the characteristic function of \([a]\) (respectively, \([a,g]\))\), one has:
\[
\delta(\chi_1)\left(1_s \otimes_r \rho(g)\right) = R^\delta(\chi_{(y,g)}), \quad \gamma(\chi_y)\left(1_r \otimes_s \lambda(g)\right) = L^Y(\chi_{(y,g)}).
\]

4. Quantum groupoids coming from generalizations of the matched pair procedure

4.1. The matched pair of groupoids situation

Now let us explain an extension of the commutative example. Let \(G\) be any groupoid and \(H, K\) be two subgroups of \(G\) such that \(G = HK = \{hk \mid h \in H, \ k \in K^{(h)}\}\) and such that \(H \cap K \subseteq G^0\). The pair \(H, K\) is called a matched pair of groupoids (see [AA] for an abstract point of view). One easily verifies that this implies that \(G^0 = H \cap K\) and that for any \(g\) in \(G\) the decomposition \(g = hk\), \(h \in H, k \in K\) is unique. Hence one can define two maps, \(p_1 : G \to H\) and \(p_2 : G \to K\) by the relation \(g = p_1(g)p_2(g)\) for any \(g \in G\). Clearly one has \(s \circ p_2 = s\) and \(r \circ p_1 = r\), but a new map appears, the middle one:

4.1.1. Notations.

1. One has \(s \circ p_1 = r \circ p_2\), this map will be denoted \(m\).
2. For any \(f \in C(G^0)\), we define \(\alpha, \beta, \hat{\beta}\) by: \(\alpha(f) = f \circ m\) (the middle representation), \(\beta(f) = f \circ r\) (the range representation) and \(\hat{\beta}(f) = f \circ s\) (the source representation).
3. With the exception of the four representations of the base \(N = C(G^0)\), we shall use the same notations as in the commutative case.

4.1.2. Lemma. For any \(h\) in \(H\), \(\text{Card}(K^{(h)}) = \text{Card}(K^{r(h)})\).

**Proof.** Let us fix \(h\) in \(H\); let \(k\) be any element of \(K^{(h)}\). As \(G = HK\), then also \(G = KH\), so there exists a single pair \((k', h')\) in \(KH\) such that \(hk = k'h'\). Let us prove that the map \(k \mapsto k'\) defines an injection from \(K^{(h)}\) into \(K^{r(h)}\); if \(k_1\) is any element of \(K^{(h)}\) such that \(tk_1 = k'\), then there exists \(h_1'\) for which one has \(hk_1 = k'h_1'\), one deduces that \(h^{-1}k' = k_1h_1'^{-1} = kh'^{-1}\) from which one deduces that \(k = k_1\) and \(h' = h_1'\). So the map is injective and \(\text{Card}(K^{(h)}) \leq \text{Card}(K^{r(h)})\), applying this to \(h^{-1}\) one also has the inverse inequality. The lemma follows. \(\Box\)

4.1.3. Lemma. For any \(u\) in \(G^0\), one has the following equalities: \(\text{Card}(m^{-1}(u)) = \text{Card}(s^{-1}(u)) = \text{Card}(r^{-1}(u)) = \text{Card}(G^u) = \text{Card}(G_u)\). One has: \(\text{tr} \circ \alpha = \text{tr} \circ \beta = \text{tr} \circ \hat{\beta}\).

**Proof.** The equality \(\text{Card}(s^{-1}(u)) = \text{Card}(r^{-1}(u))\) is well known and is due to the bijection \(g \mapsto g^{-1}\) which gives \(\text{tr} \circ \beta = \text{tr} \circ \hat{\beta}\). For every \(f\) in \(C(G^0)\) one has: \((\text{tr} \circ m)(f) = \sum_{u \in G^0} \text{Card}(m^{-1}(u))f(u)\), so the only thing to prove is that for any \(u \in G^0\), one has: \(\text{Card}(G^u) = \text{Card}(m^{-1}(u))\).

But the map \((h, k) \mapsto hk\) is a bijection between \(H_u \times K^u\) and \(m^{-1}(u)\), so
\[
\text{Card}(m^{-1}(u)) = \text{Card}(H_u) \text{Card}(K^u).
\]
On the other hand, any element $g$ in $G^u$ has a unique decomposition $g = hk$ where $h \in H^u$ and $k \in K^s(h)$, one easily gets that the image of $G^u$ by the bijection $g \mapsto (h, k)$ is equal to the disjoint union: $\bigcup_{h \in H^u} \{h\} \times K^s(h)$, so using Lemma 4.1.2 and the last equality, one has:

$$\text{Card}(G^u) = \sum_{h \in H^u} \text{Card}(K^s(h)) = \sum_{h \in H^u} \text{Card}(K^r(h)) = \text{Card}(H^u) \text{Card}(K^u)$$

$$= \text{Card}(H_u) \text{Card}(K^u) = \text{Card}(m^{-1}(u)).$$

4.1.4. Lemma. For any $x$, $y$ in $G$ such that $m(x) = r(y)$, one has:

1. the elements $p_2(x)^{-1}$ and $y$ are composable for the multiplication of $G$,
2. the same is true for $x$ and $p_1(p_2(x)^{-1}y), $$
3. $m(xp_1(p_2(x)^{-1}y)) = m(y)$.

