Available online at www.sciencedirect.com
ScienceDirect
Journal of Functional Analysis 260 (2011) 3429–3456
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Broken translation invariance in quasifree fermionic correlations out of equilibrium

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Received 2 July 2010; accepted 23 February 2011

Communicated by L. Gross

Abstract

Using the C^* algebraic scattering approach to study quasifree fermionic systems out of equilibrium in quantum statistical mechanics, we construct the nonequilibrium steady state in the isotropic XY chain whose translation invariance has been broken by a local magnetization and analyze the asymptotic behavior of the expectation value for a class of spatial correlation observables in this state. The effect of the breaking of translation invariance is twofold. Mathematically, the finite rank perturbation not only regularizes the scalar symbol of the invertible Toeplitz operator generating the leading order exponential decay but also gives rise to an additional trace class Hankel operator in the correlation determinant. Physically, in its decay rate, the nonequilibrium steady state exhibits a left mover–right mover structure affected by the scattering at the impurity.

Keywords: Nonequilibrium quantum statistical mechanics; Quasifree fermions; C^* algebraic scattering; Toeplitz theory

1. Introduction

In the mathematical study of open quantum systems, the role played by quasifree fermionic systems is an important one. Within the framework of algebraic quantum statistical mechanics, they not only allow for a powerful description by means of scattering theory on the one-particle Hilbert space over which the fermionic algebra of observables is built, being thus ideally suited for rigorous analysis on many levels, but they also represent a class of systems which are indeed realized in nature, see, for example, Culvahouse et al. [16], D'Iorio et al. [17], and Sologubenko

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2011.02.021

et al. [24]. A special instance of this class is the finite XY spin chain introduced by Lieb et al. [20] and extended to the infinite two-sided discrete line by Araki [5] in the framework of C^*-dynamical systems. As a matter of fact, this spin model can be mapped, in some precise sense, onto a gas of free fermions with the help of the Araki–Jordan–Wigner transformation. In order to study the effect of the breaking of translation invariance in this system, we choose the physically interesting and computationally convenient emptiness formation correlation observable. The so-called emptiness formation probability (EFP), i.e. the expectation value of this observable in a given state, describes, in the spin picture, the probability that all spins in a string of a given length point downwards. However, we would like to underline that the analysis is not limited to this observable but can rather be carried out for a broad class of spatial correlations.

The asymptotic behavior of the EFP in the XY chain for large string length has already been analyzed for the cases where the state is a ground state or a thermal equilibrium state at positive temperature. In both cases, the EFP can be written as the determinant of the section of a Toeplitz operator with scalar symbol. Since the higher order asymptotics of a Toeplitz determinant is highly sensitive to the regularity of the symbol of the Toeplitz operator, the asymptotic behavior of the ground state EFP is qualitatively different in the so-called critical and noncritical regimes corresponding to certain values of the anisotropy and the exterior magnetic field of the XY chain, i.e., in (19) below, the parameters γ and λ, respectively. It has been found that the EFP decays like a Gaussian in one of the critical regimes (with some additional explicit numerical prefactor and some power law prefactor), see Shiroishi et al. [23] and references therein. In a second critical regime and in all noncritical regimes, the EFP decays exponentially (in contrast to the noncritical regimes, there is an additional power law prefactor in the second critical regime whose exponent differs from the one in the first critical regime), see Abanov and Franchini [1,18]. These results have been derived by using powerful theorems of Szegő, Widom, and Fisher–Hartwig, and the yet unproven Basor–Tracy conjecture and some of its extensions, see Widom [26] and Böttcher and Silbermann [14,15]. Furthermore, in thermal equilibrium at positive temperature, the EFP can again be shown to decay exponentially by using a theorem of Szegő, see, for example, Shiroishi et al. [23] and Franchini and Abanov [18].

In contrast, out of equilibrium, the situation is more subtle. The typical open system consists of a confined sample which is coupled to extended ideal reservoirs at different temperatures. Using this paradigm, a translation invariant nonequilibrium steady state (NESS) has been constructed in Aschbacher and Pillet [13] for the XY chain using the scattering approach to algebraic quantum statistical mechanics developed by Ruelle [22] (for $\gamma = \lambda = 0$, this NESS has also been found by Araki and Ho [6] using a different method; moreover, using the latter approach, the magnetization profile at intermediate but large times has been studied by Ogata [21]). In this NESS, the EFP can still be recast into the form of a Toeplitz determinant, but now, the symbol is, in general, no longer scalar and regular. Due to the lack of control of higher order determinant asymptotics in Toeplitz theory with nontrivial irregular block symbols, we started off by studying bounds on the leading asymptotic order for a class of general block Toeplitz determinants in Aschbacher [7]. There, it turned out that suitable basic spectral information on the density of the state is sufficient to derive a bound on the rate of the exponential decay of the EFP in general translation invariant fermionic quasifree states. This bound proved to be exact not only for the decay rates of the ground states and the equilibrium states at positive temperature treated in Abanov and Franchini [1,18] and Shiroishi et al. [23] but also for the translation invariant NESS in the isotropic XY chain analyzed in Aschbacher [9].

In the present paper, new results are obtained for the asymptotic behavior of a class of spatial correlations, and in particular, for the asymptotic behavior of the EFP, in a NESS of the isotropic

XY chain whose translation invariance has been broken by a local magnetization in the form of a finite rank perturbation. Although such a spatial correlation can again be transformed from its initial Paffian form into a scalar Toeplitz determinant, the effect of the breaking of translation invariance manifests itself in a regularization of the Toeplitz symbol and the appearance of an additional Hankel operator whose symbol is smooth. Hence, due to Peller's theorem, this operator is of trace class, and the spatial asymptotics is governed by an exponential decay due to the invertibility of the Toeplitz operator at nonvanishing temperature. Moreover, the decay rate, determined by the Toeplitz symbol, exhibits the underlying left mover–right mover structure affected by the scattering at the impurity (see also Aschbacher [8] and Aschbacher and Barbaroux [10] for the left mover–right mover structure of the NESS expectation of several other types of correlation observables).

The paper is organized as follows. In Section 2, we set the stage for the nonequilibrium XY chain with impurity, construct its NESS, and derive the basic expression for the NESS EFP. Section 3 then contains the asymptotic analysis of the NESS EFP. Several ingredients of the proofs have been transferred to Appendices A–E, as, for example, the construction of the wave operators by means of stationary scattering theory or the summary of the spectral properties of the so-called magnetic Hamiltonian.

2. Nonequilibrium setting

In this section, we will shortly summarize the setting for the system out of equilibrium used in Aschbacher and Pillet [13]. In contradistinction to the presentation there, we skip the formulation of the two-sided XY chain as a spin system and rather focus directly on the underlying C^*-dynamical system structure in terms of Bogoliubov automorphisms on a selfdual CAR algebra as in Araki [5]. A C^*-dynamical system is a pair $(\mathfrak{A}, \tau)$, where $\mathfrak{A}$ is a C^* algebra and $\mathbb{R} \ni t \mapsto \tau^t \in \mathrm{Aut}(\mathfrak{A})$ a strongly continuous group of $*$-automorphism of $\mathfrak{A}$. For more information on the algebraic approach to open quantum systems, see, for example, Aschbacher et al. [11].

For some given $N \in \mathbb{N} \cup \{0\}$, the nonequilibrium configuration is set up by cutting the finite piece

$$\mathbb{Z}_{\mathcal{S}} := \{x \in \mathbb{Z} \mid -N \leqslant x \leqslant N\} \tag{1}$$

out of the two-sided discrete line $\mathbb{Z}$. This piece will play the role of the confined sample whereas the remaining parts,

$$\mathbb{Z}_L := \bigl\{x \in \mathbb{Z} \bigm| x \leqslant -(N+1)\bigr\}, \tag{2}$$

$$\mathbb{Z}_R := \{x \in \mathbb{Z} \mid x \geqslant N+1\}, \tag{3}$$

will act as infinitely extended thermal reservoirs, eventually carrying different temperatures, see Fig. 1.

The observables of the system are specified by the following selfdual CAR algebra over the wave functions on the chain.

