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Differential recursion relations for Laguerre functions on symmetric cones

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Dedicated to Jacques Faraut

Abstract

Let Ω be a symmetric cone and V the corresponding simple Euclidean Jordan algebra. In our previous papers (some with G. Zhang) we considered the family of generalized Laguerre functions on Ω that generalize the classical Laguerre functions on $\mathbb{R}^+$. This family forms an orthogonal basis for the subspace of L-invariant functions in $L^2(\Omega, \mathrm{d}\mu_\nu)$, where $\mathrm{d}\mu_\nu$ is a certain measure on the cone and where L is the group of linear transformations on V that leave the cone Ω invariant and fix the identity in Ω. The space $L^2(\Omega, \mathrm{d}\mu_\nu)$ supports a highest weight representation of the group G of holomorphic diffeomorphisms that act on the tube domain $T(\Omega) = \Omega + iV$. In this article we give an explicit formula for the action of the Lie algebra of G and via this action determine second order differential operators which give differential recursion relations for the generalized Laguerre functions generalizing the classical creation, preservation, and annihilation relations for the Laguerre functions on $\mathbb{R}^+$.

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Introduction

It is a general understanding that special functions are closely related to representation theory of special Lie groups. Special functions are realized as coefficient functions of the representation and the action of the Lie algebra is used to derive differential equations and recursion relations satisfied by those functions. Standard references to this philosophy are the works of Vilenkin and Klimyk [24,25]. We also refer the interested reader to the text [10], the recent text [1], and the work of T. Koornwinder. The present article reflects these general philosophies. In particular, we conclude our work on the connection between generalized Laguerre functions, highest weight representations and Jordan algebras, [2,5–8]. The classical Laguerre functions ℓ_n^λ form an orthogonal basis for the Hilbert space $L^2(\mathbb{R}^+, x^{\lambda-1}\mathrm{d}x)$, $\lambda > 0$. As far as we have been able to trace, the first generalizations of the Laguerre functions and polynomials is from 1935 in the work of F. Tricomi [23]. Later, in 1955, C.S. Herz [12] considered generalized Laguerre functions in the context of Bessel functions on the space of complex $(m \times m)$-matrices. The Laguerre polynomials are defined on the cone of positive definite complex matrices in terms of the generalized hypergeometric functions, also introduced in the same article. Other realizations of the Laguerre functions, using the Laplace transform, were also derived. The motivation was to construct a complete set of eigenfunctions for the Hankel transform. The generalization to all symmetric cones using Euclidean Jordan algebras was achieved almost 40 years later in the beautiful book by J. Faraut and A. Koranyi [11]. Here the Laguerre polynomials were defined in terms of certain polynomials, $\psi_{\mathbf{m}}(x)$, invariant under the action of a maximal compact subgroup L leaving the cone invariant and fixing the identity e:

$$L_{\mathbf{m}}^{\nu}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{n}| \leqslant \mathbf{m}} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{(\nu)_{\mathbf{n}}} \psi_{\mathbf{n}}(-x),$$

cf. Section 4. The Laguerre functions are defined as

$$\ell_{\mathbf{m}}^{\nu}(x) = \mathrm{e}^{-\operatorname{tr} x} L_{\mathbf{m}}^{\nu}(2x),$$

where tr is the trace in the corresponding Jordan algebra. It was shown that the Laguerre functions were orthogonal and eigenfunctions of the Hankel transform. Later, F. Ricci and A. Tabacco constructed a system of differential operators, in the context of the Jordan algebra of Hermitian symmetric matrices and real symmetric matrices, having the Laguerre functions as eigenfunctions with distinct eigenvalues, cf. [19]. In the simplest case this differential operator is nothing but the Laguerre differential operator. None of these works, however, considers the generalized differential recursion relations that correspond to raising and lower operators satisfied by the Laguerre functions.

The first time that the Laguerre polynomials were directly related to representation theory was in [24] where they were shown to be coefficient functions of representations of the group

$$\left\{ \begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R},\ c \neq 0 \right\}.$$

Later B. Kostant and N. Wallach used the recursion relations that exist amongst the Laguerre functions to construct a highest weight representation and subsequently study Whittaker vectors for some special representations [17,27]. In [17] the differential equations and recursion relations for the Laguerre functions were used to give a realization of the highest weight representations of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. The opposite point of view was taken in [7] where the authors showed how one can derive those classical relations using a highest weight representation and the Laplace transform. The classical relations were given as the action of special elements in the Lie algebra acting as second order differential operators on functions on $\mathbb{R}^+$.

The connection to the construction in [11] was established in [8] where the generalized Laguerre functions were shown to be not only orthogonal but also a basis of the space of L-invariant functions in the highest weight module realized in $L^2(\Omega, \mathrm{d}\mu_\nu)$, where Ω is a symmetric cone, and $\mathrm{d}\mu_\nu$ is a certain quasi-invariant measure on Ω. Using a certain L-invariant element in the Lie algebra, the authors showed that the Laguerre functions satisfy a first order differential recursion relation involving the Euler operator (cf. Theorem 7.9 in [8]). The terms in this relation involve a raising and lowering of indices that parameterize the Laguerre functions. In the context of a highest weight representation one deduces that the Euler operator is made up of a creation and annihilation operator derived from the action of the Lie algebra. However, no attempt was made to derive an explicit form of these operators until we considered the special cases of the cones of Hermitian symmetric matrices and real symmetric matrices in [2,5], respectively. In this article we generalize those results to the Laguerre functions related to all symmetric cones. The tools are again highest weight representations and Jordan algebras. The main results are the explicit formulas for the action of the Lie algebra in the realization of the highest weight space $L^2(\Omega, \mathrm{d}\mu_\nu)$ and then the restriction to the subalgebra of L-invariants which results in the differential equations and recursion relations in terms of explicitly constructed differential operators.

If $\mathfrak{g}$ is simple, then the subalgebra of L-invariants in $\mathfrak{g}_\mathbb{C}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. It should be noted that such a three dimensional Lie algebra of differential operators has shown up in several places in the literature. We would like to mention its important role in the study of the Huygens' principle [4,14,15], in representation theory [16] (and the references therein), and in the theory of special functions [21].

One cannot downplay the essential role that Jordan algebras play in establishing and expressing many of the fundamental results obtained about orthogonal families of special functions defined on symmetric cones. Nevertheless, the theory of highest weight representations adds fundamental new results not otherwise easily obtained. In short, our philosophy is that there is a strong interplay between Jordan algebras, highest weight representations, and special functions which has not been fully exploited.

The starting point in this project has been the representation theory, wherein the Laguerre polynomials form an orthogonal family of functions invariant under a group action. However, the Laguerre polynomials have also been introduced in the literature using several variable Jack polynomials [3,9,18]. We would like to thank M. Rösler for pointing these references out to us. To explain the connection, a little more notation is needed. Let J be an irreducible Euclidean Jordan algebra of rank r. Let $c_1, \ldots, c_r \in J$ be a Jordan frame, $\mathfrak{a} = \bigoplus_{j=1}^r \mathbb{R}c_j$ and $e = c_1 + \cdots + c_r$. Let $\Omega = \{x^2 \mid x \in J \text{ and } x \text{ regular}\}$ be the standard symmetric cone in J. Let $H = \{g \in \mathrm{GL}(J) \mid g\Omega = \Omega\}_o$ and L the maximal compact subgroup of H fixing e. Then the Laguerre functions and polynomials are L-invariant functions on Ω. Let

$$\Omega_1 = \mathfrak{a} \cap \Omega \simeq (\mathbb{R}^+)^r.$$

Then $\Omega = L \cdot \Omega_1$ and therefore the Laguerre polynomials and functions are uniquely determined by their restriction to Ω_1. Thus, the Laguerre polynomials can also be defined as polynomials on Ω_1 or the vector space $\mathfrak{a}$, invariant under the Weyl group $W_H = N_L(\mathfrak{a})/Z_L(\mathfrak{a})$. This is the way the Laguerre polynomials are defined in the above references.

