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Asymptotic Behavior of Solutions of Functional Differential Equations with Finite Delays

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DEDICATED TO THE MEMORY OF MY MOTHER HIDEKO TANIGUCHI

In this paper we consider a sufficient condition for $W(t, x(t))$ to approach zero as $t \to \infty$, where $x(t)$ is a solution of a non-autonomous functional differential equation with finite delays and $W(t, x)$ is a so-called Lyapunov function. We shall show that in the applications this provides useful information for asymptotic behavior of the solution $x(t)$. For example, we generalize examples given by J. R. Haddock and J. Terjéki (*J. Differential Equations* **48**, 1983, 95–122) to the case of non-autonomous systems.

1. INTRODUCTION

In 1967 J. K. Hale [10] extended in a natural manner LaSalle's invariance principle to autonomous functional differential equations with delays by Lyapunov functionals. However, it seems to be in general very difficult to find a Lyapunov functional with conditions which the theorem needs when one considers applications to mechanical engineering, mathematical biology, population dynamics, and the other practical sciences. In order to circumvent this difficulty, J. R. Haddock and J. Terjéki [9] developed a new invariance principle by the Lyapunov–Razumikhin method which many authors used in research of qualitative theory of solutions of functional differential equations. They presented an asymptotic stability theorem of the zero solution as a corollary of the main theorem and noted that this corollary could not be expected to non-autonomous cases without extensive modifications [9, p. 100].

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On the other hand there are various contributions by many authors to the case of non-autonomous functional differential equations. For example, T. A. Burton [1] discussed asymptotic behavior of solutions by introducing the pseudo-Lyapunov function and T. Yoshizawa [23], using this function, presented an invariance principle for the non-autonomous case, which is an extension of his own result [21]. Recently W. E. Hornor [13] discussed the Haddock−Terjéki invariance principle by limiting equations.

In this paper we discuss under what conditions $W(t, x(t))$ approaches zero as $t \to \infty$, where $x(t)$ is a solution and $W(t, x)$ is a so-called Lyapunov function, and we shall show that in the applications this result provides powerful information for asymptotic behavior of the solution $x(t)$. Since we use the Lyapunov−Razumikhin method, we can avoid the difficulty of finding Lyapunov functionals.

For example, we discuss the scalar equation

$$x'(t) = -a(t)x(t) + \int_{-r(t)}^{0} b(s)x(t+s)\,ds$$
$$- x(t)\int_{-r(t)}^{0} c(s)x^2(t+s)\,ds.$$

The case $a(t) \equiv a > 0$ and $r(t) \equiv r > 0$ was considered by Haddock and Terjéki [9]. We generalize their result (see Example 4.1). They also considered the scalar equation $x'(t) = bx(t-r)[1 - x(t)] - bx(t)$, $b, r > 0$ which was studied in detail in [5] by the new invariant theorem. We discuss the scalar equation $x'(t) = a(t)g(x(t - r(t)))(1 - x(t)) - a(t)x(t)$, where $a(t)$ is integrally positive and $u \geq g(u)$ for all $u > 0$, $g(0) \geq 0$ and $g(u)$ is continuous and monotone nondecreasing (see Example 4.2). We also consider the functional differential equation discussed by Burton [2] and Yoshizawa [23].

The contents of this paper are as follows. In Section 2 we give preliminaries. In Section 3 we discuss the asymptotic behaviors of solutions and we give two theorems. In Section 4 we give several examples which illustrate the theorems.

2. PRELIMINARIES

Let R^d be the d-dimensional Euclidean space. Let $C = C([-r, 0], R^d)$, $r > 0$, denote the space of continuous functions that map $[-r, 0]$ into R^d. If $x(t)$ is a continuous function defined on $[-r, T]$, then we define $x_t \in C$ by setting $x_t(s) = x(t+s)$, $s \in [-r, 0]$ for each $t \leq T$ where $T > 0$. For $\varphi \in C$, let $\|\varphi\|$ denote $\sup\{|\varphi(s)|: s \in [-r, 0]\}$, where $|\cdot|$ is a norm of R^d. For any fixed $B > 0$ let $C(B) = \{x_t \in C: |x(t)| \leq B \text{ for all } t \geq t_0\}$.