Definition 1 *(Observables).* Let $\mathfrak{F}(\mathfrak{h})$ denote the fermionic Fock space over the one-particle Hilbert space of wave functions on the discrete line,

$$\mathfrak{h} := \ell^2(\mathbb{Z}). \tag{4}$$

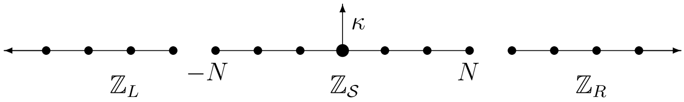

Fig. 1. The nonequilibrium setting for the XY chain.

With the help of the creation and annihilation operators $a^*(f), a(f) \in \mathcal{L}(\mathfrak{F}(\mathfrak{h}))$ with $f \in \mathfrak{h}$ (where $\mathcal{L}(\mathcal{H})$ denotes the bounded linear operators on the Hilbert space $\mathcal{H}$), the complex linear mapping $B : \mathfrak{h}^{\oplus 2} \to \mathcal{L}(\mathfrak{F}(\mathfrak{h}))$ is defined, for $F := [f_1, f_2] \in \mathfrak{h}^{\oplus 2}$, by

$$B(F) := a^*(f_1) + a(\bar{f_2}). \tag{5}$$

The observables are described by the selfdual CAR algebra over $\mathfrak{h}^{\oplus 2}$ with antiunitary involution J generated by the operators $B(F) \in \mathcal{L}(\mathfrak{F}(\mathfrak{h}))$ for all $F \in \mathfrak{h}^{\oplus 2}$, i.e. we have, for all $F, G \in \mathfrak{h}^{\oplus 2}$,

$$\big\{B^*(F), B(G)\big\} = (F, G), \tag{6}$$

$$B^*(F) = B(JF), \tag{7}$$

where $JF := [\bar{f_2}, \bar{f_1}]$ for all $F := [f_1, f_2] \in \mathfrak{h}^{\oplus 2}$, the anticommutator of $A, B \in \mathcal{L}(\mathcal{H})$ is $\{A, B\} := AB + BA$, and the scalar product in $\mathfrak{h}^{\oplus 2}$ is written as the one in $\mathfrak{h}$. We denote this algebra by $\mathfrak{A} := \mathfrak{A}(\mathfrak{h}^{\oplus 2}, J)$.

Remark 2. The concept of selfdual CAR algebras has been introduced and developed in Araki [3,4]. Here, it is just a convenient way of working with the linear combination (5). Also in view of future generalizations of the present paper, for example to the case of the truly anisotropic XY chain and other classes of correlations, we will stick to this notation in the present context.

We next specify the Bogoliubov $*$-automorphisms on the selfdual CAR algebra which describe the time evolutions used for the construction of the NESS.

Definition 3 *(Dynamics).* Let the coupling strength be $\kappa > 0$, and let $u \in \mathcal{L}(\mathfrak{h})$ be the translation given by $(uf)(x) := f(x-1)$ for all $f \in \mathfrak{h}$ and all $x \in \mathbb{Z}$. The XY, the decoupled, and the magnetic one-particle Hamiltonians $h, h_0, h_\mathrm{B} \in \mathcal{L}(\mathfrak{h})$, respectively, are defined by

$$h := \mathrm{Re}(u), \tag{8}$$

$$h_0 := h - (v_L + v_R), \tag{9}$$

$$h_\mathrm{B} := h + \kappa v, \tag{10}$$

where the decoupling operators $v_L, v_R \in \mathcal{L}^0(\mathfrak{h})$ (with $\mathcal{L}^0(\mathcal{H})$ the finite rank operators on $\mathcal{H}$) and the operator $v \in \mathcal{L}^0(\mathfrak{h})$ which breaks translation invariance have the form

$$v_L := \mathrm{Re}\big(u^{-(N+1)} p_0 u^N\big), \tag{11}$$

$$v_R := \mathrm{Re}\big(u^N p_0 u^{-(N+1)}\big), \tag{12}$$

$$v := p_0. \tag{13}$$

Here, the projection $p_0 \in \mathcal{L}^0(\mathfrak{h})$ is given by $p_0 := (\delta_0, \cdot)\delta_0$, where $\delta_x \in \mathfrak{h}$ for $x \in \mathbb{Z}$ denotes the Kronecker function (moreover, the real part of $A \in \mathcal{L}(\mathcal{H})$ is given by $\mathrm{Re}(A) := (A + A^*)/2$). For all $t \in \mathbb{R}$, the XY, the decoupled, and the magnetic time evolutions are the Bogoliubov $*$-automorphisms τ^t, τ_0^t, $\tau_{\mathrm{B}}^t \in \mathrm{Aut}(\mathfrak{A})$ defined on the generators $B(F) \in \mathfrak{A}$ with $F \in \mathfrak{h}^{\oplus 2}$ by

$$\tau^t\big(B(F)\big) := B\big(\mathrm{e}^{\mathrm{i}tH} F\big), \tag{14}$$

$$\tau_0^t\big(B(F)\big) := B\big(\mathrm{e}^{\mathrm{i}tH_0} F\big), \tag{15}$$

$$\tau_{\mathrm{B}}^t\big(B(F)\big) := B\big(\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}} F\big), \tag{16}$$

where we set $H := h \oplus -h$, $H_0 := h_0 \oplus -h_0$, and $H_{\mathrm{B}} := h_{\mathrm{B}} \oplus -h_{\mathrm{B}}$.

Remark 4. For the sake of an easy exposition, we restrict the analysis to the case $\kappa > 0$, the case $\kappa < 0$ being strictly analogous.

Remark 5. The magnetic Hamiltonian $H_{\mathrm{B}} \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ breaks translation invariance in the sense that the commutator $[H_{\mathrm{B}}, u \oplus u] = [h_{\mathrm{B}}, u] \oplus -[h_{\mathrm{B}}, u]$ is nonvanishing (where $[A, B] := AB - BA$ is the commutator of $A, B \in \mathcal{L}(\mathcal{H})$), i.e., for all $f \in \mathfrak{h}$, it holds

$$[h_{\mathrm{B}}, u]f = \kappa\big(f(-1)\delta_0 - f(0)\delta_1\big). \tag{17}$$

Remark 6. Since $H_{\mathrm{B}} \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ anticommutes with the antiunitary involution J, the magnetic Hamiltonian H_{B} generates a Bogoliubov transformation in the sense of Araki [3,4], i.e. that, for all $t \in \mathbb{R}$, we have

$$\big[\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}, J\big] = 0. \tag{18}$$

The same also holds for the XY and the decoupled Hamiltonian $H, H_0 \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$.

Remark 7. As mentioned at the beginning of this section, this model has its origin in the XY spin chain whose formal Hamiltonian is given by

$$H = -\frac{1}{4}\sum_{x\in\mathbb{Z}}\big\{(1+\gamma)\sigma_1^{(x)}\sigma_1^{(x+1)} + (1-\gamma)\sigma_2^{(x)}\sigma_2^{(x+1)} + 2\lambda\sigma_3^{(x)}\big\}, \tag{19}$$

where $\gamma \in (-1, 1)$ denotes the anisotropy, $\lambda \in \mathbb{R}$ the external magnetic field, and the Pauli basis of $\mathbb{C}^{2\times 2}$ reads

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{20}$$

The Hamiltonian h from (8) corresponds to the case of the isotropic XY chain without external magnetic field, i.e. to the case where $\gamma = 0$ and $\lambda = 0$.

The left and right reservoirs carry the inverse temperatures β_L and β_R, respectively. *Pour fixer les idées*, we assume, w.l.o.g., that they satisfy

$$0 < \beta_L \leqslant \beta_R < \infty. \tag{21}$$

Moreover, for later use, we set $\beta := (\beta_R + \beta_L)/2$ and $\delta := (\beta_R - \beta_L)/2$.

We next specify the state in which the system is prepared initially. It consists of a KMS state at the corresponding temperature for each reservoir, and, w.l.o.g., of the chaotic state for the sample. For the definition of fermionic quasifree states, see Appendix A.