Proof. For any $x$, $y$ in $G$ such that $m(x) = r(y)$, then $s(p_2(x)^{-1}) = r(p_2(x)) = m(x) = r(y)$, so $p_2(x)^{-1}$ and $y$ are composable. But one has: $r(p_1(p_2(x)^{-1}y)) = r(p_2(x)^{-1}y) = r(p_2(x)^{-1}) = s(p_2(x)) = m(x) = s(x)$, so $x$ and $p_1(p_2(x)^{-1}y)$ are composable too. As for any $(h, k)$ in $H \times K$ and $g$ in $G$, one has: $m(hgk) = m(g)$, one deduces that: $m(xp_1(p_2(x)^{-1}y)) = m(p_2(x)p_1(p_2(x)^{-1}y))$; let $(h_1, k_1)$ be in $H \times K$ and such that: $p_2(x)^{-1}y = h_1k_1$, then: $m(xp_1(p_2(x)^{-1}y)) = m(p_2(x)h_1) = m(yk_1^{-1}) = m(y)$. □

So, generalizing the case of matched pairs of groups [BS, Theorem 8.21], the following definition is relevant:

4.1.5. Definition. We shall denote by $I_{H,K}$ the linear endomorphism of $l^2(G)$ such that for any $f$ in $l^2(G)$ and $x, y$ in $G$ one has:

$$I_{H,K}(f)(x, y) = \begin{cases} f(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y) & \text{if } m(x) = r(y), \\ 0 & \text{otherwise}. \end{cases}$$

In particular: $I_{G,G^0} = I_G$ and $I_{G^0,G}$ is the mpi studied in [Val1, 4.1].

4.1.6. Proposition. $I_{H,K}$ is an mpi over the base $(C(G^0), \alpha, \beta, \beta)$. 

Proof. An easy computation gives the following formula for $I_{H,K}^*$, for any $f$ in $l^2(G)$ and $x, y$ in $G$ one has:

$$I_{H,K}^*(f)(x, y) = \begin{cases} f(xp_1(y)^{-1}, p_2(xp_1(y)^{-1})y) & \text{if } s(x) = m(y), \\ 0 & \text{otherwise}. \end{cases}$$

Hence:

$$I_{H,K}^*I_{H,K}(f)(x, y) = \begin{cases} f(x, y) & \text{if } s(x) = m(y), \\ 0 & \text{otherwise} \end{cases}$$
and:

\[
I_{H,K}I_{H,K}^*(f)(x, y) = \begin{cases} 
  f(x, y) & \text{if } m(x) = r(y), \\
  0 & \text{otherwise.}
\end{cases}
\]

This means \( I_{H,K} \) is a partial isometry the initial (respectively final) support of which is \( e_{s,m} \) (respectively \( e_{m,r} \)). Let \( f, f' \) be any element in \( C(G^0) \), \( \xi \) be any element in \( L^2(G \times G) \), \( x, y \) be any element in \( G \).

First suppose that \( s(x) = m(y) \) then:

\[
I_{H,K}(\beta(f) \otimes \hat{\beta}(f'))\xi(x, y) = (\beta(f) \otimes \hat{\beta}(f'))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
\]

\[
= f(r(xp_1(p_2(x)^{-1}y)))f'(s(p_2(x)^{-1}y))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
\]

\[
= f(r(x))f'(s(y))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
\]

\[
= (\beta(f) \otimes \hat{\beta}(f'))I_{H,K}\xi(x, y)
\]

and, using Lemma 4.1.4(3):

\[
I_{H,K}(\alpha(f) \otimes \beta(f'))\xi(x, y) = (\alpha(f) \otimes \beta(f'))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
\]

\[
= f(m(x))(xp_1(p_2(x)^{-1}y))f'(r(p_2(x)^{-1}y))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
\]

\[
= f(m(y))f'(r(p_2(x)^{-1}y))\xi(xp_1(p_2(x)^{-1}y), p_2(x)^{-1}y)
\]

\[
= f(m(y))f'(s(x))I_{H,K}\xi(x, y)
\]

\[
= (\hat{\beta}(f') \otimes \alpha(f))I_{H,K}\xi(x, y).
\]

If \( s(x) \neq m(y) \), one has: \( I_{H,K}(\beta(f) \otimes \hat{\beta}(f'))\xi(x, y) = 0 = (\beta(f) \otimes \hat{\beta}(f'))I_{H,K}\xi(x, y) \), and also: \( I_{H,K}(\alpha(f) \otimes \beta(f'))\xi(x, y) = 0 = (\hat{\beta}(f') \otimes \alpha(f))I_{H,K}\xi(x, y) \). Hence:

\[
I_{H,K}(\beta(f) \otimes \hat{\beta}(f')) = (\beta(f) \otimes \hat{\beta}(f'))I_{H,K}.
\]

\[
I_{H,K}(\alpha(f) \otimes \beta(f')) = (\hat{\beta}(f') \otimes \alpha(f))I_{H,K}.
\]