In the case of symmetric matrices, this boils down to the fact that each symmetric matrix can be diagonalized. Thus

$$\Omega_1 = \{d(\lambda_1, \dots, \lambda_n) \mid \lambda_j > 0\}$$

and the Laguerre polynomials can be viewed as polynomials in the eigenvalues, invariant under permutations.

The article is organized as follows. The necessary tools from Jordan algebra theory are introduced in Section 1. In Section 2 we introduce the tube domain $T(\Omega) = V + i\Omega$, where V is a simple Euclidean Jordan algebra. The main part of this section is devoted to describing the Lie algebra of $G = G(T(\Omega))_o$, where $G(T(\Omega))$ is the group of holomorphic automorphisms of $T(\Omega)$ and the subscript $_o$ stands for the connected component of the identity. The final result is the description of the L-invariant elements in $\mathfrak{g}$. Most of this material can be found in [11].

In Section 3 we introduce the highest weight representations and give an explicit realization of those representations in $L^2(\Omega, \mathrm{d}\mu_\nu)$ using the second order Bessel differential operator introduced in [11]:

$$\mathcal{B}_\nu = P\left(\frac{\partial}{\partial x}\right)x + \nu \frac{\partial}{\partial x},$$

where $P(a)$ denotes the quadratic representation of V. For $w \in V_{\mathbb{C}}$ we define the differential operator $\mathcal{B}_{\nu,w}$ by

$$\mathcal{B}_{\nu,w} f(x) = \big(\mathcal{B}_\nu f(x), w\big).$$

Then the following holds:

Theorem 3.4. *Let $f \in L^2(\Omega, \mathrm{d}\mu_\nu)^\infty$. The representation λ_ν of $\mathfrak{g}$ is described as follows*:

(1) $\lambda_\nu(X(iu, 0, 0)) f(x) = \mathrm{tr}(iux) f(x)$, $X(iu, 0, 0) \in \mathfrak{n}^+$,
(2) $\lambda_\nu(X(0, T, 0)) f(x) = \frac{\nu}{p} \mathrm{Tr}(T) f(x) + D_{T^t x} f(x)$, $X(0, T, 0) \in \mathfrak{h}$,
(3) $\lambda_\nu(X(0, 0, iv)) f(x) = -\mathcal{B}_{\nu,iv} f(x)$, $X(0, 0, iv) \in \mathfrak{n}^-$.

Here $\mathfrak{n}^+$, $\mathfrak{h}$, and $\mathfrak{n}^-$ are certain subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$.

In Section 4 we introduce the Laguerre functions and finally, in Section 5 we use Theorem 3.4 to derive explicit second order differential operators such that one of them has the Laguerre functions as eigenfunctions and the two others are an annihilator operator and a creation operator. Setting $B_\nu = \mathcal{B}_{\nu,e}$ we have:

Theorem 5.2. *The Laguerre functions are related by the following differential recursion relations*:

(1) $(-\mathrm{tr}\, x + B_\nu)\ell^\nu_{\mathbf{m}}(x) = -(r\nu + 2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}(x)$,
(2) $(\mathrm{tr}\, x + r\nu + 2D_x + B_\nu)\ell^\nu_{\mathbf{m}}(x) = -2\sum_{j=1}^r \binom{\mathbf{m}}{\mathbf{m}-\gamma_j}(m_j - 1 + \nu - (j-1)\frac{d}{2})\ell^\nu_{\mathbf{m}-\gamma_j}(x)$,
(3) $(\mathrm{tr}\, x - r\nu - 2D_x + B_\nu)\ell^\nu_{\mathbf{m}}(x) = -2\sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\gamma_j}(x)$, *where the constants $c_{\mathbf{m}}(j)$ are defined by*

$$c_{\mathbf{m}}(j) = \prod_{k \neq j} \frac{m_k - m_j - \frac{d}{2}(k - j + 1)}{m_k - m_j - \frac{d}{2}(k - j)}.$$

1. Jordan algebras and symmetric cones

In this section we will set down the notation and basic results concerning Jordan algebras and symmetric cones used for the remainder of this paper. We have tried to keep the notation consistent with the text by Faraut and Koranyi (cf. [11]). For proofs of results mentioned below see this text.

Let V be a real Jordan algebra. This means that V is a real vector space with a bilinear commutative product $(a,b)\mapsto ab$ such that $a^2(ab)=a(a^2b)$. In general, a Jordan algebra is not associative. Let $L(a)$ denote left multiplication by a on V. Thus $L(a)x=ax$. Since the product is bilinear, $L(a)$ is a linear operator on V. The multiplicative property given above is equivalent to $[L(a),L(a^2)]=0$, for all $a\in V$.

Let $x\in V$ and let $\mathbb{R}[x]$ be the algebra generated by x. The *rank* of V, r, is defined by

$$r=\max\bigl\{\dim\mathbb{R}[x]\mid x\in V\bigr\}.$$

An element $x\in V$ is *regular* if $\dim\mathbb{R}[x]=r$. The set of regular elements is open and dense in V. Suppose x is regular. We define $\mathrm{tr}(x)$ and $\det(x)$ as follows:

$$\mathrm{tr}(x)=\mathrm{Tr}\bigl(L(x)|_{\mathbb{R}[x]}\bigr),$$
$$\det(x)=\mathrm{Det}\bigl(L(x)|_{\mathbb{R}[x]}\bigr),$$

where Tr and Det are the usual trace and determinant of a linear operator. It is not hard to show that $\mathrm{tr}(x)$ and $\det(x)$ are polynomial functions in x and hence have polynomial extensions to all of V and $V_{\mathbb{C}}$.

Throughout, we will assume that V is finite dimensional with dimension n and contains a multiplicative identity e. Let $V_{\mathbb{C}}=V\otimes\mathbb{C}$ be the complexification of V. An element $x\in V_{\mathbb{C}}$ is said to be *invertible* if there is a $y\in\mathbb{C}[x]$ such that $xy=e$. The element y is necessarily unique, it is called the *inverse* of x, and denoted x^{-1}. We let I denote the inversion map on the set of invertible elements: $\mathrm{I}(x)=x^{-1}$.