Definition 8 *(Initial state).* The initial state $\omega_0 \in \mathcal{Q}(\mathfrak{A})$ is the quasifree state specified by the density $S_0 \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ of the form

$$S_0 := s_{0,-} \oplus s_{0,+}, \tag{22}$$

where the operators $s_{0,\pm} \in \mathcal{L}(\mathfrak{h})$ are defined by

$$s_{0,\pm} := \left(1 + \mathrm{e}^{\pm k_0}\right)^{-1}, \tag{23}$$

and $k_0 \in \mathcal{L}(\mathfrak{h} \simeq \mathfrak{h}_L \oplus \mathfrak{h}_{\mathcal{S}} \oplus \mathfrak{h}_R)$ is given by

$$k_0 := \beta_L h_L \oplus 0 \oplus \beta_R h_R. \tag{24}$$

Here, for $\alpha = L, \mathcal{S}, R$, we used the definitions $\mathfrak{h}_\alpha := \ell^2(\mathbb{Z}_\alpha)$ and $h_\alpha := i_\alpha^* h i_\alpha \in \mathcal{L}(\mathfrak{h}_\alpha)$, where $i_\alpha : \mathfrak{h}_\alpha \to \mathfrak{h}$ is the natural injection defined, for any $f \in \mathfrak{h}_\alpha$, by $i_\alpha f(x) := f(x)$ if $x \in \mathbb{Z}_\alpha$, and zero otherwise.

Remark 9. Note that $S_0 \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ is well defined, and that it satisfies the properties of a density given in Definition 32 of Appendix A.

Remark 10. The one-particle Hilbert space $\mathfrak{h}$ over $\mathbb{Z} = \mathbb{Z}_L \cup \mathbb{Z}_{\mathcal{S}} \cup \mathbb{Z}_R$ decomposes as $\mathfrak{h} \simeq \mathfrak{h}_L \oplus \mathfrak{h}_{\mathcal{S}} \oplus \mathfrak{h}_R$. It follows from (9) in Definition 3 that, w.r.t. this decomposition, the decoupled Hamiltonian h_0 does not couple the different subsystems to each other, indeed, i.e. we have $h_0 = h_L \oplus h_{\mathcal{S}} \oplus h_R$.

As discussed in the Introduction, we pick the EFP correlation observable in order to study the effect of the breaking of translation invariance on nonequilibrium expectation values. This observable is defined as follows.

Definition 11 *(EFP).* Let $x_0 \in \mathbb{Z}$ and $n \in \mathbb{N}$. The EFP observable $A_n \in \mathfrak{A}$ is defined by

$$A_n := \prod_{i=1}^{2n} B(F_i), \tag{25}$$

where, for all $i \in \mathbb{N}$, the form factors $F_i \in \mathfrak{h}^{\oplus 2}$ are given by

$$F_{2i-1} := u^i \oplus u^i G_1, \tag{26}$$

$$F_{2i} := u^i \oplus u^i G_2, \tag{27}$$

and the initial form factors $G_1, G_2 \in \mathfrak{h}^{\oplus 2}$ look like

$$G_1 := JG_2 := [0, \delta_{x_0-1}]. \tag{28}$$

Moreover, the expectation value $\mathrm{P} : \mathbb{N} \to [0,1]$ of the EFP observable $A_n \in \mathfrak{A}$ in the NESS $\omega_{\mathrm{B}} \in \mathcal{E}(\mathfrak{A})$ constructed in Theorem 17 below is denoted by

$$\mathrm{P}(n) := \omega_{\mathrm{B}}(A_n). \tag{29}$$

Remark 12. As for the name EFP, note that $A_n = \prod_{x=x_0}^{x_0+n-1} a_x a_x^*$, and that, with $B_n := \prod_{x=x_0}^{x_0+n-1} a_x$, we have, for any state $\omega \in \mathcal{E}(\mathfrak{A})$,

$$0 \leqslant \omega(A_n) = \omega\big(B_n B_n^*\big) \leqslant \|B_n\|^2 \leqslant \prod_{x=x_0}^{x_0+n-1} \|\delta_x\|^2 = 1. \tag{30}$$

Remark 13. The analysis of this paper can also be carried out for different form factors. If we choose the initial form factors $G_i =: [g_{i,1}, g_{i,2}] \in \mathfrak{h}^{\oplus 2}$ for $i = 1, 2$ to be of the completely localized form $g_{i,l} = a_{il}\delta_{x_{il}}$ for $a_{il} \in \mathbb{C}$ and $x_{il} \in \mathbb{Z}$ with $l = 1, 2$, we cover the case $G_1 = [-\delta_{-1}, \delta_{-1}]$ and $G_2 = [\delta_0, \delta_0]$. This choice describes the prominent spin-spin correlations $\sigma_1^{(0)}\sigma_1^{(n)}$, see, for example, Aschbacher and Barbaroux [10].

The following definition from Ruelle [22] introduces the concept of nonequilibrium steady state (NESS) in the framework of C^*-dynamical systems. For the situation at hand, the C^*-dynamical system is given in terms of the magnetic Bogoliubov $*$-automorphism group τ_{B} on the selfdual CAR algebra $\mathfrak{A}$.

Definition 14 *(NESS).* A NESS associated with the C^*-dynamical system $(\mathfrak{A}, \tau_{\mathrm{B}})$ and the initial state $\omega_0 \in \mathcal{E}(\mathfrak{A})$ is a weak-$*$ limit point for $T \to \infty$ of the net

$$\left\{ \frac{1}{T} \int_0^T \mathrm{d}t\, \omega_0 \circ \tau_{\mathrm{B}}^t \;\middle|\; T > 0 \right\}. \tag{31}$$

Next, we define the time dependent correlation matrix of the EFP observable $A_n \in \mathfrak{A}$ w.r.t. the initial state $\omega_0 \in \mathcal{E}(\mathfrak{A})$ and the magnetic dynamics $\tau_{\mathrm{B}}^t \in \mathrm{Aut}(\mathfrak{A})$.

Definition 15 *(Correlation matrix).* Let $F_i \in \mathfrak{h}^{\oplus 2}$ for $i \in \mathbb{N}$ be the form factors of Definition 11. For all $t \in \mathbb{R}$, the skew-symmetric correlation matrix $\Omega_n(t) \in \mathbb{C}_a^{2n\times 2n} := \{A \in \mathbb{C}^{2n\times 2n} \mid A^{\mathrm{t}} = -A\}$ (where A^{t} is the transpose of A) is defined, for all $i, j = 1, \ldots, 2n$, by its entries

$$\Omega_{ij}(t) := \begin{cases} \omega_0(B^*(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} JF_i)B(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} F_j)), & \text{if } i < j, \\ 0, & \text{if } i = j, \\ -\Omega_{ji}(t), & \text{if } i > j. \end{cases} \tag{32}$$

Moreover, the matrices $\Omega_n^{\mathrm{aa}}(t), \Omega_n^{\mathrm{ap}}(t), \Omega_n^{\mathrm{pa}}(t), \Omega_n^{\mathrm{pp}}(t) \in \mathbb{C}_a^{2n\times 2n}$ are defined, for $i, j = 1, \dots, 2n$ and $i < j$, by

$$\Omega_{ij}^{\mathrm{aa}}(t) := \omega_0\big(B^*\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{ac}}(H_\mathrm{B})JF_i\big)B\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{ac}}(H_\mathrm{B})F_j\big)\big), \tag{33}$$

$$\Omega_{ij}^{\mathrm{ap}}(t) := \omega_0\big(B^*\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{ac}}(H_\mathrm{B})JF_i\big)B\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{pp}}(H_\mathrm{B})F_j\big)\big), \tag{34}$$

$$\Omega_{ij}^{\mathrm{pa}}(t) := \omega_0\big(B^*\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{pp}}(H_\mathrm{B})JF_i\big)B\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{ac}}(H_\mathrm{B})F_j\big)\big), \tag{35}$$

$$\Omega_{ij}^{\mathrm{pp}}(t) := \omega_0\big(B^*\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{pp}}(H_\mathrm{B})JF_i\big)B\big(\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{pp}}(H_\mathrm{B})F_j\big)\big), \tag{36}$$

and are to be completed as in (32) for $i \geqslant j$. Here, $1_{\mathrm{ac}}(H_\mathrm{B}), 1_{\mathrm{pp}}(H_\mathrm{B}) \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ are the spectral projections onto the absolutely continuous and the pure point subspaces of H_B, respectively.

The contributions which will play a role in the large time limit are defined as follows.