Now let us prove the pentagonal relation for \( I_{H,K} \). Let us fix some notation: for any \( x, y \) in \( G \) such that \( m(x) = r(y) \) then one can define: \( V = p_2(x)^{-1}y \) and \( X = xp_1(V) \), if moreover \( z \) is any element of \( G \) such that \( m(y) = r(z) \), then \( m(p_2(x)^{-1}y) = r(p_2(X)^{-1}z) \) and, by two routine calculations the following relations are true:

\[
(I_{H,K})_{12}(I_{H,K})_{13}(I_{H,K})_{23}\xi(x, y, z)
\]

\[
= \xi(xp_1(V)p_1(p_2(X)^{-1}z), p_2(x)^{-1}yp_1(p_2(V)^{-1}p_2(X)^{-1}z), p_2(V)^{-1}p_2(X)^{-1}z),
\]
\[(I_{\mathcal{H},\mathcal{K}})_{23}(I_{\mathcal{H},\mathcal{K}})_{12}\xi(x, y, z)\]
\[= \xi(xp_1(p_2(x)^{-1}yp_1(p_2(y)^{-1}z)), p_2(x)^{-1}yp_1(p_2(y)^{-1}z), p_2(y)^{-1}z).\]

Let \((h, k)\) be in \(\mathcal{H} \times \mathcal{K}\) such that: 
\[p_2(x)^{-1}p_1(y) = hk\], then \(V = hkp_2(y)\) and \(X = p_1(x)p_2(x)h = p_1(x)p_1(y)k^{-1}\) hence one has:

\[p_2(V)^{-1}p_2(X)^{-1} = p_2(y)^{-1}, \quad (1)\]
\[p_2(X)^{-1} = k, \quad (2)\]
\[p_1(V) = h. \quad (3)\]

So, using (3) and the notation: \(k' = kp_2(y)\), one has:
\[p_1(p_2(x)^{-1}yp_1(p_2(y)^{-1}z)) = p_1(hkp_2(y)p_1(p_2(y)^{-1}z)) = p_1(V)p_1(k)p_1(k^{-1}kz)).\]

Now let us define \((h', k'')\) in \(\mathcal{H} \times \mathcal{K}\) such that: 
\[k''^{-1}kz = h'k''\], hence using (2), one has:
\[p_1(p_2(x)^{-1}yp_1(p_2(y)^{-1}z)) = p_1(V)p_1(k'p_2(V)p_1(p_2(X)^{-1}z).\]

This last equality and (1) give that for any triple \((x, y, z)\) in \(G^3\) such that \(m(x) = r(y)\) and \(m(y) = r(z)\):
\[(I_{\mathcal{H},\mathcal{K}})_{12}(I_{\mathcal{H},\mathcal{K}})_{13}(I_{\mathcal{H},\mathcal{K}})_{23}\xi(x, y, z) = (I_{\mathcal{H},\mathcal{K}})_{23}(I_{\mathcal{H},\mathcal{K}})_{12}\xi(x, y, z)\]
but for all the other triples \((x, y, z)\) in \(G^3\) the two sides of this equality are equal to 0, hence, \(I_{\mathcal{H},\mathcal{K}}\) is an mpi. \(\square\)

4.2. Crossed products and matched pairs of groupoids

The situation of a matched pair of groupoids \(G = \mathcal{H}\mathcal{K}\), leads to a natural right action of the groupoid \(\mathcal{H}\) on the fibered space \(\mathcal{K}\) and a left action of the groupoid \(\mathcal{K}\) on the fibered space \(\mathcal{H}\). Using the inverse map, one has \(G = \mathcal{H}\mathcal{K} = \mathcal{K}\mathcal{H}\). Hence, for any \(k \in \mathcal{K}\) and \(h \in \mathcal{H}^{s(k)}\), there exist a unique \(h' \in H\) and a unique \(k' \in K^{s(h')}\) such that \(kh = h'k'\).

4.2.1. Lemma and definition. Let \(G = \mathcal{H}\mathcal{K}\) be a matched pair of groupoids, and for any \(k \in \mathcal{K}\) and \(h \in \mathcal{H}^{s(k)}\), let us denote by \(k \triangleright h\) (respectively \(k \triangleleft h\)) the unique element in \(H\) (respectively \(K^{s(h)}\)) such that:

\[kh = (k \triangleright h)(k \triangleleft h),\]
then \(\triangleright\) (respectively \(\triangleleft\)) is a left action of the groupoid \(\mathcal{K}\) on the fibered space \(\mathcal{H}\) (respectively right action of the groupoid \(\mathcal{H}\) on the fibered space \(\mathcal{K}\)).
Proof. Left to the reader. □

Let \( \mathcal{G}/\mathcal{K} \) (respectively \( \mathcal{H}\backslash \mathcal{G} \)) be the set of right (respectively left) classes in \( \mathcal{G} \) modulo \( \mathcal{K} \) (respectively \( \mathcal{H} \)), that is \( \{ gK^{x(g)}/g \in \mathcal{G} \} \) (respectively \( \{ Hr(g)g/g \in \mathcal{G} \} \)). In that case, the map: \( h \rightarrow hK^{x(h)} \) (respectively \( k \rightarrow Hr(k)k \)) is a natural bijection between \( \mathcal{H} \) and \( \mathcal{G}/\mathcal{K} \). Using these maps, \( \mathcal{G}/\mathcal{K} \) and \( \mathcal{H}\backslash \mathcal{G} \) are fibered by \( \mathcal{G}^0 \): for any \( u \in \mathcal{G}^0 \), one can define \( (G/\mathcal{K})^u = \{ gK^{x(g)}/g \in \mathcal{G} \} \) and \( (H\backslash \mathcal{G})^u = \{ Hr(g)g/s(g) = u \} \). Also \( \mathcal{K} \) (respectively \( \mathcal{H} \)) has a left (respectively right) action on \( \mathcal{G}/\mathcal{K} \) (respectively \( \mathcal{H}\backslash \mathcal{G} \)) by multiplication: for any \( h \in \mathcal{H} \), \( k \in \mathcal{K} \), \( g \in \mathcal{G}_{r(h)} \) and \( g' \in \mathcal{G}^{s(k)} \), one can define \( \delta_k(Hr(g)g) = Hr(g)gh \) and \( \gamma_k(gK^{x(g')}) = kg'K^{x(g')} \). Using the natural bijections above, one easily sees that the actions considered in Lemma 4.2.1 also come from \( \delta \) and \( \gamma \).