We consider the functional differential equation

$$dx/dt = F(t, x_t), \qquad x_{t_0} = \psi \in C, \; t_0 \geq 0, \tag{1}$$

where $F\colon [0,\infty) \times C \to R^d$ is continuous. Assume that any solution with any initial value (t_0, ψ), $t_0 \geq 0$, $\psi \in C$, exists in the future. Let $V(t, x)$ be a continuous scalar function defined on $[-r,\infty) \times R^d$ which satisfies locally a Lipschitz condition with respect to x. Then, for any $\varphi \in C$ we define the function

$$V'_{(1)}(t, \varphi) = \limsup_{h \to 0+} \{V(t + h, \varphi(0) + hF(t, \varphi)) - V(t, \varphi(0))\}/h.$$

Let $x(t) = x(t; t_0, \psi)$ denote a solution of (1) with an initial value (t_0, ψ), $t_0 \geq 0$, $\psi \in C$ from now on in this paper.

The following definition was given by L. Hatvani [12].

Definition 1. A continuous function $p\colon [0,\infty) \to [0,\infty)$ is said to be *integrally positive* if $\int_J p(s)\, ds = \infty$ holds on every set $J = \bigcup_{m=1}^{\infty}(a_m, b_m)$ such that $0 \leq a_1$, $a_m < b_m \leq a_{m+1}$, $b_m - a_m \geq \delta > 0$ $(m = 1, 2, \ldots)$ for some fixed $\delta > 0$.

For example, it is known that the function $t^2 \sin^2 t$, $t \geq 0$ is integrally positive.

We give the definition of stability of the zero solution of (1), where $F(t, 0) \equiv 0$ for all $t \geq 0$ and the one of boundedness of solutions.

Definition 2. The zero solution of (1) is said to be *stable* if for any $\varepsilon > 0$ and any initial time $t_0 \geq 0$ there exists a $\delta(t_0, \varepsilon) > 0$ such that $|x(t; t_0, \varphi)| < \varepsilon$, $t \geq t_0$ for any $\varphi \in C$ with $\|\varphi\| < \delta(t_0, \varepsilon)$.

Definition 3. The zero solution of (1) is said to be *asymptotically stable* if it is stable and furthermore, there exists a $\sigma(t_0) > 0$ such that $|x(t; t_0, \varphi)| \to 0$ as $t \to \infty$ for any $\varphi \in C$ with $\|\varphi\| < \sigma(t_0)$. If $\sigma(t_0) = \infty$, then it is said to be *globally asymptotically stable*.

Definition 4. The solutions of (1) are said to be *equi-bounded* if for any $\alpha > 0$ and any initial time $t_0 \geq 0$, there exists a $\beta(t_0, \alpha) > 0$ such that $|x(t; t_0, \varphi)| < \beta(t_0, \alpha)$ for any $\varphi \in C$ with $\|\varphi\| < \alpha$.

The following theorem is known and useful [11].

Theorem 2.1. *For the functional differential equation* (1) *suppose that the following condition is satisfied*:

(1) *there exists a positive definite, continuous function* $V(t, x)$ *defined on* $[0, \infty) \times R^d$ *which is locally Lipschitzian with respect to x such that*

$$V'_{(1)}(t, \varphi) \le 0, \qquad t \ge t_0$$

whenever $\varphi \in C$, $\sup\{V(t+s, \varphi(s)): s \in [-r, 0]\} \le h(V(t, \varphi(0)))$, *where* h: $[0, \infty) \to [0, \infty)$ *is continuous with* $h(0) = 0$ *and* $h(u) > u$ *for all* $u > 0$.

Then, solutions are equi-bounded. Furthermore, if $F(t, 0) \equiv 0$ *for all* $t \ge 0$, *then the zero solution is stable.*

3. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

Let $V(t, x)$ defined on $[-r, \infty) \times R^d$ be a real valued, continuous function which is locally Lipschitzian with respect to x. Then, M. E. Parrott [19] considered under what conditions $V(t, x(t))$ approaches some finite value as $t \to \infty$.