Definition 16 *(Asymptotic correlation matrix).* Let $F_i \in \mathfrak{h}^{\oplus 2}$ for $i \in \mathbb{N}$ be the form factors of Definition 11. The matrices $\Omega_n^{\mathrm{aa}}, \Omega_n^{\mathrm{pp}} \in \mathbb{C}_a^{2n\times 2n}$ are defined, for $i, j = 1, \dots, 2n$ and $i < j$, by

$$\Omega_{ij}^{\mathrm{aa}} := \omega_0\big(B^*\big(W(H_0, H_\mathrm{B})JF_i\big)B\big(W(H_0, H_\mathrm{B})F_j\big)\big), \tag{37}$$

$$\Omega_{ij}^{\mathrm{pp}} := \sum_{e\in \mathrm{spec}_{\mathrm{pp}}(H_\mathrm{B})} \omega_0\big(B^*\big(1_e(H_\mathrm{B})JF_i\big)B\big(1_e(H_\mathrm{B})F_j\big)\big), \tag{38}$$

and are to be completed as in (32) for $i \geqslant j$. Here, $1_e(H_\mathrm{B}) \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ denotes the spectral projection onto the eigenspace associated with the eigenvalue e in the set of eigenvalues $\mathrm{spec}_{\mathrm{pp}}(H_\mathrm{B})$ of H_B, and the wave operator $W(H_0, H_\mathrm{B}) \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ is defined by

$$W(H_0, H_\mathrm{B}) := \operatorname*{s-lim}_{t\to\infty} \mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}tH_\mathrm{B}} 1_{\mathrm{ac}}(H_\mathrm{B}). \tag{39}$$

The following theorem establishes the existence and uniqueness of the NESS and yields an expression for the EFP in this NESS.

From now on, whenever an entry of a skew-symmetric matrix is written down, we always assume that the row index is strictly smaller than the column index. Moreover, for the definition of the Pfaffian, see Appendix A.

Theorem 17 *(NESS and NESS EFP). There exists a unique quasifree NESS $\omega_\mathrm{B} \in \mathcal{Q}(\mathfrak{A})$ associated with the C^*-dynamical system $(\mathfrak{A}, \tau_\mathrm{B})$ and the initial state $\omega_0 \in \mathcal{E}(\mathfrak{A})$ whose density $S_\mathrm{B} \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ has the form*

$$S_\mathrm{B} = W^*(H_0, H_\mathrm{B})S_0W(H_0, H_\mathrm{B}) + \sum_{e\in \mathrm{spec}_{\mathrm{pp}}(H_\mathrm{B})} 1_e(H_\mathrm{B})S_0 1_e(H_\mathrm{B}). \tag{40}$$

Moreover, the expectation value of the EFP observable in this NESS is given by

$$\mathrm{P}(n) = \mathrm{pf}\big(\Omega_n^{\mathrm{aa}} + \Omega_n^{\mathrm{pp}}\big). \tag{41}$$

Proof. We proceed similarly to the proof of Theorem 3.2 in Aschbacher et al. [12]. To this end, we note that the expectation value in the quasifree initial state $\omega_0 \in \mathcal{Q}(\mathfrak{A})$ of the correlation observable $A_n \in \mathfrak{A}$ propagated in time with the magnetic dynamics $\tau_{\mathrm{B}}^t \in \mathrm{Aut}(\mathfrak{A})$ can be written, for all $t \in \mathbb{R}$, as the Pfaffian of the correlation matrix $\Omega_n(t) \in \mathbb{C}_a^{2n\times 2n}$ from Definition 15,

$$\omega_0\big(\tau_{\mathrm{B}}^t(A_n)\big) = \mathrm{pf}\big(\Omega_n(t)\big), \tag{42}$$

where we used (18) in Remark 6 to commute the antiunitary involution J across the unitary group generated by $H_{\mathrm{B}} \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$. In order to treat the argument of the Pfaffian, we make use of assertion (a) in Theorem 36 of Appendix B which states that, for the singular continuous spectrum, we have

$$\mathrm{spec}_{\mathrm{sc}}(H_{\mathrm{B}}) = \emptyset. \tag{43}$$

Hence, injecting $1_{\mathrm{ac}}(H_{\mathrm{B}}) + 1_{\mathrm{pp}}(H_{\mathrm{B}}) = 1 \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ to the left of JF_i and F_j in the correlation matrix entry

$$\begin{aligned}\Omega_{ij}(t) &= \omega_0\big(B^*\big(\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}JF_i\big)B\big(\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}F_j\big)\big)\\ &= \big(\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}JF_i, S_0\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}F_j\big),\end{aligned} \tag{44}$$

the correlation matrix can be decomposed as

$$\Omega_n(t) = \Omega_n^{\mathrm{aa}}(t) + \Omega_n^{\mathrm{ap}}(t) + \Omega_n^{\mathrm{pa}}(t) + \Omega_n^{\mathrm{pp}}(t), \tag{45}$$

where the matrices on the r.h.s. of (45) are given in Definition 15. Since the NESS is constructed in the large time limit, we separately study this limit for all the terms in (45). So, using that the initial state is invariant under the decoupled time evolution, i.e. $[H_0, S_0] = 0$, the first term can be written as

$$\Omega_{ij}^{\mathrm{aa}}(t) = \big(\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}1_{\mathrm{ac}}(H_{\mathrm{B}})JF_i, S_0\mathrm{e}^{-\mathrm{i}tH_0}\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}1_{\mathrm{ac}}(H_{\mathrm{B}})F_j\big). \tag{46}$$

Thus, with the help of the Kato–Rosenblum theorem from scattering theory for perturbations of trace class type (see, for example, Yafaev [27]), we find

$$\lim_{t\to\infty}\Omega_n^{\mathrm{aa}}(t) = \Omega_n^{\mathrm{aa}}, \tag{47}$$

where we used that $H_0 - H_{\mathrm{B}} \in \mathcal{L}^0(\mathfrak{h}^{\oplus 2})$, and the r.h.s. is given in Definition 16. For the second term on the r.h.s. of (45), we have the bound

$$\big|\Omega_{ij}^{\mathrm{ap}}(t)\big| \leqslant \big\|1_{\mathrm{pp}}(H_{\mathrm{B}})S_0\mathrm{e}^{\mathrm{i}tH_{\mathrm{B}}}1_{\mathrm{ac}}(H_{\mathrm{B}})JF_i\big\|\,\|F_j\|. \tag{48}$$

Since the pure point spectrum of H_{B} consists of the two simple eigenvalues $\pm e_{\mathrm{B}}$, where e_{B} is given in assertion (c) of Theorem 36 in Appendix B, we have $1_{\mathrm{pp}}(H_{\mathrm{B}}) \in \mathcal{L}^0(\mathfrak{h}^{\oplus 2})$, and, hence, it follows that

$$\lim_{t\to\infty} \Omega_n^{\mathrm{ap}}(t) = 0. \tag{49}$$

The same holds for $\Omega_n^{\mathrm{pa}}(t)$, of course. For the last term on the r.h.s. of (45), using $1_{\mathrm{pp}}(H_{\mathrm{B}}) = 1_{e_{\mathrm{B}}}(H_{\mathrm{B}}) + 1_{-e_{\mathrm{B}}}(H_{\mathrm{B}})$, we get

$$\Omega_{ij}^{\mathrm{pp}}(t) = \sum_{e,e' \in \{\pm e_{\mathrm{B}}\}} \mathrm{e}^{-\mathrm{i}t(e'-e)} \big(1_e(H_{\mathrm{B}}) J F_i, S_0 1_{e'}(H_{\mathrm{B}}) F_j\big). \tag{50}$$

Moreover, since assertion (c) of Theorem 36 also states that $\mathrm{ran}(1_{e_{\mathrm{B}}}(H_{\mathrm{B}})) \subset \mathfrak{h} \oplus 0$ and $\mathrm{ran}(1_{-e_{\mathrm{B}}}(H_{\mathrm{B}})) \subset 0 \oplus \mathfrak{h}$, and since the density of the initial state $S_0 \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ has the block diagonal form given in (22) of Definition 8, the terms in (50) for different energies vanish, and, hence, the time dependence drops out of (50). This leads to

$$\Omega_n^{\mathrm{pp}}(t) = \Omega_n^{\mathrm{pp}} \tag{51}$$

for all $t \in \mathbb{R}$, where the r.h.s. is given in Definition 16. Finally, since the Pfaffian $\mathrm{pf} : \mathbb{C}_a^{2n\times 2n} \to \mathbb{C}$ is a continuous mapping, we get

$$\begin{aligned} \mathrm{P}(n) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, \omega_0\big(\tau_{\mathrm{B}}^t(A_n)\big) \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, \mathrm{pf}\big(\Omega_n(t)\big) \\ &= \mathrm{pf}\big(\Omega_n^{\mathrm{aa}} + \Omega_n^{\mathrm{pp}}\big). \end{aligned} \tag{52}$$