The *quadratic representation* P of V is defined by

$$P(a)=2L(a)^2-L(a^2)$$

and plays a pivotal role in all that follows. If $F:V\to V$ is a differentiable map, we denote by $DF:V\to\mathrm{End}(V)$ the derivative of F. For $u,x\in V$ we set

$$D_uF(x)=DF(x)u=\lim_{t\to 0}\frac{F(x+tu)-F(x)}{t}.$$

Lemma 1.1. *An element x is invertible if and only if $P(x)$ is invertible as a linear operator on V. In this case*

$$P(x)(x^{-1})=x,$$
$$P(x)^{-1}=P(x^{-1}).$$

The set of invertible elements is an open set in V given by $\{x\in V\mid \mathrm{Det}(P(x))\neq 0\}$. *The derivative of the inversion map,* I, *is given by*

$$D\mathrm{I}(x)=-P(x)^{-1},$$

and, in particular, for all $u\in V$, we have

$$D_u\mathrm{I}(x)=-P(x)^{-1}u.$$

The polarization of P is given by

$$\begin{aligned} P(x, y) &:= \frac{1}{2} D_y\big(P(x)\big) \\ &= \frac{1}{2}\big(P(x+y) - P(x) - P(y)\big) \\ &= L(x)L(y) + L(y)L(x) - L(xy). \end{aligned}$$

A real Jordan algebra is *Euclidean* if there is an associative inner product on V. In other words, there is an inner product $(\cdot, \cdot)$ satisfying

$$(xu, v) = (u, xv),$$

for all $x, u, v \in V$. This is equivalent to saying that $L(x)$ is symmetric for all $x \in V$. A Jordan algebra is simple if there are no nontrivial ideals.

Proposition 1.2. *Suppose V is a simple Euclidean Jordan algebra of dimension n and rank r. For $x, y \in V$ we have*

$$\begin{aligned} &\operatorname{Tr} L(x) = \frac{n}{r} \operatorname{tr}(x), \\ &\operatorname{Det} P(x) = (\det x)^{2n/r}, \\ &\det(P(y)x) = (\det y)^2 \det x. \end{aligned}$$

Henceforth, we will assume V is a simple Euclidean Jordan algebra of dimension n and rank r. Let Ω be the interior of the set of all squares x^2, $x \in V$. Let $G(\Omega)$ be the group of all invertible linear transformations on V which leave Ω invariant. We will also use the notation $H = G(\Omega)_o$, where the subscript $_o$ denotes the connected component containing the identity.

Proposition 1.3. *The set Ω is a symmetric cone. This means that Ω is an open convex cone in V, self-dual in the sense that*

$$\Omega = \big\{ y \in V \mid (x, y) > 0, \forall x \in \overline{\Omega} \setminus \{0\} \big\},$$

and $G(\Omega)$ and H acts transitively on Ω. Furthermore, Ω is the connected component of e in the set of invertible elements of V and

$$\Omega = \big\{ x \in V \mid L(x) \text{ is positive definite} \big\}.$$

2. The tube domain $T(\Omega)$

Let V be a simple Euclidean Jordan algebra and $T(\Omega) = \Omega + iV$.[3] We note that $T(\Omega)$ is contained in the set of invertible elements in $V_{\mathbb{C}}$ and $\imath : z \mapsto z^{-1}$ is an involutive holomorphic automorphism of $T(\Omega)$ having e as its unique fixed point, cf. [11, Theorem X.1.1]. We note that $V_{\mathbb{C}}$ is a complex Jordan algebra. The multiplication, trace, and determinant formulas all extend from V to $V_{\mathbb{C}}$ in the usual way. We extend the bilinear form $(\cdot, \cdot)$ on V to a complex bilinear form on $V_{\mathbb{C}}$ and denote it in the same way.

[3] We choose the right half plane for $T(\Omega)$ instead of the upper half plane, $V + i\Omega$, given in [11], for example, and usually referred to as the Siegel upper half plane.

Let $G(T(\Omega))$ be the group of holomorphic automorphisms of $T(\Omega)$ and $G = G(T(\Omega))_o$. We describe elements in G as follows: Let $(iu, T, iv) \in iV \times H \times iV$ and define

$$\tau_{iu}(z) = z + iu,$$
$$\rho_T(z) = Tz,$$
$$\sigma_{iv}(z) = (z^{-1} + iv)^{-1}.$$

We observe that

$$\sigma_{iv} = \mathrm{I}\tau_{iv}\mathrm{I}^{-1}$$

Let $N^+ = \{\tau_{iu} \mid u \in V\}$ and $N^- = \{\sigma_{iv} \mid v \in V\}$. We identify H with $\{\rho_T \mid T \in H\}$. It is well known that the map, $(iu, T, iv) \mapsto \tau_{iu}\rho_T\sigma_{iv}$, is a diffeomorphism of $N^+ \times H \times N^-$ onto an open dense subset of G. Furthermore, if $K = G_e$, the stabilizer of e, i.e., the set of all $g \in G$ such that $ge = e$, then

$$G = N^+HK,$$

cf. [11, pp. 205–207] for details. We set $L = K \cap H$ and note that K is a maximal compact subgroup of G and L is a maximal compact subgroup of H.

Let $\mathfrak{n}^+$, $\mathfrak{n}^-$, $\mathfrak{h}$, $\mathfrak{g}$, and $\mathfrak{k}$ be the Lie algebras corresponding to N^+, N^-, H, G, and K, respectively. The one parameter subgroups

$$z \mapsto z + itu \in N^+ \quad (u \in V),$$
$$z \mapsto \exp(tT)z \in H \quad (T \in \mathfrak{h}),$$
$$z \mapsto (z^{-1} + itv)^{-1} \in N^- \quad (v \in V)$$

induce the corresponding vector fields

$$X(z) = iu \in \mathfrak{n}^+,$$
$$X(z) = Tz \in \mathfrak{h},$$
$$X(z) = -P(z)iv \in \mathfrak{n}^-.$$

As $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ it follows, that every vector field $\mathfrak{g}$ is of the form

$$X(z) = iu + T(z) - P(z)(iv)$$

and we will denote it by the triple $X(iu, T, iv)$. For $x, y \in V_{\mathbb{C}}$ set

$$x \square y = L(xy) + \big[L(x), L(y)\big].$$

Proposition 2.1. *Let $X(iu_1, T_1, iv_1)$ and $X(iu_2, T_2, iv_2)$ be two vector fields in $\mathfrak{g}$. Then the Lie bracket is given by*

$$\big[X(iu_1, T_1, iv_1), X(iu_2, T_2, iv_2)\big] = X(iu, T, iv),$$

where

$$u = T_1u_2 - T_2u_1,$$
$$T = [T_1, T_2] - 2\big((u_1 \square v_2) - (u_2 \square v_1)\big),$$
$$v = T_2^t v_1 - T_1^t v_2.$$

Proof. The proof is just as is found in [11, p. 209]. □

We note that $\mathfrak{l}$, the Lie algebra of L is given by

$$\mathfrak{l} = \{X(0, T, 0) \mid \mathfrak{h} \ni T = -T^t\},$$

where ^{t} denotes the transpose of T.

Proposition 2.2. *The Lie algebra of K is given by*

$$\mathfrak{k} = \{X(iu, T, iu) \mid u \in V,\ T \in \mathfrak{l}\}.$$

Proof. The map $s(X) = -iXi$ takes vector fields acting in the upper half plane to those acting on the right half plane and vice versa. The vector fields of the form $X(-u, T, u)$, $T \in \mathfrak{l}$, with the obvious notation, form the Lie algebra for the group acting on the upper half plane that fixes i.e. (cf. [11, p. 210]). Furthermore,

$$\begin{aligned} s\big(X(-u, T, u)\big) &= -iX(-u, T, u)i(z) \\ &= -i\big(-u + T(iz) - P(iz)u\big) \\ &= iu + Tz - P(z)iu \\ &= X(iu, T, iu) \end{aligned}$$

and this implies the proposition. □

Proposition 2.3. *The Killing form, B, on $\mathfrak{g}$ is given by*

$$B\big(X(iu_1, T_1, iv_1), X(iu_2, T_2, iv_2)\big) = B_\circ(T_1, T_2) + 2\,\mathrm{Tr}(T_1T_2) - 4\frac{n}{r}\big((u_1, v_2) + (u_2, v_1)\big),$$

where $B_\circ(\cdot,\cdot)$ is the Killing form on $\mathfrak{h}$. It is nondegenerate on $\mathfrak{g}$.