THEOREM 3.1 (Parrott). *Suppose that there exists an integrable function p*: $[0, \infty) \to [0, \infty)$ *such that* $V'_{(1)}(t, \varphi) \le p_1(t)$ *whenever* $t \ge 0$, $\varphi \in C(B)$, $\sup\{V(t+s, \varphi(s)): s \in [-r, 0]\} = V(t, \varphi(0))$. *If for every* $\varepsilon > 0$ *there exist a* $K = K(\varepsilon) > 0$, *an integrable function* p_2: $[0, \infty) \to [0, \infty)$, *a continuous function* h:$[0, \infty) \to [0, \infty)$ *with* $h(u) = h(\varepsilon)(u) > u$ *for* $u > 0$, $h(0) = 0$, *and a time* $T = T(\varepsilon) > t_0$ *such that*

$$V'_{(1)}(t, \varphi) \le Kp_2(t), \qquad t \ge T,$$

whenever $\varphi \in C(B)$, $\sup\{V(t+s, \varphi(s)): s \in [-r, 0]\} \le 2\varepsilon$, $V(t, \varphi(0)) \ge \varepsilon$, *and* $\sup\{V(t+s, \varphi(s)): s \in [-r, 0]\} \le h(V(t, \varphi(0)))$ *for all* $t \ge T$, *then for any solution* $x(t)$ *of* (1) *such that* $x_t \in C(B)$ *for all* $t \ge t_0$, $\lim_{t \to \infty} V(t, x(t))$ *exists.*

But we note that from Theorem 3.1 one cannot know whether $V(t, x(t))$ approaches zero as $t \to \infty$ or not. Thus, consider another so-called Lyapunov function $W(t, x)$ for discussing asymptotic behavior of the solution $x(t)$ and, noting that $\limsup_{t \to \infty}\{V(t+s, x(t+s)): s \in [-r, 0]\} = \limsup_{t \to \infty} V(t, x(t))$, we shall give a sufficient condition for $W(t, x(t))$ to approach zero as $t \to \infty$.

THEOREM 3.2. *For the functional differential equation* (1), *assume that there exist continuous functions* $V(t, x)$, $W(t, x)$ *on* $[-r, \infty) \times R^d$ *which are locally Lipschitzian with respect to* x, *an integrally positive function* p: $[0, \infty) \to [0, \infty)$ *and an integrable function* e: $[0, \infty) \to [0, \infty)$. *And suppose that there exists an integrable function* p_1: $[0, \infty) \to [0, \infty)$ *such that* $V'_{(1)}(t, \varphi) \le$

$p_1(t)$ *whenever* $t \geq 0$, $\varphi \in C(B)$, $\sup\{V(t+s, \varphi(s))\colon\ s \in [-r, 0]\} = V(t, \varphi(0))$, *and furthermore, for every* $\varepsilon > 0$ *and any fixed* $B > 0$ *there exists a* $H = H(\varepsilon) > 0$, *a continuous* $h = h(\varepsilon)\colon [0,\infty) \to [0,\infty)$ *with* $h(u) > u$ *for* $u > 0$ *and a time* $T = T(\varepsilon) > t_0$ *such that*

(3.2a) *there exists an* $M > 0$ *such that* $W'_{(1)}(t, \varphi) \leq M$, $t \geq T$,

(3.2b) $V'_{(1)}(t, \varphi) \leq -Hp(t)W(t, \varphi(0)) + e(t)$, $t \geq T$,

whenever $\varphi \in C(B)$, $\sup\{V(t+s, \varphi(s))\colon s \in [-r, 0]\} \leq 2\varepsilon$, $V(t, \varphi(0)) \geq \varepsilon$, *and* $\sup\{V(t+s, \varphi(s))\colon s \in [-r, 0]\} \leq h(V(t, \varphi(0)))$ *for all* $t \geq T$. *Furthermore, assume that if* $\lim_{t\to\infty} V(t, x(t)) = 0$, *then* $\lim_{t\to\infty} x(t) = 0$ *and if* $\lim_{t\to\infty} W(t, x(t)) \neq 0$, *then* $\lim_{t\to\infty} x(t) \neq 0$. *Then,* $W(t, x(t)) \to 0$ *as* $t \to \infty$ *for any solution* $x(t)$ *of* (1) *such that* $x_t \in C(B)$ *for all* $t \geq t_0$.