Note that we didn't make use of the specific structure of the form factors F_i. Hence, since the algebra of observables $\mathfrak{A}$ is generated by the operators $B(F)$ for $F \in \mathfrak{h}^{\oplus 2}$, and since the mapping $\mathfrak{A} \ni A \to \omega_0(\tau_{\mathrm{B}}^t(A)) \in \mathbb{C}$ is continuous uniformly in $t \in \mathbb{R}$, the relation (52) defines the unique NESS $\omega_{\mathrm{B}} \in \mathcal{Q}(\mathfrak{A})$. The form (40) of the density S_{B} follows from (37) and (38). Moreover, due to the completeness of the wave operator and Remark 6, S_{B} has the defining properties of a density given in Definition 32 of Appendix A. This is the assertion. □

3. NESS correlation asymptotics

In order to approach the asymptotic behavior of the NESS EFP from Theorem 17, we start off by studying more closely the two pieces of the asymptotic correlation matrix given in Definition 16. For this purpose, besides the position space, we will use the momentum space and the energy space defined before Theorem 35 of Appendix A and in Definition 39 of Appendix C, respectively.

Lemma 18 *(ac-structure). The asymptotic correlation matrix $\Omega_n^{\mathrm{aa}} \in \mathbb{C}_a^{2n\times 2n}$ has the decomposition*

$$\Omega_n^{\mathrm{aa}} = \sum_{\sigma=\pm} \Omega_n^{\mathrm{aa},\sigma}, \tag{53}$$

where the matrices $\Omega_n^{\mathrm{aa},\pm} \in \mathbb{C}_a^{2n\times 2n}$ are defined, for all $i, j = 1, \dots, n$, by $\Omega_{2i-1\,2j-1}^{\mathrm{aa},\pm} := \Omega_{2i\,2j}^{\mathrm{aa},\pm} := \Omega_{2i\,2j-1}^{\mathrm{aa},-} := \Omega_{2i-1\,2j}^{\mathrm{aa},+} = 0$, and the nonvanishing entries are given by

$$\Omega_{2i-1\,2j}^{\mathrm{aa},-} := \big(w_-(h,h_{\mathrm{B}})\delta_{i+x_0-1}, s_- w_-(h,h_{\mathrm{B}})\delta_{j+x_0-1}\big), \tag{54}$$

$$\Omega_{2i\,2j-1}^{\mathrm{aa},+} := \big(w_+(h,h_{\mathrm{B}})\delta_{i+x_0-1}, s_+ w_+(h,h_{\mathrm{B}})\delta_{j+x_0-1}\big). \tag{55}$$

Here, $s_\pm \in \mathcal{L}(\mathfrak{h})$ are the density components of the translation invariant XY NESS given in Theorem 35 *of Appendix* A*, and the wave operators $w_\pm(h,h_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h})$ are defined by*

$$w_\pm(h,h_{\mathrm{B}}) := \operatorname*{s\text{-}lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}th}\mathrm{e}^{-\mathrm{i}th_{\mathrm{B}}} 1_{\mathrm{ac}}(h_{\mathrm{B}}), \tag{56}$$

where, from now on, all the spectral projections of h_{B} are denoted as the ones for H_{B} given in the Definitions 15 *and* 16 *with H_{B} replaced by h_{B}.*

Proof. In order to rewrite the absolutely continuous contribution to the asymptotic correlation matrix from Definition 16, we want to take advantage of the fact that the operator

$$S = W^*(H_0,H)S_0W(H_0,H) \tag{57}$$

is the known density of the translation invariant XY NESS (i.e. the NESS for $\kappa = 0$) given in Theorem 35 of Appendix A. For this purpose, we use the chain rule $W(H_0,H_{\mathrm{B}}) = W(H_0,H)W(H,H_{\mathrm{B}})$ which is permissible since $H - H_0, H_{\mathrm{B}} - H \in \mathcal{L}^0(\mathcal{H})$ (the wave operators $W(H_0,H), W(H,H_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ are defined as in (39) with the appropriate replacements). Hence, the absolutely continuous contribution becomes

$$\Omega_{ij}^{\mathrm{aa}} = \big(W(H,H_{\mathrm{B}})JF_i, SW(H,H_{\mathrm{B}})F_j\big). \tag{58}$$

Using the block diagonal structure of the operators $H, H_{\mathrm{B}}, S \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ and plugging the explicit form of the form factors from Definition 11 into (58) leads to the assertion. □

In order to evaluate the nonvanishing entries (54) and (55) from Lemma 18, we determine the action of the wave operators on completely localized wave functions. The main computations are carried out in Appendix C.

Proposition 19 *(Wave operators). Let $x \in \mathbb{Z}$ be any site. Then, in momentum space $\hat{\mathfrak{h}} = L^2(\mathbb{T})$, the wave operators $w_\pm(h,h_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h})$ act on the completely localized wave function $\delta_x \in \mathfrak{h}$ as*

$$\hat{w}_\pm(h,h_{\mathrm{B}})\mathrm{e}_x(k) = \mathrm{e}_x(k) \mp \mathrm{i}\kappa \frac{\mathrm{e}_{|x|}(\mp|k|)}{\sin(|k|) \pm \mathrm{i}\kappa}, \tag{59}$$

where we set $\mathrm{e}_x(k) := \hat{\delta}_x(k) = \mathrm{e}^{\mathrm{i}kx}$ for all $x \in \mathbb{Z}$ and for all $k \in (-\pi, \pi]$.

Proof. Plugging (141) into (131) in Appendix C and applying $\tilde{\mathfrak{f}} : \hat{\mathfrak{h}} \to \tilde{\mathfrak{h}}$ from (124) to e_x, the action of the wave operator is expressed in energy space $\tilde{\mathfrak{h}}$ as

$$\tilde{w}_\pm(h, h_\mathrm{B})\tilde{\delta}_x(e) = (2\pi)^{-1/2}\big(1-e^2\big)^{-1/4}\bigg(\big[\big(e+\mathrm{i}\sqrt{1-e^2}\big)^x, \big(e-\mathrm{i}\sqrt{1-e^2}\big)^x\big] \mp \mathrm{i}\kappa\frac{(e\mp\mathrm{i}\sqrt{1-e^2})^{|x|}}{\sqrt{1-e^2}\pm\mathrm{i}\kappa}[1,1]\bigg). \tag{60}$$

Applying $\tilde{\mathfrak{f}}^* : \tilde{\mathfrak{h}} \to \hat{\mathfrak{h}}$ from (126) in Appendix C to (60) yields the assertion. □

Remark 20. The action (59) relates to the action of the wave operator for the one-center δ-interaction on the continuous line by replacing $\sin(|k|)$ by $|k|$, see, for example, Albeverio et al. [2].

We next turn to the pure point contribution.

Lemma 21 (pp-*structure*). *The asymptotic correlation matrix $\Omega_n^{\mathrm{pp}} \in \mathbb{C}_a^{2n\times 2n}$ has the decomposition*

$$\Omega_n^{\mathrm{pp}} = \sum_{\sigma=\pm}\Omega_n^{\mathrm{pp},\sigma}, \tag{61}$$

where the matrices $\Omega_n^{\mathrm{pp},\pm} \in \mathbb{C}_a^{2n\times 2n}$ are defined, for all $i, j = 1,\dots,n$, by $\Omega^{\mathrm{pp},\pm}_{2i-1\,2j-1} := \Omega^{\mathrm{pp},\pm}_{2i\,2j} := \Omega^{\mathrm{pp},-}_{2i\,2j-1} := \Omega^{\mathrm{pp},+}_{2i-1\,2j} := 0$, and the nonvanishing entries are given by

$$\Omega^{\mathrm{pp},-}_{2i-1\,2j} := \big(1_{\mathrm{pp}}(h_\mathrm{B})\delta_{i+x_0-1}, s_{0,-}1_{\mathrm{pp}}(h_\mathrm{B})\delta_{j+x_0-1}\big), \tag{62}$$

$$\Omega^{\mathrm{pp},+}_{2i\,2j-1} := \big(1_{\mathrm{pp}}(h_\mathrm{B})\delta_{i+x_0-1}, s_{0,+}1_{\mathrm{pp}}(h_\mathrm{B})\delta_{j+x_0-1}\big). \tag{63}$$

Here, $s_{0,\pm} \in \mathcal{L}(\mathfrak{h})$ are the density components of the initial state given in Definition 8.