Proof. Cf. [22, p. 28]. □

We now define

$$\mathfrak{p} = \{X(iu, T, -iu) \mid u \in V,\ T = T^t \in \mathfrak{h}\}.$$

It is not difficult to see that the Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. Moreover,

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

is the Cartan decomposition of $\mathfrak{g}$ corresponding to the Cartan involution $\Theta : \mathfrak{g} \to \mathfrak{g}$ given by

$$\Theta\big(X(iu, T, iv)\big) = X(iv, -T^t, iu).$$

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ which we will identify with the set of all vector fields of the form $X(z, T, w)$, where $z, w \in V_{\mathbb{C}}$ and $T \in \mathfrak{h}_{\mathbb{C}}$. We will let $[\cdot,\cdot]$ denote the complex linear extension of the bracket given in Proposition 2.1. Specifically, we have

$$\big[X(z_1, T_1, w_1), X(z_2, T_2, w_2)\big] = X(z, T, w),$$

where

$$z = T_1 z_2 - T_2 z_1,$$
$$T = [T_1, T_2] + 2\big((z_1 \Box w_2) - (z_2 \Box w_1)\big),$$
$$w = T_2^t w_1 - T_1^t w_2.$$

Let $T \in \mathfrak{h}_{\mathbb{C}}$ and write $T = T_1 + iT_2$ where $T_1, T_2 \in \mathfrak{h}$. If $T = T^t$ then T_1 and T_2 are likewise self adjoint. Any self adjoint operator in $\mathfrak{h}$ is a left translation operator $L(x)$, for some $x \in V$. It follows then that $T = L(x) + iL(y) = L(x + iy)$, for some $x, y \in V$. Therefore, the self adjoint operators in $\mathfrak{h}_{\mathbb{C}}$ are left multiplication operators on $V_{\mathbb{C}}$ by elements in $V_{\mathbb{C}}$.

Let $\mathrm{Z} = X(-e, 0, -e)$. Then an easy calculation shows that Z is in the center of $\mathfrak{k}_{\mathbb{C}}$. Furthermore, $\mathrm{ad}(\mathrm{Z})$ has eigenvalues ± 2 on $\mathfrak{p}_{\mathbb{C}}$, the complexification of $\mathfrak{p}$ in $\mathfrak{g}_{\mathbb{C}}$. Indeed, let

$$\mathfrak{p}_+ = \big\{X(z, L(2z), -z) \mid z \in V_{\mathbb{C}}\big\} \quad \text{and} \quad \mathfrak{p}_- = \big\{X(z, -L(2z), -z) \mid z \in V_{\mathbb{C}}\big\}. \tag{2.1}$$

Then for $X(z, L(2z), -z) \in \mathfrak{p}_+$

$$\big[X(-e, 0, -e), X(z, L(2z), -z)\big] = 2X\big(z, L(2z), -z\big)$$

and for $X(z, -L(2z), -z) \in \mathfrak{p}_-$

$$\big[X(-e, 0, -e), X(z, -L(2z), -z)\big] = -2X\big(z, -L(2z), -z\big).$$

Since

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$$

it follows that $\mathfrak{p}_+$ is the $+2$-eigenspace of $\mathrm{ad}(\mathrm{Z})$ and $\mathfrak{p}_-$ is the -2-eigenspace of $\mathrm{ad}(\mathrm{Z})$. Note, that both $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are Abelian subalgebras of $\mathfrak{p}_{\mathbb{C}}$.

2.1. *L*-fixed vectors

The group K (and its Lie algebra $\mathfrak{k}$) naturally acts on $\mathfrak{g}_{\mathbb{C}}$. We are interested in the set of vectors, $\mathfrak{g}_{\mathbb{C}}^L$, fixed by the action of the subgroup L or, equivalently, those vectors annihilated by $\mathfrak{l}$ via the adjoint representation. First of all, since

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-$$

is a decomposition into $\mathfrak{k}_{\mathbb{C}}$-invariant subspaces it follows that

$$\mathfrak{g}_{\mathbb{C}}^L = \mathfrak{p}_+^L \oplus \mathfrak{k}_{\mathbb{C}}^L \oplus \mathfrak{p}_-^L.$$

Let $\mathrm{X} = \frac{1}{2}X(e, 2L(e), -e)$, $\mathrm{Y} = \frac{1}{2}X(-e, 2L(e), e)$, and $\mathrm{Z} = X(-e, 0, -e)$ as above. Then $\mathrm{X} \in \mathfrak{p}_+$, $\mathrm{Y} \in \mathfrak{p}_-$, and $\mathrm{Z} \in \mathfrak{k}_{\mathbb{C}}$ and each are fixed by L. Furthermore, if $\mathfrak{s}$ is the $\mathbb{C}$-span of $\{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ then $\mathfrak{s}$ is a Lie subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. Indeed, we need only observe that

$$[\mathrm{X}, \mathrm{Y}] = \mathrm{Z},$$
$$[\mathrm{Z}, \mathrm{X}] = 2\mathrm{X},$$
$$[\mathrm{Z}, \mathrm{Y}] = -2\mathrm{Y}.$$

Proposition 2.4. *With the notation established above we have*

$$\mathfrak{g}_{\mathbb{C}}^L = \mathfrak{s}.$$

Proof. It follows by [13, Theorem 1.3.11], that $\dim_{\mathbb{R}} \mathfrak{g}^L = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^L = 3$. The claim follows as $\dim_{\mathbb{C}} \mathfrak{s} = 3$. $\square$

3. Highest weight representations of G

In this section we will discuss a well know series of representations of G acting on spaces, $\mathcal{H}_\nu(T(\Omega))$, of holomorphic functions defined on the tube domain $T(\Omega) = \Omega + iV$. The Laplace transform, $\mathcal{L}_\nu$, is a unitary isomorphism of $L^2(\Omega, \mathrm{d}\mu_\nu)$ onto $\mathcal{H}_\nu(T(\Omega))$. We use this isomorphism to define an equivalent representation of G on $L^2(\Omega, \mathrm{d}\mu_\nu)$.