Proof. Suppose that there exists a solution $x(t)$ of (1) such that $W(t, x(t)) \nrightarrow 0$ as $t \to \infty$, $x_t \in C(B)$. Then, since $x(t) \nrightarrow 0$ as $t \to \infty$, we have $\lim_{t\to\infty} V(t, x(t)) = \beta > 0$ by Theorem 3.1 and the hypothesis. First assume that there exist two sequences $\{t'_j\}$, $\{t_j\}$ with t'_j, $t_j \to \infty$ as $j \to \infty$, $t_j < t'_j < t_{j+1}$ $(j = 1, 2, \ldots)$ and some $\delta > 0$ such that

$$W\big(t_j, x(t_j)\big) = \delta/2, \qquad W\big(t'_j, x(t'_j)\big) = \delta,$$

$$\delta/2 < W(t, x(t)) < \delta, \qquad t \in \big(t_j, t'_j\big).$$

Here, because of $\beta > 0$, we can take a $\lambda > 0$ with $\lambda < \beta < 2\lambda$. Let $\sigma(\lambda) = \min\{|h(u) - u|\colon\ \lambda \leq u \leq 2\lambda\}$ where $h = h(\lambda)$ and choose a time $T_0 > T = T(\lambda)$ such that $\sup\{V(t+s, x(t+s))\colon\ s \in [-r, 0]\} < \beta + \sigma(\lambda)/2$, $\sup\{V(t+s, x(t+s))\colon\ s \in [-r, 0]\} < 2\lambda$, $V(t, x(t)) \geq \lambda$, and $|V(t, x(t))) - \beta| < \sigma(\lambda)/2$ for all $t \geq T_0$. Then, since $\lambda \leq V(t, x(t)) \leq 2\lambda$ and $\beta \leq \sigma(\lambda)/2 + V(t, x(t))$, we obtain that $\sup\{V(t+s, x(t+s))\colon s \in [-r, 0]\} < \beta + \sigma(\lambda)/2 \leq \sigma(\lambda)/2 + \sigma(\lambda)/2 + V(t, x(t)) = \min\{|h(u) - |\colon\ \lambda \leq u \leq 2\lambda\} + V(t, x(t)) \leq h(V(t, x(t)))$ for all $t \geq T_0$. Therefore, by condition (3.2a),

$$\begin{aligned}\delta/2 &= W\big(t'_j, x(t'_j)\big) - W\big(t_j, x(t_j)\big)\\ &\leq \int_{t_j}^{t'_j} W'_{(1)}(s, x_s)\, ds\\ &\leq M\big(t'_j - t_j\big).\end{aligned}$$

Thus, we have $t'_j \geq t_j + \delta/[2M]$ $(j = 1, 2, 3, \ldots)$. Furthermore, by condition (3.2b), we have that

$$V'_{(1)}(t, x_t) \leq -Hp(t)W(t, x(t)) + e(t), \qquad t \geq T_0.$$

We may suppose $t_j > T_0$ $(j = 1, 2, \dots)$ by taking a subsequence of $\{t_j\}$, if necessary. Thus,

$$\begin{aligned} V(t'_k, x(t'_k)) &\le V(t_1, x(t_1)) + \int_{t_1}^{t'_k} V'_{(1)}(s, x_s)\, ds \\ &\le V(t_1, x(t_1)) + \sum_{j=1}^{k} \int_{t_j}^{t'_j} [-Hp(s)W(s, x(s))]\, ds + \int_{t_1}^{t'_k} e(s)\, ds \\ &\le V(t_1, x(t_1)) + \sum_{j=1}^{k} \int_{t_j}^{t'_j} [-Hp(s)\delta/2]\, ds + \int_0^{\infty} e(s)\, ds, \end{aligned}$$

which tends to $-\infty$ as $k \to \infty$ since $p(t)$ is integrally positive and $t'_j \ge t_j + \delta/[2M]$ $(j = 1, 2, 3, \dots)$. This is a contradiction. Next, suppose that there exists a time $T_1 > t_0$ and $\gamma > 0$ such that $W(t, x(t)) \ge \gamma$ for all $t \ge T_1$. Then, since x_t satisfies the condition of the theorem, we have that $V(t, x(t)) - V(T_1, x(T_1)) \le -H\gamma \int_{T_1}^{t} p(s)\, ds + \int_{T_1}^{t} e(s)\, ds$. Thus, it follows that $V(t, x(t)) \to -\infty$ as $t \to \infty$. This is a contradiction. Consequently, the proof of the theorem is complete.

In some cases the next theorem will be useful.