Proof. Using the block diagonal structures of $H_\mathrm{B}, S_0 \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ and plugging the explicit form of the form factors from Definition 11 into (38) leads to the assertion. □

In order to evaluate the nonvanishing entries (62) and (63) from Lemma 21, we determine the form of the projections onto the pure point subspaces of the magnetic Hamiltonian. A summary of its spectral properties is given in Appendix B.

Lemma 22 *(Pure point projection). The projection onto the pure point subspace of the magnetic Hamiltonian h_B satisfies*

$$\dim\big(\mathrm{ran}\big(1_{\mathrm{pp}}(h_\mathrm{B})\big)\big) = 1, \tag{64}$$

and its range is spanned by an exponentially localized eigenfunction $f_\mathrm{B} \in \mathfrak{h}$ of $h_\mathrm{B} \in \mathcal{L}(\mathfrak{h})$ with eigenvalue $e_\mathrm{B} > 1$.

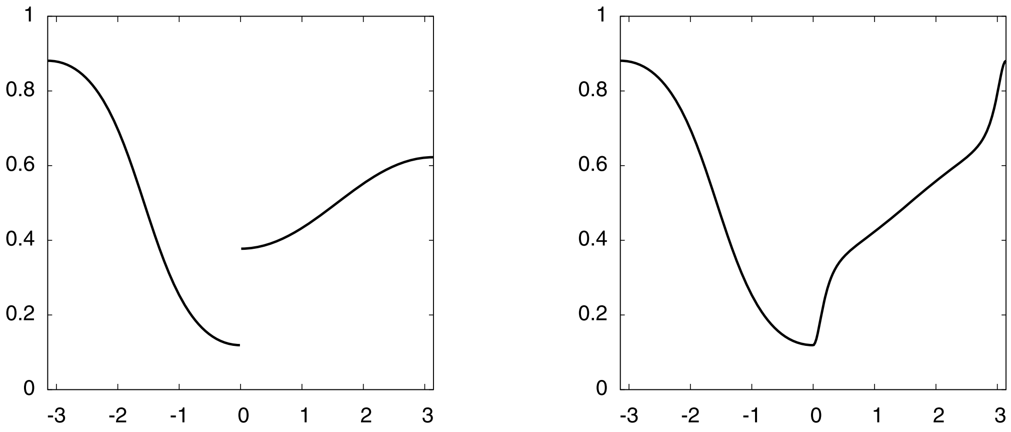

Fig. 2. The symbol $a(k) > 0$ with $k \in (-\pi, \pi]$ for $\beta_R = 2$, $\beta_L = \frac{1}{2}$, and $\kappa = 0$ to the left and $\kappa = \frac{1}{5}$ to the right. The nonvanishing magnetic field regularizes the symbol.

Proof. See Theorem 36 in Appendix B. $\square$

Collecting the properties of the absolutely continuous and the pure point contributions to the asymptotic correlation matrix from Lemma 18 to Lemma 22, we get the following structural assertion.

For the ingredients from Toeplitz theory referred to in the remainder of the present section, see, for example, Böttcher and Silbermann [14,15]. Moreover, we denote by $\chi_A : \mathbb{R} \to \{0, 1\}$ the characteristic function of the set $A \subset \mathbb{R}$.

Proposition 23 *(Determinantal structure). The NESS EFP is the determinant of the finite section of a Toeplitz operator, a Hankel operator, and an operator of finite rank. The symbol $a \in L^\infty(\mathbb{T})$ of the Toeplitz operator reads*

$$a = \varphi_{\mathrm{B}} \hat{s}_{-,L} + (1 - \varphi_{\mathrm{B}}) \hat{s}_{-,R}, \tag{65}$$

where the functions $\varphi_{\mathrm{B}}, \hat{s}_{\pm,\alpha} \in L^\infty(\mathbb{T})$ with $\alpha = L, R$, are defined, for $k \in (-\pi, \pi]$, by

$$\hat{s}_{\pm,\alpha}(k) := \frac{1}{2}\big(1 \pm \tanh\big[\tfrac{1}{2}\beta_\alpha \cos(k)\big]\big), \tag{66}$$

$$\varphi_{\mathrm{B}}(k) := \chi_{[0,\pi]}(k) \frac{\sin^2(k)}{\sin^2(k) + \kappa^2}, \tag{67}$$

see Fig. 2. Moreover, the symbol of the Hankel operator is smooth.

Remark 24. In the limit $\kappa \to 0$, we recover the symbol derived in Aschbacher [7,9] for the translation invariant case.

Remark 25. Note that for nonvanishing coupling, the characteristic function in (67) is smoothed out. This will play an essential role in the asymptotic analysis of the corresponding Toeplitz determinant.

Proof. The total asymptotic correlation matrix, defined by $\Omega_n := \Omega_n^{\mathrm{aa}} + \Omega_n^{\mathrm{pp}} \in \mathbb{C}_a^{2n\times 2n}$, has the 2×2 block substructure $\Omega_n = [A_{ij}]_{i,j=1,\dots,n}$, where the matrices $A_{ij} \in \mathbb{C}^{2\times 2}$ are defined, for $i, j = 1, \dots, n$, by

$$A_{ij} := \begin{cases} \begin{bmatrix} 0 & b_{ij} \\ c_{ij} & 0 \end{bmatrix}, & \text{if } i < j, \\ \begin{bmatrix} 0 & b_{ii} \\ -b_{ii} & 0 \end{bmatrix}, & \text{if } i = j, \\ -A_{ji}^{\mathrm{t}}, & \text{if } i > j, \end{cases} \tag{68}$$

and the entries are given by

$$b_{ij} := \Omega^{\mathrm{aa},-}_{2i-1\,2j} + \Omega^{\mathrm{pp},-}_{2i-1\,2j}, \quad \text{if } i \leqslant j, \tag{69}$$

$$c_{ij} := \Omega^{\mathrm{aa},+}_{2i\,2j-1} + \Omega^{\mathrm{pp},+}_{2i\,2j-1}, \quad \text{if } i < j. \tag{70}$$

In order to rewrite the argument of the Pfaffian in $\mathrm{P}(n) = \mathrm{pf}(\Omega_n)$ from Theorem 17 in a form more suited for the subsequent analysis, we want to apply a similarity transformation to Ω_n. To this end, for any $i, j = 1, \dots, 2n$ with $i < j$, we denote by $R^{[ij]} \in O(2n)$ the elementary matrix whose left multiplication with any matrix $A \in \mathbb{C}^{2n\times 2n}$ exchanges the ith and jth row of A (where $O(n)$ stands for the orthogonal matrices in $\mathbb{R}^{n\times n}$). Then, using the matrix $R \in O(2n)$ defined by $R := \prod_{k=1}^{n-1}\prod_{l=0}^{k-1} R^{[2(n-k)+l,2(n-k)+l+1]}$, we can transform Ω_n into off-diagonal block form,

$$R^{\mathrm{t}}\Omega_n R = \begin{bmatrix} 0 & \Theta_n \\ -\Theta_n^{\mathrm{t}} & 0 \end{bmatrix}, \tag{71}$$

where the matrix $\Theta_n \in \mathbb{C}^{n\times n}$, called the reduced correlation matrix, is defined by its entries $\Theta_{ij} := \theta_{ij}$, and, for all $i, j \in \mathbb{N}$, the numbers θ_{ij} are given by

$$\theta_{ij} := \begin{cases} b_{ij}, & \text{if } i \leqslant j, \\ -c_{ji}, & \text{if } i > j. \end{cases} \tag{72}$$

Hence, using assertions (a) and (b) from Lemma 34 of Appendix A, we get

$$\begin{aligned} \mathrm{P}(n) &= \mathrm{pf}(\Omega_n) \\ &= (-1)^{\frac{n(n-1)}{2}}\,\mathrm{pf}\left(\begin{bmatrix} 0 & \Theta_n \\ -\Theta_n^{\mathrm{t}} & 0 \end{bmatrix}\right) \\ &= \det(\Theta_n). \end{aligned} \tag{73}$$

Let us next analyze the structure of Θ_n. In order to do so, we subdivide the discussion into the following two cases w.r.t. the starting site $x_0 \in \mathbb{Z}$ of the EFP string.