In all what follows, let \( \chi_p \) (respectively \( \chi_Z \)) be the characteristic function of the singleton \( \{ p \} \) (respectively set \( Z \)).

### 4.2.2. Proposition and notations.

The mpi \( I_{\mathcal{H},\mathcal{K}} \) is regular and the \( \mathcal{C}^* \)-algebra \( S \) (respectively \( \tilde{S} \)) associated to \( I_{\mathcal{H},\mathcal{K}} \) is isomorphic to the crossed product \( C(\mathcal{K}) \rtimes_{\gamma} \mathcal{K} \mathcal{C}(\mathcal{H}) \) (respectively \( C(\mathcal{K}) \rtimes_{\delta} \mathcal{H} \mathcal{C}(\mathcal{H}) \)), where \( \gamma \mathcal{K} \) is the left action of \( C(\mathcal{K}) \) on \( C(\mathcal{H}) \) associated with \( \triangleright \) (respectively \( \delta \mathcal{H} \) is the right action of \( C(\mathcal{H}) \) on \( C(\mathcal{K}) \) associated with \( \triangleleft \). Hence \( C(\mathcal{K}) \rtimes_{\gamma} \mathcal{K} \mathcal{C}(\mathcal{H}) \) and \( C(\mathcal{K}) \rtimes_{\delta} \mathcal{H} \mathcal{C}(\mathcal{H}) \) have weak Hopf \( \mathcal{C}^* \)-algebras structures in duality, we shall note them \( (C(\mathcal{K}) \rtimes_{\gamma} \mathcal{K} \mathcal{C}(\mathcal{H}), \Gamma\gamma, \kappa\gamma, \epsilon\gamma) \) and \( (C(\mathcal{K}) \rtimes_{\delta} \mathcal{H} \mathcal{C}(\mathcal{H}), \Gamma\delta, \kappa\delta, \epsilon\delta) \).

Proof. For any \( g, g', p, q \in \mathcal{G} \) and \( \xi \) in \( I^2(\mathcal{G}) \), one has:

\[
(i \otimes \omega_{\chi_p, \chi_q})(I_{\mathcal{H}, \mathcal{K}})(\xi)(g) = ((i \otimes \omega_{\chi_p, \chi_q})(I_{\mathcal{H}, \mathcal{K}})(\xi), \chi_g) = (\omega_{\xi, \chi_g} \otimes \omega_{\chi_p, \chi_q})(I_{\mathcal{H}, \mathcal{K}})
\]

\[
= (I_{\mathcal{H}, \mathcal{K}}(\xi \otimes \chi_p), \chi_g \otimes \chi_q) = I_{\mathcal{H}, \mathcal{K}}(\xi \otimes \chi_p)(g, q).
\]

Hence, \( (i \otimes \omega_{\chi_p, \chi_q})(I_{\mathcal{H}, \mathcal{K}})\xi(g) = 0 \) if \( m(g) \neq r(q) \); otherwise:

\[
(i \otimes \omega_{\chi_p, \chi_q})(I_{\mathcal{H}, \mathcal{K}})(\xi)(g) = I_{\mathcal{H}, \mathcal{K}}(\xi \otimes \chi_p)(g, q) = \chi_p(p_2(g)^{-1}q)\xi(gp_1(p_2(g)^{-1}q)).
\]

This also implies that \( (i \otimes \omega_{\chi_p, \chi_q})(I_{\mathcal{H}, \mathcal{K}})(\xi)(g) \neq 0 \) only if there exists \( k \in \mathcal{K} \) such that \( q = kp \) and \( p_2(g) = k \); one can see that these two conditions imply that \( m(g) = r(q) \).

To summarize, for any \( k \in \mathcal{K}, p \in \mathcal{G}^{s(k)} \) and \( g \in \mathcal{G} \), one has: \( (i \otimes \omega_{\chi_p, \chi_k})(I_{\mathcal{H}, \mathcal{K}})(\xi)(g) = 1_{p_2^{-1}(k)}(g)\xi(gp_1(p)) \), and if \( q \) is not in \( \mathcal{K}p \), one has: \( (i \otimes \omega_{\chi_p, \chi_k})(I_{\mathcal{H}, \mathcal{K}})=0 \). So \( \tilde{S} \) is generated by the operators: \( \xi \mapsto (g \mapsto \chi_{\mathcal{H}k}(g)\xi(gh)) \), for any \( (k, h) \) in \( \mathcal{K}_s \times_r \mathcal{H} \), up to the natural identification of \( \mathcal{G} \) with \( \mathcal{K}_s \times_r \mathcal{H} \), this is \( \delta^H(\chi_k)(1_s \otimes_r \rho(h)) \) (or \( \delta^H(\chi_k, h) \)) so \( \tilde{S} \) is isomorphic to the crossed product \( C(\mathcal{K}) \rtimes_{\delta^H} \mathcal{H} \mathcal{C}(\mathcal{H}) \), hence \( I_{\mathcal{H}, \mathcal{K}} \) is regular. In a very similar way, \( S \) is generated by the operators \( (\omega_{\chi_k, \chi_k} \otimes i)(I_{\mathcal{H}, \mathcal{K}})(k, h) \) in \( \mathcal{K}_s \times_r \mathcal{H} \) which appear, up to the identification of \( \mathcal{H}_r \otimes_r \mathcal{K} \) with \( \mathcal{G} \), to be equal to \( \gamma^H(\chi_{kh})(1_r \otimes_s \lambda(k)) \) (\( = \gamma^H(\chi_{kh}, h) \)) (observe that \( r(k) = r(k \triangleright h) \)); hence \( S \) is isomorphic to \( C(\mathcal{K}) \rtimes_{\gamma} \mathcal{K} \mathcal{C}(\mathcal{H}) \). □
Let us compare these structures to ones defined by N. Andruskiewitsch and S. Natale in [AN].