THEOREM 3.3. *For any fixed bounded solution* $\theta(t) = (\theta_1(t), \theta_2(t), \dots, \theta_d(t))$ *of* (1), *set*

$$F(t) = F_\theta(t) = \sum_{i=1}^{d} b_i \int_{-r}^{0} \left\{ \int_s^0 \theta_i(t+u)^2\, du \right\} ds, \qquad b_i \ge 0.$$

Assume that there exist continuous functions $V(t, x)$, $W(x)$, *and* $U(t, x)$ *on* $[-r, \infty] \times R^d$ *which are locally Lipschitzian with respect to* x, *an integrally positive function* p: $[0, \infty) \to [0, \infty)$ *and an integrable function* e: $[0, \infty) \to [0, \infty)$ *such that*

$$V(t, x) = U(t, x) + F(t), \qquad x \in R^d, t \ge t_0.$$

Here, consider $V'_{(1)}(t, x_t)$ *for any solution* $x(t)$ *of* (1). *If for* $x_t = \theta_t$, $t \ge t_0$, *the following inequality is satisfied*:

(3.3a) $V'_{(1)}(t, \theta_t) \le -p(t)W(\theta(t)) + e(t)$, $t \ge t_0$,

then $W(\theta(t)) \to 0$ *as* $t \to \infty$.

Proof. Suppose that there exists a solution $\theta(t)$ of (1) such that $W(\theta(t)) \nrightarrow 0$ as $t \to \infty$, where $\theta_t \in C(B)$. First, assume that there exist two sequences $\{t'_j\}$, $\{t_j\}$ with t'_j, $t_j \to \infty$ as $j \to \infty$, $t_j < t'_j < t_{j+1}$ $(j = 1, 2, \dots)$ and some $\delta > 0$ such that $W(\theta(t_j)) = \delta/2$, $W(\theta(t'_j)) = \delta$, $\delta/2 <$

$W(\theta(t)) < \delta$ for all $t \in (t_j, t_j')$. Since $\theta(t)$ is the bounded solution of (1), there exists an $M > 0$ such that $W'_{(1)}(\theta_t) \leq M$. Thus, $t_j' \geq t_j + \delta/[2M]$ ($j = 1, 2, 3, \ldots$). On the other hand, since condition (3.3a) holds, by the same method as in the proof of Theorem 3.2, we have a contradiction. Furthermore, by continuing the same discussion as in the proof of Theorem 3.2, the proof of the theorem is complete.

4. APPLICATIONS

In this section we present several examples which illustrate Theorem 3.2. Let $B > 0$ denote any fixed real number. First, we generalize the example by Haddock and Terjéki [9].

EXAMPLE 4.1. Consider the scalar differential equation

$$x'(t) = -a(t)x(t) + \int_{-r(t)}^{0} b(s)x(t+s)\,ds - x(t)\int_{-r(t)}^{0} c(s)x^2(t+s)\,ds, \tag{2}$$

where $a, b, c, r\colon [-r, \infty) \to R^1$ are real valued and continuous functions such that $0 \leq r(t) < r$ for some $r > 0$, $\int_{-r(t)}^{0} |b(s)|\,ds \leq a(t)$, and $c(t) \geq 0$ for all $t \in [-r, \infty)$. Let $V(t, x) = x^2/2$. Let $x(t)$ denote any solution of (2). Then,

$$V'_{(2)}(t, x_t) = x(t)\left\{-a(t)x(t) + \int_{-r(t)}^{0} b(s)x(t+s)\,ds - x(t)\int_{-r(t)}^{0} c(s)x^2(t+s)\,ds\right\}.$$

If $\alpha(t) := a(t) - q\int_{-r(t)}^{0} |b(s)|\,ds \geq 0$ with $q > 1$ is integrally positive, then any solution of (2) approaches zero as $t \to \infty$. To see this, let $h(u) = q^2 u$ for any $u \geq 0$. Then

$$V'_{(2)}(t, x_t) \leq -\left\{a(t) - q\int_{-r(t)}^{0} |b(s)|\,ds\right\}x(t)^2,$$

whenever $t \geq t_0$, $V(t+s, x(t+s)) \leq h(V(t, x(t)))$ for any $s \in [-r, 0]$. Then we note that all solutions of (2) are bounded by Theorem 2.1. Let $W(t, x) = x^2/2$. There exists an $M > 0$ such that $W'_{(2)}(t, x_t) \leq M$ whenever $x_t \in C(B)$, $t \geq t_0$, $V(t+s, x(t+s)) \leq h(V(t, x(t)))$ for any $s \in [-r, 0]$.