Case 1: $x_0 \geqslant 0$.

With the help of Lemma 44 of Appendix D, we make the decomposition $\Theta_n = \Theta_{T,n} + \Theta_{H,n}$, where the matrix $\Theta_{T,n} \in \mathbb{C}^{n\times n}$ has the entries $\Theta_{T,ij} := \theta_{T,ij}$ given by $\theta_{T,ij} := b_{T,ij}$ if $i \leqslant j$, and $\theta_{T,ij} := -c_{T,ji}$ if $i > j$. Here, for all $i, j \in \mathbb{N}$, we define

$$b_{T,ij} := (\mathrm{e}_{i-j}, \hat{s}_-\mathrm{e}_0) + (\mathrm{e}_{i-j}, a_-\mathrm{e}_0), \quad \text{if } i \leqslant j, \tag{74}$$

$$-c_{T,ji} := (\mathrm{e}_{j-i}, \hat{s}_+\mathrm{e}_0) + (\mathrm{e}_{i-j}, a_+\mathrm{e}_0), \quad \text{if } i > j, \tag{75}$$

where $\hat{s}_\pm, a_\pm \in L^\infty(\mathbb{T})$ are given in Theorem 35 of Appendix A and Lemma 44 of Appendix D, respectively. Similarly, the matrix $\Theta_{H,n} \in \mathbb{C}^{n\times n}$ has the entries $\Theta_{H,ij} := \theta_{H,ij}$ given by $\theta_{H,ij} := b_{H,ij}$ if $i \leqslant j$, and $\theta_{H,ij} := -c_{H,ji}$ if $i > j$. Here, for all $i, j \in \mathbb{N}$, we define

$$b_{H,ij} := (e_{i+j}, b_- \mathrm{e}_0), \quad \text{if } i \leqslant j, \tag{76}$$

$$-c_{H,ji} := (e_{i+j}, b_+ \mathrm{e}_0), \quad \text{if } i > j, \tag{77}$$

where $b_\pm \in L^\infty(\mathbb{T})$ are given in Lemma 44 of Appendix D. Using (66) and (67) in (74), and $\hat{s}_{+,R} - \hat{s}_{+,L} = -(\hat{s}_{-,R} - \hat{s}_{-,L})$ and $(\mathrm{e}_{i-j}, \hat{s}_+ \mathrm{e}_0) = -(\mathrm{e}_{j-i}, \hat{s}_- \mathrm{e}_0)$ in (75), where the last identity is due to Definition 32 of the density in Appendix A, we have $b_{T,ij} = (\mathrm{e}_{i-j}, a\mathrm{e}_0)$ for $i \leqslant j$ and $-c_{T,ji} = (\mathrm{e}_{i-j}, a\mathrm{e}_0)$ for $i > j$. Hence, $\Theta_{T,n}$ is the finite section of the Toeplitz operator $T[a] \in \mathcal{L}(\ell^2(\mathbb{N}))$ generated by the symbol $a \in L^\infty(\mathbb{T})$, i.e.

$$\Theta_{T,n} = T_n[a]. \tag{78}$$

Moreover, as for (76) and (77), using (119) in Remark 37 of Appendix B, we find that $b_{H,ij} = (\mathrm{e}_{i+j-1}, b\mathrm{e}_0)$ for $i \leqslant j$ and $-c_{H,ji} = (\mathrm{e}_{i+j-1}, b\mathrm{e}_0)$ for $i > j$, where the function $b \in C^\infty(\mathbb{T})$ is defined by

$$b(k) := \mathrm{i}\kappa \frac{\mathrm{e}^{-\mathrm{i}k(2x_0-1)}}{\sin(k) + \mathrm{i}\kappa} \left[\frac{(f_B, s_{0,-} f_B)}{e_{\mathrm{B}}^2} - \hat{s}_{-,R}(k) \right]. \tag{79}$$

Hence, $\Theta_{H,n}$ is the finite section of the Hankel operator $H[b] \in \mathcal{L}(\ell^2(\mathbb{N}))$ generated by the symbol $b \in C^\infty(\mathbb{T})$, i.e.

$$\Theta_{H,n} = H_n[b]. \tag{80}$$

Therefore, it follows from (73) that the NESS EFP is the determinant of the finite section of the sum of a Toeplitz and a Hankel operator,

$$\mathrm{P}(n) = \det\bigl(T_n[a] + H_n[b]\bigr), \tag{81}$$

where, in this case, the finite rank operator from the formulation of the assertion vanishes.

Let us now turn to the case where the EFP string starts to the left of the origin.

Case 2: $x_0 < 0$.

For $n \gg 1 + n_0$, where we set $n_0 := -x_0$, we again have from Lemma 44, that, for all $i, j = 1, \ldots, n - n_0$,

$$\Theta_{n,i+n_0\, j+n_0} = T_{n-n_0,ij}[a] + H_{n-n_0,ij}[c], \tag{82}$$

where we set $c := \mathrm{e}_{-(2n_0+1)} b$. Defining the operator $\Theta : \mathbb{C}^{n_0} \oplus \ell^2(\mathbb{N}) \to \mathbb{C}^{n_0} \oplus \ell^2(\mathbb{N})$ on $[\xi, f] \in \mathbb{C}^{n_0} \oplus \ell^2(\mathbb{N})$ by the matrix multiplication with the infinite matrix θ_{ij} from (72), we have

$$M := \Theta - 0 \oplus \bigl(T[a] + H[c]\bigr) \in \mathcal{L}^0\bigl(\mathbb{C}^{n_0} \oplus \ell^2(\mathbb{N})\bigr). \tag{83}$$

Since the reduced correlation matrix satisfies $\Theta_n = R_n \Theta R_n \restriction_{\mathrm{ran}(R_n)}$ with $R_n := 1 \oplus P_n \in \mathcal{L}(\mathbb{C}^{n_0} \oplus \ell^2(\mathbb{N}))$, it follows from (73) that

$$\mathrm{P}(n) = \det\bigl(0 \oplus \bigl(T_{n-n_0}[a] + H_{n-n_0}[c]\bigr) + M_n\bigr), \tag{84}$$

where $M_n := R_n M R_n \restriction_{\mathrm{ran}(R_n)}$. Hence, we arrive at the assertion. □

We are now ready to formulate our main result on the behavior of the NESS EFP for large string lengths.

Theorem 26 *(Exponential decay). For $n \to \infty$, the NESS EFP $\mathrm{P} : \mathbb{N} \to [0, 1]$ has an exponentially decaying bound,*

$$\mathrm{P}(n) = \mathcal{O}\bigl(\mathrm{e}^{-\Gamma n}\bigr). \tag{85}$$

The decay rate $\Gamma := \Gamma_R + \Gamma_{\mathrm{B}} > 0$ contains the two parts

$$\Gamma_R := -\frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \log\bigl[\hat{s}_{-,R}(k)\bigr], \tag{86}$$

$$\Gamma_{\mathrm{B}} := -\frac{1}{2} \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \log\bigl[\sigma_{\mathrm{B}}(k)\hat{s}_{-,L}(k) + \bigl(1 - \sigma_{\mathrm{B}}(k)\bigr)\hat{s}_{-,R}(k)\bigr], \tag{87}$$

where the function $\sigma_{\mathrm{B}} \in L^\infty(\mathbb{T})$ is given by

$$\sigma_{\mathrm{B}}(k) := \frac{\sin^2(k)}{\sin^2(k) + \kappa^2}. \tag{88}$$

Remark 27. Note that Theorem 26 holds for any coupling strength. In the small coupling limit, we recover the exact decay rate from Aschbacher [9], and the bound derived for general quasifree systems in Aschbacher [7].