Let us use the notations of [AN, Theorem 3.1], and let us identify $K_s \times_r \mathcal{H}$ and $T$, the double groupoid associated with the matched pair $\mathcal{KH}$ by [AN, Proposition 2.9], using the bijection:

$$(k, h) \mapsto \begin{array}{c} k \\ h \end{array}.$$  

The vector spaces $C(K) \ltimes_{\delta} C(\mathcal{H})$ and $\mathbb{C}T$ are equal. As one can define on $T$ a horizontal and a vertical product (see Lemma 1.5 of [AN]), the identification above makes it possible to define composition laws on $K_s \times_r \mathcal{H}$.

4.2.3. Definition and notations. We shall denote by $\sqcap$ the horizontal product defined for every $(k, h), (k', h')$ in $K_s \times_r \mathcal{H}$ such that $h' = k \triangleright h$ by:

$$(k, h) \sqcap (k', h') = (kk', h').$$

We shall denote by $\sqcup$ the vertical product defined for every $(k, h), (k', h')$ in $K_s \times_r \mathcal{H}$ such that $k' = k \triangleleft h$ by:

$$(k, h) \sqcup (k', h') = (k, hh').$$

Now we shall give a complete description of the quantum groupoid structure given by Proposition 4.2.2 to $C(K) \ltimes_{\delta} C(\mathcal{H})$, which proves that it is isomorphic to $\mathbb{C}T$.

4.2.4. Lemma. One has:

$$\Gamma^\delta \delta^\mathcal{H} \Gamma = (\delta^\mathcal{H} \otimes \delta^\mathcal{H}) \Gamma_K,$$

so for any $k$ in $\mathcal{K}$ and any $g, g'$ in $G$, one gets:

$$\Gamma^\delta (\delta^\mathcal{H}(\chi_k)) \xi(g, g') = \chi_k(p_2(g)p_2(g')) \xi(g, g').$$

Proof. Left to the courageous reader. $\square$

4.2.5. Theorem.

(1) For every $(k, h), (k', h')$ in $K_s \times_r \mathcal{H}$, one has:

$$\chi(k, h) \ast_{\delta} \chi(k', h') = \begin{cases} \chi(k, h) \sqcup (k', h') & \text{if } k' = k \triangleleft h, \\ 0 & \text{otherwise}, \end{cases}$$

and $$(\chi(k, h))^\#_{\delta} = \chi(k \triangleleft h, h^{-1}).$$

The identification of $K_s \times_r \mathcal{H}$ with $T$ given by the map:

$$(k, h) \mapsto \begin{array}{c} k \\ h \end{array},$$

leads to a $C^*$-isomorphism between $C(K) \ltimes_{\delta} C(\mathcal{H})$ and $\mathbb{C}T$.

(2) For every $(k, h)$ in $K_s \times_r \mathcal{H}$, one has:
\[ \Gamma^\delta (R^\delta (\chi(k,h))) = \sum_{(k_1,h_1) \sqcap (k_2,h_2) = (k,h)} R^\delta (\chi(k_1,h_1)) \otimes R^\delta (\chi(k_2,h_2)), \]

\[ k^\delta (R^\delta (\chi(k,h))) = R^\delta (\chi((k \cdot h)^{-1}, (k \cdot h)^{-1})), \]

\[ \epsilon (R^\delta (\chi(k,h))) = \begin{cases} 1 & \text{if } k = r(h), \\ 0 & \text{otherwise}. \end{cases} \]

\[ R^\delta \] is an isomorphism of \( C^* \)-quantum groupoids between \( CT \) associated with the matched pair \( KH \) by Proposition 3.4 of [AN] and \( (C(K) \rtimes_{\delta^H} C(H), \Gamma^\delta, \kappa^\delta, \epsilon^\delta) \).