Thus, all conditions of Theorem 3.2 are satisfied. Therefore, since B is any fixed real number, any solution $x(t)$ of (2) approaches zero as $t \to \infty$. And, since the zero solution of (2) is stable and solutions are equi-bounded by Theorem 2.1, the zero solution of (2) is globally asymptotically stable.

If $a(t) = \int_{-r(t)}^{0} |b(s)|\, ds$, $t \geq 0$, and there exists an $L > 1$ such that $a(t) < L\int_{-r(t)}^{0} c(s)\, ds$ for all $t \geq t_0 \geq 0$, and $\int_{-r(t)}^{0} c(s)\, ds - a(t)/L$ is integrally positive, then any solution $x(t)$ of (2) approaches zero as $t \to \infty$. To see this, for any $\varepsilon > 0$, let $h(u) = q^2 u$, $u \geq 0$, where $q = (1 + 2\varepsilon/L)$. Then, we obtain that

$$V'_{(2)}(t, x_t) \leq -\left\{a(t) - (1 + 2\varepsilon/L)\int_{-r(t)}^{0} |b(s)|\, ds\right\} x^2(t)$$

$$-\left\{\int_{-r(t)}^{0} 2\varepsilon c(s)\, ds\right\} x^2(t)$$

$$= -2\varepsilon\left\{\int_{-r(t)}^{0} c(s)\, ds - a(t)/L\right\} x^2(t),$$

whenever $\sup\{V(t+s, x(t+s)) \colon s \in [-r, 0]\} \leq h(V(t, x(t)), V(t, x(t)) \geq \varepsilon$ for all $t \geq T$ where T is a sufficiently large time. Therefore, by Theorems 2.1 and 3.2 the proof of the example is complete.

Remark 4.1. Let $a(t) \equiv a > 0$, $r(t) \equiv r > 0$ for all $t \geq 0$. If $\int_{-r}^{0} |b(s)|\, ds < a$, then there exists a $q > 1$ such that $a - q\int_{-r}^{0} |b(s)|\, ds > 0$. If $\int_{-r}^{0} |b(s)|\, ds = a$ and $c(0) > 0$, then there exists an $L > 1$ such that $a < L\int_{-r}^{0} c(s)\, ds$, and $\int_{-r}^{0} c(s) - a/L > 0$ is integrally positive since it is a constant. Therefore, Example 4.1 is a generalization of Example 2.1 in [9, p. 101].

Remark 4.2. T. A. Burton [2] and T. Yoshizawa [23] discussed the next scalar differential equation

$$x'(t) = -\left[a + (t \sin t)^2\right] x(t) + bx(t - r(t)),$$

where $a > 0$, b are constants. If $a > b$, $0 \leq r(t) < r$, then any solution $x(t) \to 0$ as $t \to \infty$ by Theorem 3.2. However, they used a Lyapunov functional $V(t, x(\cdot)) = x(t)^2/2 + k\int_{t-r(t)}^{t} x(s)^2\, ds$ where $k > 0$ and under conditions $b^2 < a^2(1 - \alpha)$, $-M \leq r'(t) \leq \alpha$, $0 < \alpha < 1$ and $b^2 \leq a^2(1 - \alpha)$, $r'(t) \leq \alpha$, $0 < \alpha < 1$, respectively, they proved $x(t) \to 0$ as $t \to \infty$.

Next, we consider a generalization of Cooke's model [5, p. 41]. See also Haddock and Terjéki [9, p. 103].

EXAMPLE 4.2. Consider the scalar differential equation

$$x'(t) = a(t)g\big(x(t-r(t))\big)(1-x(t)) - a(t)x(t), \qquad t \geq 0, \quad (3)$$

where $a(t) \geq 0$ is integrally positive, $g(u)$ is continuous, monotone nondecreasing with $g(0) \geq 0$ and $u \geq g(u)$ for any $u \geq 0$. For every $\varepsilon > 0$ choose a positive real number q such that $1 + g(\sqrt{2\varepsilon}) > q > 1$. Let $h(u) = q^2 u$ for any $u \geq 0$ and let $V(t,x) = x^2/2$. $r(t)$ is the same one as in Example 4.1. Then,

$$\begin{aligned} V'_{(3)}(t, x_t) = x(t)\{&-a(t)x(t) + a(t)g\big(x(t-r(t))\big) \\ &-a(t)x(t)g\big(x(t-r(t))\big)\}. \end{aligned}$$