3.1. Representation on $\mathcal{H}_\nu(T(\Omega))$

Let $\tilde{G}$ be the universal covering group of G and $\kappa : \tilde{G} \to G$ the covering map. Then $\tilde{G}$ acts on $T(\Omega)$ via the covering map, i.e., $g \cdot z = \kappa(g)z$. For $\nu > 1 + n(r-1)$ let $\mathcal{H}_\nu(T(\Omega))$ be the space of holomorphic functions $F : T(\Omega) \to \mathbb{C}$ such that

$$\|F\|_\nu^2 := \alpha_\nu \int\limits_{T(\Omega)} \left|F(ix+y)\right|^2 \Delta(y)^{\nu - 2n/r} \,\mathrm{d}x\,\mathrm{d}y < \infty,$$

where

$$\alpha_\nu = \frac{2^{r\nu}}{(4\pi)^n \Gamma_\Omega(\nu - n/r)}.$$

(See Section 4.2 for the definition of Γ_Ω.) Then $\mathcal{H}_\nu(T(\Omega))$ is a nontrivial Hilbert space. For $\nu \leqslant 1 + n(r-1)$ this space reduces to $\{0\}$. If $\nu = 2n/r$ this is the *Bergman space*. For $g \in \tilde{G}$ and $z \in T(\Omega)$, let $J(g, z)$ be the *complex* Jacobian determinant of the action of $g \in \tilde{G}$ on $T(\Omega)$ at the point z. Then

$$J(ab, z) = J(a, b \cdot z) J(b, z)$$

for all $a, b \in \tilde{G}$ and $z \in T(\Omega)$. Recall that the *genus* of $T(\Omega)$ is $p = \frac{2n}{r}$. It is well known that for $\nu > 1 + n(r-1)$ that

$$\pi_\nu(g)F(z) = J(g^{-1}, z)^{\nu/p} F(g^{-1} \cdot z)$$

defines a unitary irreducible representation of $\tilde{G}$, cf. [11,20,26], for example.

For the following we need the explicit form of $J(g, z)$ on the generators τ_{iu}, ρ_T, and σ_{iv}.

Lemma 3.1. *The multiplier, J, satisfies*

(1) $J(\tau_{iu}, z) = 1$, $u \in V$;
(2) $J(\rho_T, z) = \operatorname{Det} T$, $T \in \mathfrak{h}$;
(3) $J(\sigma_{iv}, z) = \det(e + izv)^{-p}$, $v \in V$.

Proof. In the following w denotes arbitrary element in $V_\mathbb{C}$ and $z \in T(\Omega)$.

(1) Let $u \in V$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \tau_{iu}(z + tw)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(z + tw + iu)|_{t=0} = w.$$

Hence $J(\tau_{iu}, z) = 1$.

(2) Let $T \in H$. We then have $\frac{\mathrm{d}}{\mathrm{d}t}\rho_T(z + tw)|_{t=0} = Tw$. Hence $J(\rho_T, z) = \operatorname{Det} T$.

(3) For $v \in V$ we get by Lemma 1.1 and the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} \sigma_{iv}(z + tw)|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left((z + tw)^{-1} + iv\right)^{-1}\Big|_{t=0} = \left(P(z)P(z^{-1} + iv)\right)^{-1} w.$$

Hence $J(\sigma_{iv}, z) = ((\det z)\det(z^{-1} + iv))^{-2n/r} = \det(e + izv)^{-2n/r}$. □

Recall that the space of smooth vectors $\mathcal{H}_\nu(T(\Omega))^\infty$ in $\mathcal{H}_\nu(T(\Omega))$ is the space of all $F \in \mathcal{H}_\nu(T(\Omega))$ such that

$$\mathbb{R} \ni t \mapsto \pi_\nu(\exp tX)F \in \mathcal{H}_\nu\big(T(\Omega)\big)$$

is smooth for all $X \in \mathfrak{g}$. We denote also by π_ν the action of the Lie algebra $\mathfrak{g}$ and the complex linear extension to $\mathfrak{g}_{\mathbb{C}}$. For $F \in \mathcal{H}_\nu(T(\Omega))$ and $X \in \mathfrak{g}$ this action is given by

$$\pi_\nu(X)F = \lim_{t\to 0} \frac{\pi_\nu(\exp tX)F - F}{t}.$$

If F is a complex valued holomorphic function on $T(\Omega)$ we let, as before,

$$D_w F(z) = DF(z)w = \frac{\mathrm{d}}{\mathrm{d}t} F(z + tw)|_{t=0}$$

be the (nonnormalized) directional derivative of F in the direction $w \in V_{\mathbb{C}}$. As the point evaluation maps $F \mapsto F(z)$ are continuous linear functionals in $\mathcal{H}_\nu(T(\Omega))$ it follows easily that

$$\begin{aligned}
\pi_\nu(X)F(z) &= \frac{\mathrm{d}}{\mathrm{d}t} J\big(\exp(-tX), z\big)^{\nu/p} F\big(\exp(-tX)z\big)\big|_{t=0} \\
&= J\big(\exp(-tX), z\big)^{\nu/p}\big|_{t=0} F(z) + \frac{\mathrm{d}}{\mathrm{d}t} F\big(\exp(-tX)z\big)\big|_{t=0},
\end{aligned}$$

for all $z \in T(\Omega)$, $X \in \mathfrak{g}_{\mathbb{C}}$, and $F \in \mathcal{H}_\nu(T(\Omega))^\infty$. The following proposition gives the action of $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{h}$, and hence the full Lie algebra, on $\mathcal{H}_\nu(T(\Omega))$:

Proposition 3.2. *Let $F \in \mathcal{H}_\nu(T(\Omega))^\infty$. Then the subalgebras $\mathfrak{n}^+$, $\mathfrak{h}$, and $\mathfrak{n}^-$ act by the following formulas*:

(1) $\pi_\nu(X(iu, 0, 0))F(z) = -D_{iu}F(z)$, $X(iu, 0, 0) \in \mathfrak{n}^+$;
(2) $\pi_\nu(X(0, T, 0))F(z) = -\frac{\nu}{p}\operatorname{tr}(T)F(z) - D_{Tz}F(z)$, $X(0, T, 0) \in \mathfrak{h}$;
(3) $\pi_\nu(X(0, 0, iv))F(z) = i\nu \operatorname{tr}(zv)F(z) + D_{P(z)iv}F(z)$, $X(0, 0, iv) \in \mathfrak{n}^-$.

Proof. (1) Let $u \in V$. The formula $t \mapsto \tau_{tiu}(z) = z + itu$ defines the one parameter subgroup in the direction $X(iu, 0, 0) \in \mathfrak{n}^+$. By Lemma 3.1, we then have

$$\begin{aligned}
\pi_\nu\big(X(iu, 0, 0)\big)F(z) &= \frac{\mathrm{d}}{\mathrm{d}t} J\big(\tau_{tiu}^{-1}, z\big)^{\nu/p} F\big(\tau_{tiu}^{-1} z\big)\big|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} J(\tau_{-tiu}, z)^{\nu/p}|_{t=0} F(z) + \frac{\mathrm{d}}{\mathrm{d}t} F(\tau_{-tiu} z)|_{t=0} \\
&= D_{-iu}F(z).
\end{aligned}$$

(2) Let $T \in \mathfrak{h}$. Then $t \mapsto \rho_{\exp(tT)}(z) = \exp(tT)z$ defines the one parameter subgroup in the direction $X(0, T, 0) \in \mathfrak{h}$. By Lemma 3.1, we have

$$\begin{aligned}
\pi_\nu\big(X(0, T, 0)\big)F(z) &= \frac{\mathrm{d}}{\mathrm{d}t} J(\rho_{\exp(-tT)}, z)^{\nu/p} F(\rho_{\exp(-tT)} z)|_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} J(\rho_{\exp(-tT)}, z)^{\nu/p}|_{t=0} F(z) + \frac{\mathrm{d}}{\mathrm{d}t} F(\rho_{\exp(-tT)} z)|_{t=0} \\
&= \frac{-\nu}{p}\operatorname{tr}(T)F(z) - D_{Tz}F(z).
\end{aligned}$$