**Proof.** An easy computation gives that, for all \((k, h), (k', h')\) in \( K_s \times_r H\), one has:

\[ \chi(k,h) \star^\delta \chi(k',h') = \begin{cases} \chi(k,h) \sqcap (k',h') & \text{if } k' = k \cdot h, \\ 0 & \text{otherwise}, \end{cases} \]

and \((h_1, k_1) \sqcap (k_2, h_2) = (k, s(k))\) if and only if \(k_1 \cdot k_2 = k\) and \(h_1 = s(k_1), h_2 = s(k_2)\), hence for any \(g, g'\) in \( G\) and \( \xi \) in \( L^2(G \times G)\), by Lemma 4.2.4 one has:

\[ \Gamma^\delta (R^\delta (\chi(k,h))) = \sum_{(k_1,h_1) \sqcap (k_2,h_2) = (k,h)} R^\delta (\chi(k_1,h_1)) \otimes R^\delta (\chi(k_2,h_2)). \]

Let us first prove the first formula of the theorem when \( h = s(k) \) (i.e. \( \Gamma^\delta (\delta^H(\chi_k)) = \Theta(\delta^H(\chi_k)) \)). One can easily observe that for any \((k_1,h_1), (k_2,h_2)\) in \( K_s \times_r H\), one has \((h_1, k_1) \sqcap (k_2, h_2) = (k, s(k))\) if and only if \(k_1 \cdot k_2 = k\) and \(h_1 = s(k_1), h_2 = s(k_2)\), hence for any \(g, g'\) in \( G\) and \( \xi \) in \( L^2(G \times G)\), by Lemma 4.2.4 one has:
by definition, for any \( r(k) \) if and only if \( ρ(h) ξ(h) \notin H^∗ \)

Now let us denote by \( \Gamma δ(χ(k)) \) the right regular representation of \( H \), so for any \( k, h \) in \( H \), \( χ(k) \in l^2(H) \):

\[
\rho(h) ξ(h') = ξ(h'h).
\]

Then in \( C(K) \times l^2(H) \), one has: \( 1 \otimes ρ(h) = \sum_{s(k)=r(h)} R^δ(χ(k,h)) \).

As by definition, for any \( h \) in \( H \), \( χ \in l^2(G \times G) \), \( g, g' \) in \( G \), one has:

\[
Γ^δ(1_k \otimes ρ(h)) ξ(g, g') = I_{H,K}^δ(1_g \otimes (1_k \otimes ρ(h))) I_{H,K} ξ(g, g'),
\]

an easy computation gives that:

\[
Γ^δ(1_k \otimes ρ(h)) ξ(g, g') = \begin{cases} ξ(gp_1(p_2(g')h), g'h') & \text{if } s(g) = m(g') \text{ and } s(g') = r(h), \\ 0 & \text{otherwise}. \end{cases}
\]

On the other hand, for any \( k_1, k_2 \) in \( K \), \( h_1, h_2 \) in \( H \), one has \( (h_1, k_1) \circ (k_2, h_2) = (k, h) \) if and only if \( r(k_1) = r(k), h_1 = k_1^{-1} k \triangleright h, k_2 = k_1^{-1} k, h_2 = h \). Hence, we can write that:

\[
θ(1_k \otimes ρ(h)) ξ(g, g') = \sum_{s(k)=r(h)} \sum_{r(k)=r(h)} R^δ(χ(k_1^{-1} k, h)) \otimes R^δ(χ(k_1^{-1} k, h)) ξ(g, g')
\]

\[
= \sum_{s(k)=r(h)} \sum_{r(k)=r(h)} \sum_{r(h')=s(g)} \sum_{r(h'')=s(g')} \chi_{k_1}(p_2(g)) \chi_{k_1^{-1} k}(h') \chi_{k_1^{-1} k}(p_2(g')) \times \chi_h(h'') ξ(g'h', g''').
\]

All terms of the sum are zero except when:

\[
\begin{align*}
{k_1} &= p_2(g), \\
{k_2}^{-1} k \triangleright h &= h', \\
{k_2}^{-1} k &= p_2(g'), \\
h'' &= h.
\end{align*}
\]

To have non-null terms, it is necessary that: \( k = p_2(g)p_2(g') \), hence \( s(g) = m(g') \), and also \( s(g') = s(k) \), which implies \( s(g') = r(h) \). In these conditions there is a single term in the sum,
it is obtained for: \( k = p_2(g) p_2(g'), k_1 = p_2(g), h' = p_2(g) \triangleright h = p_1(p_2(g)h) \), hence: \( \Theta(1_K \otimes \rho(h)) \xi(g, g') = \xi(gp_1(p_2(g')h), g'h) \). One deduces that:

\[
\Gamma^\delta(1_K \otimes \rho(h)) = \Theta(1_K \otimes \rho(h)).
\] (5)

As \( \Gamma^\delta \) and \( \Theta \) are both multiplicative, using (4) and (5), one deduces that for any \((k, h)\) in \( K_s \times_r H\):

\[
\Theta(R^\delta(\chi(k,h))) = \sum_{(k_1, h_1)} R^\delta(\chi(k_1,h_1)) \otimes R^\delta(\chi(k_2,h_2)).
\]

Now for any \( g \in G \), any \((k, h)\) in \( K_s \times_r H\), and any \( \xi \) in \( l^2(G) \), one has:

\[
\kappa^\delta(R^\delta(\chi(k,h)))\xi(g) = (i \otimes \omega_{\chi_h, \chi_{kh}})(I^*_H, K) \xi(g) = I^*_H, K(\xi \otimes \chi_h)(g, kh)
\]

\[
= \xi(gp_1(kh)^{-1}) \chi_h(p_2(gp_1(kh)^{-1})kh).
\]