Whenever $x_t \in C(B)$, $x(t) \geq 0$ for all $t \geq t_0$ and $x(t)$ satisfies $\sup\{V(t+s, x(t+s))\colon s \in [-r, 0]\} \leq h(V(t, x(t)))$, $V(t, x(t)) \geq \varepsilon$, and $\sup\{V(t+s, x(t+s))\colon s \in [-r, 0]\} \leq 2\varepsilon$ for all $t \geq T$ where T is a sufficiently large time,

$$\begin{aligned} V'_{(3)}(t, x_t) &\leq -a(t)x(t)^2 + qa(t)x(t)^2 - g\big(\sqrt{2\varepsilon}\big)a(t)x(t)^2 \\ &= -\big(1 + g\big(\sqrt{2\varepsilon}\big) - q\big)a(t)x(t)^2, \qquad t \geq T + r. \end{aligned}$$

Let $W(t,x) = x^2/2$. Then, all conditions of Theorem 3.2 are satisfied. Therefore, since B is any fixed real number, any positive bounded solution $x(t)$ of (3) approaches zero as $t \to \infty$.

Finally, we give an example which illustrates Theorem 3.3. The next type of system was discussed by Burton [2] and Yoshizawa [23].

EXAMPLE 4.3. Consider the 2-dimensional differential equation

$$\begin{aligned} x'(t) &= y(t) \qquad (4) \\ y'(t) &= -c(t, x(t), y(t))y(t) - g(x(t)) \\ &\quad + \int_{-r(t)}^{0} g'(x(t+s))y(t+s)\,ds, \end{aligned}$$

where $r(t)$ is the same one as in Example 4.1, $g\colon (-\infty, \infty) \to (-\infty, \infty)$ is continuously differentiable, and $|g'(u)| \leq L$ for some $L > 0$ and $ug(u) > 0$ for any $u \neq 0$. Suppose that there exists a function $a(t) > 0$ such that $c(t, x, y) \geq a(t)$, $t \geq 0$ and the function $a(t) - rL \geq 0$ is integrally positive. Now, for any fixed bounded solution $\theta(t) = (\alpha(t), \beta(t))$ of (4) let $F(t) = (L/2)\int_{-r}^{0}(\int_s^0 \beta(t+u)^2\,du)\,ds$. And set $V(t, x, y) = G(x) + y^2/2 + F(t)$, and $W(x, y) = y^2/2$ for all $x, y \in R^1$, where $G(x) = \int_0^x g(s)\,ds$. Then, for any solution $(x(t), y(t))$ of (4)

$$V'_{(4)}(t, x_t, y_t) \leq -a(t)y(t)^2 + (L/2)\int_{-r}^{0} 2|y(t)||y(t+s)|\,ds$$

$$+(L/2)\int_{-r}^{0}\left(\beta(t)^2 - \beta(t+s)^2\right)ds.$$

If $(x(t), y(t)) = (\alpha(t), \beta(t))$, $t \geq t_0$, then

$$V'_{(4)}(t, \alpha_t, \beta_t) \leq -(a(t) - rL)\beta(t)^2 \qquad (2ab \leq a^2 + b^2)$$

and there exists an $M > 0$ such that $W'_{(4)}(\theta_t) \leq M$. Therefore, by Theorem 3.3 we have that $W(\alpha(t), \beta(t)) = \beta^2(t)/2$ approaches zero as $t \to \infty$. Thus, all bounded solutions approach the x-axis as $t \to \infty$. Thus, if $G(u) \to \infty$ as $|u| \to \infty$, then any solution $(x(t), y(t))$ of (4) approaches the x-axis and $G(x(t))$ approaches a constant as $t \to \infty$.

Remark 4.3. T. Yoshizawa [23] proved the same conclusion under the conditions $0 \leq r(t) \leq \beta(t)$, $\beta'(t) \leq \beta_0 < 1$, $h(t, x, y) \geq b\beta(t)$, $b > 0$ and $L^2 < b^2(1 - \beta_0)$ via a Lyapunov functional

$$V(t, x(\cdot), y(\cdot)) = 2G(x(t)) + y(t)^2 + b\int_{-\beta(t)}^{0}\left(\int_{s}^{0} y(t+u)^2\,du\right)ds$$

assuming that $\beta(t)$ is integrally positive. See also T. A. Burton [2].

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