(3) Finally, let $v \in V$. Then $z \mapsto \sigma_{tiv}(z) = (z^{-1} + itv)^{-1}$ defines the one parameter subgroup in the direction $X(0,0,iv)$. Again, by Lemma 3.1 and by Lemma 1.1, we have

$$\begin{aligned}\pi_\nu\big(X(0,0,iv)\big)F(z) &= \frac{\mathrm{d}}{\mathrm{d}t} J\big(\sigma_{itv}^{-1}, z\big)^{\nu/p} F\big(\sigma_{itv}^{-1} z\big)\big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} J(\sigma_{-itv}, z)^{\nu/p}|_{t=0} F(z) + \frac{\mathrm{d}}{\mathrm{d}t} F(z^{-1} - itv)^{-1}|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \det(e - itzv)^{-\nu}|_{t=0} F(z) + DF(z)\big[-P(z^{-1})^{-1}(-iv)\big] \\ &= i\nu\,\mathrm{tr}(zv)F(z) + D_{P(z)iv}F(z). \qquad \square\end{aligned}$$

3.2. The Laplace transform

Let $L^2(\Omega, \mathrm{d}\mu_\nu)$ be the Hilbert space of all Lebesgue measurable functions on Ω such that

$$\|f\|^2 = \int_\Omega \big|f(x)\big|^2 \,\mathrm{d}\mu_\nu(x) < \infty,$$

where $\mathrm{d}\mu_\nu(x) = \Delta^{\nu - n/r}(x)\,\mathrm{d}x$. For $f \in L^2(\Omega, \mathrm{d}\mu_\nu)$ the Laplace transform is defined by the formula

$$\mathcal{L}_\nu(f)(z) = \int_\Omega \mathrm{e}^{-(z,x)} f(x)\,\mathrm{d}\mu_\nu(x).$$

Proposition 3.3. *Let $f \in L^2(\Omega, \mathrm{d}\mu_\nu)$. Then $\mathcal{L}_\nu f \in \mathcal{H}_\nu(T(\Omega))$. Furthermore, the map*

$$\mathcal{L}_\nu : L^2(\Omega, \mathrm{d}\mu_\nu) \to \mathcal{H}_\nu\big(T(\Omega)\big)$$

is a unitary isomorphism.

Proof. Cf. [8,20]. $\square$

3.3. Representation on $L^2(\Omega, \mathrm{d}\mu_\nu)$

By Proposition 3.3 we can define an equivalent representation, λ_ν, of G on $L^2(\Omega, \mathrm{d}\mu_\nu)$ so that $\mathcal{L}_\nu$ is an intertwining operator. Specifically,

$$\lambda_\nu(g)f = \mathcal{L}_\nu^{-1}\pi_\nu(g)\mathcal{L}_\nu f,$$

for $g \in G$. We denote by $L^2(\Omega, \mathrm{d}\mu_\nu)^\infty$ the space of smooth vectors. As usual we will let λ_ν also denote the action of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$ on $L^2(\Omega, \mathrm{d}\mu_\nu)^\infty$. Note that this representation is not geometric in the sense that $\tilde{G}$ does not act naturally on Ω, only the subgroup – with the obvious notation – $\widetilde{G(\Omega)} \cap G$ acts on Ω. We follow [11] to define the Bessel operator $\mathcal{B}_\nu : C^\infty(V) \to C^\infty(V) \otimes V_\mathbb{C}$ formally by

$$\mathcal{B}_\nu = P\left(\frac{\partial}{\partial x}\right)x + \nu\frac{\partial}{\partial x}. \tag{3.1}$$

If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V and $(x_1, \ldots, x_n)$ the corresponding coordinate functions, then

$$\mathcal{B}_\nu f(x) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(x)P(e_i, e_j)x + \nu\sum_i \frac{\partial f}{\partial x_i}(x)e_i.$$

By the definition given in Eq. (3.1) this formula is obviously basis independent. We refer to [11, p. 322] for more details. For $w \in V_{\mathbb{C}}$ we define the differential operator $\mathcal{B}_{\nu,w}$ by

$$\mathcal{B}_{\nu,w} f(x) = \big(\mathcal{B}_\nu f(x), w\big). \tag{3.2}$$

Theorem 3.4. *Let $f \in L^2(\Omega, \mathrm{d}\mu_\nu)^\infty$. The representation λ_ν of $\mathfrak{g}$ is described as follows*:

(1) $\lambda_\nu(X(iu,0,0)) f(x) = \mathrm{tr}(iux) f(x)$, $X(iu,0,0) \in \mathfrak{n}^+$;
(2) $\lambda_\nu(X(0,T,0)) f(x) = \frac{\nu}{p}\,\mathrm{Tr}(T) f(x) + D_{T^t x} f(x)$, $X(0,T,0) \in \mathfrak{h}$;
(3) $\lambda_\nu(X(0,0,iv)) f(x) = -\mathcal{B}_{\nu,iv} f(x)$, $X(0,0,iv) \in \mathfrak{n}^-$.

Proof. (1) Let $u \in V$ and for convenience let $m = \nu - n/r$. Let $w = \frac{1}{2} z \in \Omega$. Then $w + \Omega$ is an open neighborhood of z and for $f \in L^2(\Omega, \mathrm{d}\mu_\nu)^\infty$ we have

$$\big|\mathrm{e}^{-(y,x)}(iu,x) f(x) \Delta^m(x)\big| \leqslant \big|\mathrm{e}^{-(w,x)}(iu,x) f(x) \Delta^m(x)\big|,$$

for all $y \in w + \Omega$. As $\mathrm{e}^{-(w,\cdot)}(iu,\cdot) f \Delta^m \in L^1(\Omega, \mathrm{d}\mu_\nu)$ we can interchange the integration and differentiation in the following to get

$$\begin{aligned}
\pi_\nu\big(X(iu,0,0)\big)\mathcal{L}_\nu f(z) &= -D_{iu}\mathcal{L}_\nu f(z) \\
&= -\int_\Omega D_{iu}\big(\mathrm{e}^{-(z,x)}\big) f(x) \Delta^m(x)\,\mathrm{d}x \\
&= \int_\Omega \mathrm{e}^{-(z,x)}(iu,x) f(x) \Delta^m(x)\,\mathrm{d}x \\
&= \mathcal{L}_\nu\big(\mathrm{tr}(iux) f(x)\big)(z).
\end{aligned}$$

(2) In [8, p. 191] we determine the action of H on $L^2(\Omega, \mathrm{d}\mu_\nu)$ as follows:

$$\lambda_\nu(h) f(x) = \mathrm{Det}(h)^{\frac{\nu}{p}} f(h^t x),$$

$h \in H$. Differentiation of this formula gives (2).

(3) By Proposition XV.2.4 of [11] we have

$$\begin{aligned}
\mathcal{L}_\nu(\mathcal{B}_{\nu,iv} f)(z) &= -\left(P(z)\frac{\partial}{\partial z} + \nu z, iv\right)\mathcal{L}_\nu f(z) \\
&= -\big(D_{P(z)iv} + \nu\,\mathrm{tr}(izv)\big)\mathcal{L}_\nu f(z) \\
&= -\pi_\nu\big(X(0,0,iv)\big)\mathcal{L}_\nu f(z),
\end{aligned}$$

from which the result follows. □

Remark 3.5. Each of these formulas extend to the complexification in an obvious way:

(1) $\lambda_\nu(X(w,0,0)) f(x) = \mathrm{tr}(wx) f(x)$, $X(w,0,0) \in \mathfrak{n}^+_{\mathbb{C}}$;
(2) $\lambda_\nu(X(0,T,0)) f(x) = \frac{\nu}{p}\,\mathrm{Tr}(T) f(x) + D_{T^t x} f(x)$, $X(0,T,0) \in \mathfrak{h}_{\mathbb{C}}$;
(3) $\lambda_\nu(X(0,0,w)) f(x) = -\mathcal{B}_{\nu,w} f(x)$, $X(0,0,w) \in \mathfrak{n}^-_{\mathbb{C}}$.