One easily sees that \( p_2(gp_1(kh)^{-1})kh = h \) if and only if \( p_2(g) = (k \triangleleft h)^{-1} \). So:

\[
\kappa^\delta(R^\delta(\chi(k,h)))\xi(g) = \xi(g(k \triangleright h)^{-1}) \chi_{H(k \triangleright h)}(g) = R^\delta(\chi((k \triangleright h)^{-1},(k \triangleright h)^{-1}))\xi(g).
\]

Hence:

\[
\kappa^\delta(R^\delta(\chi(k,h))) = R^\delta(\chi((k \triangleright h)^{-1},(k \triangleright h)^{-1})).
\]

For any \((k, h)\) in \( K_s \times_r H\), one has:

\[
e(R^\delta(\chi(k,h))) = e((i \otimes \omega_{\chi_h, \chi_{kh}})(I^*_H, K)) = \omega_{\chi_h, \chi_{kh}}(1) = \sum_{g \in G} \chi_h(g) \chi_{kh}(g).
\]

Hence:

\[
e(R^\delta(\chi(k,h))) = \begin{cases} 1 & \text{if } k = r(h), \\ 0 & \text{otherwise}. \end{cases}
\]

The theorem follows immediately.  \( \Box \)

4.2.6. Remark. Using the natural identification of \( G \) and \( K_s \times_r H\), one can also express the \( C^*\)-quantum groupoid structure \((C(K) \rtimes_{\gamma^\delta} C(H), \Gamma^\delta, \kappa^\delta, \epsilon^\delta)\) by the following formulae:

- \( \Gamma^\delta(R^\delta(\chi_g)) = \sum_{(k_1, h_1) \circ (k_2, h_2) = (p_2(g), p_1(g))} R^\delta(\chi_{h_1 k_1}) \otimes R^\delta(\chi_{h_2 k_2}). \)

- \( \kappa^\delta(R^\delta(\chi_g)) = R^\delta(\chi_{g^{-1}}). \)

- \( \epsilon(R^\delta(\chi_g)) = \begin{cases} 1 & \text{if } g \in H, \\ 0 & \text{otherwise}. \end{cases} \)
Naturally, using the identification of $G$ and $H \times_r K$, one also has a characterization of the $C^*$-quantum groupoid structure for $(C(K) \ltimes_{\gamma,K} C(H), \Gamma^\gamma, \kappa^\gamma, \epsilon^\gamma)$ dual to $(C(K) \times_{\delta,K} C(H), \Gamma^\delta, \kappa^\delta, \epsilon^\delta)$. Let us recall that $T^t$ (the transpose of $T$) is by definition equal to $T$ as a set but the horizontal and vertical laws are exchanged and due to [AN, Proposition 3.11] $CT^t$ is the dual of $CT$.

The map

\[(h, k) \mapsto h \begin{array}{c} k \end{array}\]

gives a bijection between $H_r \times_r K$ and $T^t$ (equal to $T$ as a set).

The same calculations as above give the following result:

**4.2.7. Proposition.** There exists an isomorphism of $C^*$-quantum groupoids between $(C(K) \ltimes_{\gamma,K} C(H), \Gamma^\gamma, \kappa^\gamma, \epsilon^\gamma)$ and $CT^t$.

One can give explicitly this isomorphism. If

\[\text{char} \left( h \begin{array}{c} k \end{array} \right) \]

is the characteristic function of $h \begin{array}{c} k \end{array}$ in $CT^t$, then:

\[L^\gamma(\chi(h,k)) \mapsto \text{char} \left( h \begin{array}{c} k \end{array} \right).\]

**4.3. $C^*$-quantum groupoids coming from relative matched pairs of groups**

We suppose known, in this section, the Jones’ tower theory (see [GHJ]). Let $H$ and $K$ be subgroups of a group $G$ acting properly and outerly on the hyperfinite type II$_1$ factor $R$, in such a way we can identify $G$ with a subgroup of $\text{Out} R$, in particular there is here no ambiguity for $H \cap K \subset \text{Out} R$: let us call $\alpha$ the action of $K$ and $\beta$ the one of $H$ here these actions coincide on $H \cap K$. In [BH] is given a very deep study of inclusions of the form $R^H \subset R \ltimes K$, where $R^H$ is the fixed points algebra of $\beta$ and $R \ltimes K$ is the crossed product of $R$ and $H$ under the action $\alpha$, it is proven that this inclusion is finite depth if and only if the group generated by $H$ and $K$ in $\text{Out} R$ is finite, and, in that situation, it is irreducible and depth two when $H, K$ is a matched pair, which means that $G = HK = \{hk \mid h \in H, k \in K \}$ and $\text{card}(H \cap K) = 1$. We shall now prove that we still obtain depth two but no more irreducible inclusions when $\text{card}(H \cap K) \neq 1$ (such a pair will be called a relatively matched pair) and using [NV2] or [EV], these inclusions come from quantum groupoids actions.
Let us give some facts about inclusions of the form $R^H \subset R \rtimes K$, when $H$ and $K$ are finite. First, one can observe that $\text{Out} R$ can be identified, using Sauvageot Connes fusion multiplication and the contragradient procedure, to a group of $R-R$ bimodules over $L^2(R)$; and using the sum operation on bimodules, $\text{Out} R$ has also a second operation with a distributivity property. We shall follow the same notations than in [BH] and assimilate any element of $\text{Out} R$ to the bimodule associated with. Let $\gamma =_{RH} L^2(R)_R$ and $\delta =_R L^2(R \rtimes K)_{R \rtimes K}$, then for any $h \in H$ and $k \in K$ one has:

$$\gamma h = \gamma, \quad k\delta = \delta$$

(6)

$$\bigoplus_{h \in H} h, \quad \bigoplus_{k \in K} k.$$  

(7)

4.3.1. Lemma. Let $H, K$ be two finite subgroups of $\text{Out} R$ such that $HK = \{hk / h \in H, \ k \in K\}$ is a subgroup of $\text{Out} R$, then for any $g$ in $G = HK$, one has $\gamma g \delta = \gamma \delta$.