4. Laguerre functions

We continue with the assumption that V is a simple Euclidean Jordan algebra with rank r, dimension n, and degree d; cf. [11, pp. 71 and 98] for the definition of the degree of a Jordan algebra. Let c_i, $i = 1, \dots, r$, be a fixed Jordan frame and $V^{(k)}$ the $+1$ eigenspace of the operator $L(c_1 + \cdots + c_k)$. Then each $V^{(i)}$ is a Jordan subalgebra of V and we have the following inclusions:

$$V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(r)} = V.$$

Let $\det_i$, $i = 1, \dots, r$, be the determinant function on $V^{(i)}$ and define $\Delta_i(x) = \det(P_i x)$, where P_i is orthogonal projection of V onto $V^{(i)}$. These are the principal minors, they are homogeneous polynomials of degree i, and $\Delta_r(x) = \det x$. For convenience we write $\Delta = \Delta_r$, cf. [11, p. 114] for details. For $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^r$ define

$$\Delta_{\mathbf{s}} = \Delta_1^{s_1 - s_2} \Delta_2^{s_2 - s_3} \cdots \Delta_r^{s_r}.$$

For $\mathbf{m} = (m_1, \dots, m_r)$ a sequence on nonnegative integers we write $\mathbf{m} \geqslant 0$ to mean $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r \geqslant 0$. Let

$$\Lambda = \{\mathbf{m} \mid \mathbf{m} \geqslant 0\}.$$

Then $\Delta_{\mathbf{m}}$ are the *generalized power functions* of degree $|\mathbf{m}| = m_1 + \cdots + m_r$. Since $\Delta_{\mathbf{m}}$ is a polynomial function on V it extends uniquely to a holomorphic polynomial function on $V_{\mathbb{C}}$.

4.1. *L-invariant polynomials*

We define $\psi_{\mathbf{m}}$ by the following formula

$$\psi_{\mathbf{m}}(x) = \int_L \Delta_{\mathbf{m}}(lx)\, \mathrm{d}l, \quad x \in V,$$

where $\mathrm{d}l$ is normalized Haar measure on L. The function $\psi_{\mathbf{m}}$ is a nonzero L-invariant polynomial on V, for each $\mathbf{m} \in \Lambda$, which also extends uniquely to a holomorphic function on $V_{\mathbb{C}}$. Furthermore, the set of L-invariant polynomials is spanned by the set of all $\psi_{\mathbf{m}}$, $\mathbf{m} \in \Lambda$. Moreover, if $\mathcal{P}_k(V)$ denotes the set of L-invariant polynomials on V of degree at most k then $\mathcal{P}_k(V)$ is spanned by all those $\psi_{\mathbf{m}}$ with $|\mathbf{m}| \leqslant k$. The function $\psi_{\mathbf{m}}(e + x)$ is also an L-invariant polynomial of degree $|\mathbf{m}|$ and has an expansion that defines the generalized binomial coefficients $\binom{\mathbf{m}}{\mathbf{n}}$:

$$\psi_{\mathbf{m}}(e + x) = \sum_{|\mathbf{n}| \leqslant |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \psi_{\mathbf{n}}(x).$$

4.2. *The gamma function*

For convenience we also reproduce the gamma function of the symmetric cone Ω: Let $\mathbf{s} \in \mathbb{C}^r$ and define

$$\Gamma_\Omega(\mathbf{s}) = \int_\Omega \mathrm{e}^{-\operatorname{tr} x} \Delta_{\mathbf{s}}(x) \Delta^{-n/r}(x)\, \mathrm{d}x,$$

where $\Delta(x) = \Delta_r(x)$ as before. For ν a real number and $\mathbf{m} \in \Lambda$ define

$$(\nu)_{\mathbf{m}} = \frac{\Gamma_\Omega(\nu + \mathbf{m})}{\Gamma_\Omega(\nu)},$$

where $\nu + \mathbf{m}$ means to add ν to each component of $\mathbf{m}$.

4.3. The generalized Laguerre functions

The Laguerre polynomials are defined by

$$L^\nu_{\mathbf{m}}(x) = (\nu)_{\mathbf{m}} \sum_{|\mathbf{n}| \leqslant |\mathbf{m}|} \binom{\mathbf{m}}{\mathbf{n}} \frac{1}{(\nu)_{\mathbf{n}}} \psi_{\mathbf{n}}(-x), \tag{4.1}$$

and the generalized Laguerre functions by

$$\ell^\nu_{\mathbf{m}}(x) = \mathrm{e}^{-\operatorname{tr} x} L^\nu_{\mathbf{m}}(2x), \tag{4.2}$$

cf. [11, p. 343].

Remark 4.1. Let $\mathbf{0}$ denote the multi-index with entries 0. Then $\ell^\nu_{\mathbf{0}} = \mathrm{e}^{-\operatorname{tr} x}$. Let $X(z, 2L(z), -z) \in \mathfrak{p}_+$, cf. Eq. (2.1). Then a straightforward calculation gives

$$\lambda_\nu\big(X(z, 2L(z), -z)\big)\ell^\nu_{\mathbf{0}} = 0,$$

for all $z \in V_{\mathbb{C}}$. Thus $\ell^\nu_{\mathbf{0}}$ is the highest weight vector for λ_ν.

Theorem 4.2. *Let $L^2(\Omega, \mathrm{d}\mu_\nu)^L$ be the space of L-invariant function in $L^2(\Omega, \mathrm{d}\mu_\nu)$. Then the Laguerre functions form an orthogonal basis of $L^2(\Omega, \mathrm{d}\mu_\nu)^L$. Moreover,*

$$\big\|\ell^\nu_{\mathbf{m}}(x)\big\|^2 = \frac{1}{2^{r\nu}} \frac{1}{d_{\mathbf{m}}} \left(\frac{n}{r}\right)_{\mathbf{m}} \Gamma_\Omega(\nu + \mathbf{m}).$$

Proof. This is Theorem 4.1 in [8]. See also [11, p. 344]. □

5. Differential recursion relations

Recall that $\mathfrak{g}^L_{\mathbb{C}}$ is the set of vector fields in $\mathfrak{g}_{\mathbb{C}}$ invariant under the adjoint action of L. Proposition 2.4 establishes that $\mathfrak{s} = \mathfrak{g}^L_{\mathbb{C}}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ and is spanned be the vector fields

$$\mathrm{X} = \frac{1}{2} X\big(e, 2L(e), -e\big) \in \mathfrak{p}^+,$$
$$\mathrm{Y} = \frac{1}{2} X\big(-e, 2L(e), e\big) \in \mathfrak{p}^-,$$
$$\mathrm{Z} = X(-e, 0, -e) \in \mathfrak{k}_{\mathbb{C}}.$$

Our main theorem generalizes the classical differential recursion relations on Laguerre functions by way of the explicit action of $\mathfrak{s}$ on $L^2(\Omega, \mathrm{d}\mu_\nu)^{\infty L}$. Set $B_\nu = \mathcal{B}_{\nu,e}$.