Proof. For any $g$ in $G = HK$, there exist $h \in H$ and $k \in K$ such that $g = hk$, hence due to (6), one has $\gamma g \delta = \gamma hk \delta = \gamma \delta$. ⊓⊔

4.3.2. Lemma. Let $M_0 \subset M_1 \subset M_2 \subset \cdots$ be a Jones tower of type $II_1$ factors such that the bimodule $M_0 L^2(M_2)_{M_1}$ is an amplification of $M_0 L^2(M_1)_{M_1}$, i.e. there exists an integer $n$ such that $M_0 L^2(M_2)_{M_1}$ is isomorphic to $M_0 \otimes (L^2(M_1) \otimes \mathbb{C}^n)_{M_1}$. Then the inclusion $M_0 \subset M_1$ is depth two.

Proof. As $M_0' \cap M_1$ (respectively $M_0' \cap M_3$) is equal to $\text{Hom}_{M_0, M_1}(M_0 L^2(M_1)_{M_1})$ (respectively $\text{Hom}_{M_0, M_1}(M_0 L^2(M_2)_{M_1})$), in the conditions of the lemma the inclusion $M_0' \cap M_1 \subset M_0' \cap M_3$ is isomorphic to the amplification inclusion: $M_0' \cap M_1 \subset (M_0' \cap M_1) \otimes M_3(C)$, so the principal graph of the inclusion $M_0 \subset M_1$ ends at the second floor, this is depth two definition. ⊓⊔

4.3.3. Theorem. Let $H, K$ be two finite subgroups of $\text{Out} R$ such that $HK = \{hk / h \in H, \ k \in K\}$ is a subgroup of $\text{Out} R$, let $M_2$ be the third element of Jones’ tower of inclusion $R^H \subset R \rtimes K$. There exists a $C^*$-quantum groupoid structure on $(R^H)' \cap M_2$ over the base $(R^H)' \cap R \rtimes K$ and an action $\alpha$ of $(R^H)' \cap M_2$ on $R \rtimes K$ in such a way that inclusion $R^H \subset R \rtimes K \subset M_2$ is isomorphic to $(R \rtimes K)^\alpha \subset R \rtimes K \subset (R \rtimes K) \rtimes \alpha ((R^H)' \cap M_2)$.

Proof. Let $(M_k)_{k \in K}$ be Jones’ tower of inclusion $R^H \subset R \rtimes K$, due to Chapters 2 and 3 of [BH], the bimodule $M_0 L^2(M_2)_{M_1}$ is equal to $\gamma (\delta \bar{\gamma} \gamma) \delta$, but using (7) and Lemma 4.3.1, one has

$$\gamma (\delta \bar{\gamma} \gamma) \delta = \gamma \left( \bigoplus_{h \in H, k \in K} hk \right) \delta = \text{Card}(H \cap K) \gamma \left( \bigoplus_{g \in HK} g \right) \delta$$

$$= \text{Card}(H \cap K) \text{Card}(HK) \gamma \delta.$$  

So, due to Lemma 4.3.2, the inclusion $R^H \subset R \rtimes K$ is depth two and one can apply [NV2, Chapters 5 and 6] or [E2, Theorem 9.2] to conclude. ⊓⊔
4.3.4. Remarks.

(1) The algebra \((RH)′ \cap R \rtimes K\) is isomorphic to the group algebra \(L(H \cap K)\): if \(u_k\) and \(v_h\) are canonical implementations of \(\alpha\) and \(\beta\) on \(L^2(R)\), one can suppose \(u_x = v_x\) for any \(x\) in \(H \cap K\), these \(u_x\) generate a \(*\)-algebra isomorphic to \(L(H \cap K)\) and are clearly in \((RH)′ \cap R \rtimes K\), on the other hand, using the calculation in the proof of 4.1 in [BH], one has: \(\dim((RH)′ \cap R \rtimes K) = \text{card}(H \cap K)\). The base \((RH)′ \cap R \rtimes K\) of the \(C^\ast\)-quantum groupoid \((RH)′ \cap R \rtimes K\) is commutative if and only if \(H \cap K\) is abelian. Hence, when \(H \cap K\) is non-abelian, this \(C^\ast\)-quantum groupoid structure does not come from a matched pair of groupoids.

(2) One can construct a lot of examples of “relatively matched pairs,” using Frattini Argument [G, Theorem 1.11.8]: let \(G\) be a finite group, let \(N\) be a normal subgroup of \(G\), let \(P\) be a Sylow \(p\)-subgroup of \(N\), then \(G = NNG(P)\), where \(NG(P)\) is the normalizer of \(P\) in \(G\).

Acknowledgments

I want to thank a lot L. Vainerman who suggested a good part of the study, D. Bisch for some explanations about proper and outer actions, and also M. Enock, S. Baaj and M.C. David for the great number of discussions we had about all the subject.

References