Proposition 5.1. *Let $f \in L^2(\Omega, \mathrm{d}\mu_\nu)^\infty$. With notation as above we have*

(1) $\lambda_\nu(\mathrm{X}) f(x) = \frac{1}{2}(\operatorname{tr} x + r\nu + 2D_x + B_\nu) f(x)$,

(2) $\lambda_\nu(\mathrm{Y})f(x)=\frac{1}{2}(-\operatorname{tr}x+r\nu+2D_x-B_\nu)f(x)$,
(3) $\lambda_\nu(\mathrm{Z})f(x)=(-\operatorname{tr}x+B_\nu)f(x)$.

Proof. These formulas follow directly from Remark 3.5. □

Theorem 5.2. *The Laguerre functions are related by the following differential recursion relations*:

(1) $(-\operatorname{tr}x+B_\nu)\ell^\nu_{\mathbf{m}}(x)=-(r\nu+2|\mathbf{m}|)\ell^\nu_{\mathbf{m}}(x)$,
(2) $(\operatorname{tr}x+r\nu+2D_x+B_\nu)\ell^\nu_{\mathbf{m}}(x)=-2\sum_{j=1}^r\binom{\mathbf{m}}{\mathbf{m}-\gamma_j}(m_j-1+\nu-(j-1)\frac{d}{2})\ell^\nu_{\mathbf{m}-\gamma_j}(x)$,
(3) $(\operatorname{tr}x-r\nu-2D_x+B_\nu)\ell^\nu_{\mathbf{m}}(x)=-2\sum_{j=1}^r c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\gamma_j}(x)$, *where the constants* $c_{\mathbf{m}}(j)$ *are defined by*

$$c_{\mathbf{m}}(j)=\prod_{k\neq j}\frac{m_k-m_j-\frac{d}{2}(k-j+1)}{m_k-m_j-\frac{d}{2}(k-j)}.$$

Remark 5.3. In the formulas above γ_j is the multi-index with 1 in the jth position and 0's elsewhere. It should be understood that if $\mathbf{m}+\gamma_{\mathbf{j}}$ or $\mathbf{m}-\gamma_{\mathbf{j}}$ is not in Λ then the corresponding Laguerre function does not appear in the sum.

Proof. Let $\xi=X(e,0,e)=e-P(z)e$. Then $\xi=-\mathrm{Z}$ is the vector field given by the same symbol in [8]. By Lemma 5.5 of [8]

$$\pi_\nu(\xi)q_{\mathbf{m},\nu}=\big(r\nu+2|\mathbf{m}|\big)q_{\mathbf{m},\nu},$$

where $\Gamma_\Omega(\mathbf{m}+\nu)q_{\mathbf{m},\nu}=\mathcal{L}_\nu(\ell^\nu_{\mathbf{m}})$. By the unitary equivalence of π_ν and λ_ν we correspondingly have

$$\lambda_\nu(\xi)\ell^\nu_{\mathbf{m}}=\big(r\nu+2|\mathbf{m}|\big)\ell^\nu_{\mathbf{m}}.$$

On the other hand,

$$\lambda_\nu(\xi)\ell^\nu_{\mathbf{m}}=-\lambda_\nu(\mathrm{Z})\ell^\nu_{\mathbf{m}}=(\operatorname{tr}x-B_\nu)\ell^\nu_{\mathbf{m}}$$

by Proposition 5.1. Part (1) now follows.

Let

$$L^2_k(\Omega,\mathrm{d}\mu_\nu)=\big\{f\in L^2(\Omega,\mathrm{d}\mu_\nu)^\infty\mid\lambda_\nu(z)f=-(r\nu+2k)f\big\}.$$

Since λ_ν is an irreducible highest weight representation it is well known that $L^2_k(\Omega,\mathrm{d}\mu_\nu)$ is finite dimensional, nonzero if $k\geqslant 0$, and

$$L^2(\Omega,\mathrm{d}\mu_\nu)=\bigoplus_{k=0}^{\infty}L^2_k(\Omega,\mathrm{d}\mu_\nu).$$

Furthermore, part (1) implies that $\ell^\nu_{\mathbf{m}}\in L^2_{|\mathbf{m}|}(\Omega,\mathrm{d}\mu_\nu)$. For $\mathrm{W}=X(w,2L(w),-w)\in\mathfrak{p}^+$ and $f\in L^2_k(\Omega,\mathrm{d}\mu_\nu)$ we have

$$\begin{aligned}\lambda_\nu(\mathrm{Z})\lambda_\nu(\mathrm{W})f&=\lambda_\nu(\mathrm{W})\lambda(\mathrm{Z})f+\lambda_\nu\big([\mathrm{Z},\mathrm{W}]\big)f\\&=-(r\nu+2k)\lambda_\nu(\mathrm{W})f+2\lambda_\nu(\mathrm{W})f\\&=-\big(r\nu+2(k-1)\big)\lambda_\nu(\mathrm{W})f.\end{aligned}$$

This implies that $\lambda_\nu(\mathrm{W})f \in L^2_{k-1}(\Omega, \mathrm{d}\mu_\nu)$. Similarly, for $\mathrm{W} \in \mathfrak{p}^-$, we have $\lambda_\nu(\mathrm{W})f \in L^2_{k+1}(\Omega, \mathrm{d}\mu_\nu)$.

Now let $Z_0 = \frac{1}{2}(\mathrm{X}+\mathrm{Y})$. Then $Z_0 = X(0, I, 0)$ is the Euler vector field $z\frac{\partial}{\partial z}$ given in [8, p. 161]. By Theorem 7.9 of [8] (and its proof) we have

$$-2\lambda_\nu(Z_0)\ell^\nu_{\mathbf{m}} = \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\left(m_j - 1 + \nu - (j-1)\frac{a}{2}\right)\ell^\nu_{\mathbf{m}-\gamma_j} - \sum_{j=1}^{r} c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\gamma_j}.$$

If P_k denotes orthogonal projection of $L^2(\Omega, \mathrm{d}\mu_\nu)$ onto $L^2_k(\Omega, \mathrm{d}\mu_\nu)$ then

$$-\lambda_\nu(\mathrm{X})\ell^\nu_{\mathbf{m}} = P_{|\mathbf{m}|-1}\big(-2\lambda_\nu(Z_0)\ell^\nu_{\mathbf{m}}\big) = \sum_{j=1}^{r} \binom{\mathbf{m}}{\mathbf{m}-\gamma_j}\left(m_j - 1 + \nu - (j-1)\frac{a}{2}\right)\ell^\nu_{\mathbf{m}-\gamma_j}$$

and

$$-\lambda_\nu(\mathrm{Y})\ell^\nu_{\mathbf{m}} = P_{|\mathbf{m}|+1}\big(-2\lambda_\nu(Z_0)\ell^\nu_{\mathbf{m}}\big) = -\sum_{j=1}^{r} c_{\mathbf{m}}(j)\ell^\nu_{\mathbf{m}+\gamma_j}.$$

We obtain formulas (2) and (3) by again applying Proposition 5.1. □

Remark 5.4. Observe that one-half the difference between formula (2) and (3) gives Theorem 7.9 of [8].

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