Remark 28. As can be seen in (87), the NESS EFP decay rate displays a left mover – right mover structure. It is composed of a left mover carrying temperature β_R and coming from $+\infty$, a left mover carrying temperature β_R having been reflected at the perturbation at the origin, and a right mover carrying temperature β_L having been transmitted through the origin. This left mover–right mover structure has already been observed in translation invariant systems for several types of correlation functions, see Aschbacher [8,9] and Aschbacher and Barbaroux [10].

Remark 29. Defining Γ_L analogously to (86), we have the rewritings, for $\alpha = L, R$,

$$\Gamma_\alpha = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}k}{2\pi} \log\bigl[\tfrac{1}{4}\bigl(1 - \tanh^2\bigl[\tfrac{1}{2}\beta_\alpha \cos(k)\bigr]\bigr)\bigr], \tag{89}$$

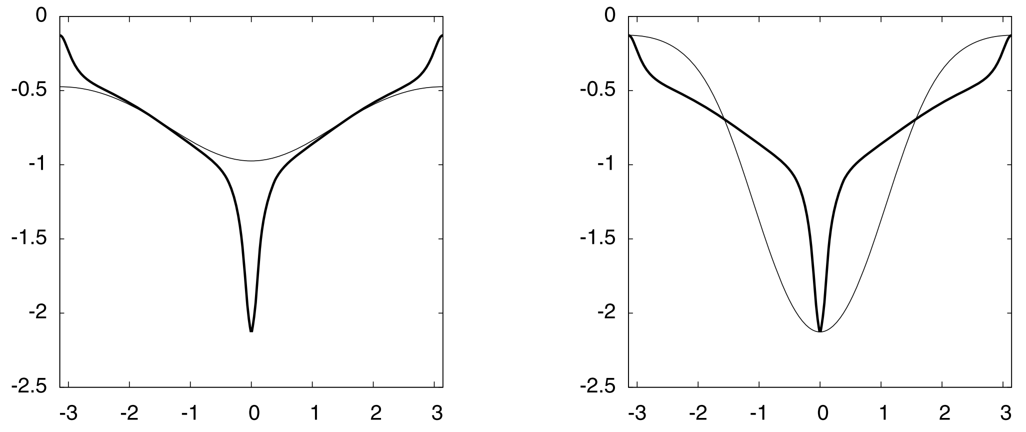

Fig. 3. For $\beta_R = 2$ and $\beta_L = \frac{1}{2}$, the integrand of Γ_L (left, thin line) and Γ_R (right, thin line) compared to Γ_{B} with $\kappa = \frac{1}{5}$ (left and right, thick line).

$$\Gamma_{\mathrm{B}} = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}k}{2\pi}\, \log\Big[\tfrac{1}{4}\Big(1 - \Big[\big(1-\sigma_{\mathrm{B}}(k)\big)\tanh\big[\tfrac{1}{2}\beta_R\cos(k)\big]$$

$$+\,\sigma_{\mathrm{B}}(k)\tanh\big[\tfrac{1}{2}\beta_L\cos(k)\big]\Big]^2\Big)\Big]. \tag{90}$$

From (89) and (90), it immediately follows that, if the system is truly out of equilibrium, i.e. if $\delta > 0$, the decay rates are ordered as

$$0 < \Gamma_L < \Gamma_{\mathrm{B}} < \Gamma_R, \tag{91}$$

see also Fig. 3.

Remark 30. It follows from assertion (a) in Proposition 45 of Appendix E that the nonvanishing coupling regularizes the underlying Toeplitz theory in the sense that the symbol which determines the decay rate is smoother than in the case $\kappa = 0$, see Fig. 2. Namely, the latter case requires Fisher–Hartwig theory and, if $\delta > 0$, leads to a strictly positive power law subleading order as given in Aschbacher [9].

Proof. Since the Toeplitz symbol $a \in L^\infty(\mathbb{T})$ from (65) is real-valued, we can make use of the Hartman–Wintner theorem in order to control the spectrum of the selfadjoint Toeplitz operator $T[a] \in \mathcal{L}(\ell^2(\mathbb{N}))$. Moreover, due to Proposition 45 of Appendix E, we have

$$a \in C(\mathbb{T}), \tag{92}$$

and, hence, $\mathrm{spec}(T[a]) = \mathrm{ran}(a) = [\hat{s}_{-,R}(0), \hat{s}_{+,R}(0)]$, where $0 < \hat{s}_{-,R}(0) < \hat{s}_{+,R}(0) < 1$ in the temperature range $0 < \beta_L \leqslant \beta_R < \infty$. Therefore, $T[a] \in \mathcal{L}(\ell^2(\mathbb{N}))$ is invertible,

$$0 \notin \mathrm{spec}\big(T[a]\big), \tag{93}$$

and the spectrum is independent of the coupling strength. Moreover, since (92) and (93) hold, the Gohberg–Feldman theorem implies that the sequence $\{T_n[a]\}_{n\in\mathbb{N}}$ is stable,

$$\limsup_{n\to\infty}\|T_n^{-1}[a]\| < \infty. \tag{94}$$

For the following analysis, as in the proof of Proposition 23, we discuss the cases $x_0 \geqslant 0$ and $x_0 < 0$ separately. For convenience of exposition, we start with the second case.

Case 2: $x_0 < 0$.

Since we want to analyze the asymptotic behavior for large n with the help of Szegő's strong limit theorem, we write, using (94),

$$\frac{\mathrm{P}(n)}{G(a)^n} = \frac{\mathrm{P}(n)}{\det(T_{n-n_0}[a])}\frac{\det(T_{n-n_0}[a])}{\det(T_n[a])}\frac{\det(T_n[a])}{G(a)^n}, \tag{95}$$

where $G(a)$ is the exponential of the 0th Fourier coefficient of $\log(a)$, and $n_0 := -x_0$ as before. Due to (84) and (94), the first factor on the r.h.s. of (95) can be written as

$$\frac{\mathrm{P}(n)}{\det(T_{n-n_0}[a])} = \det\bigl(1 + 1 \oplus T_{n-n_0}^{-1}[a]\bigl((-1) \oplus H_{n-n_0}[c] + M_n\bigr)\bigr). \tag{96}$$

Moreover, since we know from Proposition 23 that $c \in C^\infty(\mathbb{T})$, we also have

$$c \in L^\infty(\mathbb{T}) \cap B_1^1(\mathbb{T}), \tag{97}$$

where $B_p^\alpha(\mathbb{T})$ are the usual Besov spaces. Therefore, Peller's theorem allows us to conclude that ($\mathcal{L}^1(\mathcal{H})$ are the trace class operators on the Hilbert space $\mathcal{H}$)

$$H[b] \in \mathcal{L}^1\bigl(\ell^2(\mathbb{N})\bigr). \tag{98}$$

Due to (83), (84), and (98), the r.h.s. of (96) converges to the constant $K(a,b) := \det(1 + 1 \oplus T^{-1}[a](-1 \oplus H[b] + M))$. In order to treat the second factor on the r.h.s. of (95), we apply Szegő's first limit theorem which is applicable due to (92) and (93). Hence, if we factorize the quotient as

$$\frac{\det(T_{n-n_0}[a])}{\det(T_n[a])} = \prod_{i=1}^{n_0}\frac{\det(T_{n-i}[a])}{\det(T_{n+1-i}[a])}, \tag{99}$$

each factor on the r.h.s. of (99) converges to $1/G(a)$. In order to treat the third factor on the r.h.s. of (95), we make use of Szegő's strong limit theorem. This theorem states that, since $a(t) > 0$ for all $t \in \mathbb{T}$ s.t. $\mathrm{ind}(a) = 0$, and since

$$a \in W(\mathbb{T}) \cap B_2^{1/2}(\mathbb{T}), \tag{100}$$

which follows from $a \in C^1(\mathbb{T}) \cap PC^\infty(\mathbb{T})$ in Proposition 45 of Appendix E ($W(\mathbb{T})$ is the Wiener algebra and $PC^\infty(\mathbb{T})$ are the piecewise smooth functions), the quotient converges to a constant usually denoted by $E(a)$. Plugging the foregoing three limits into the r.h.s. of (95), we get

$$\lim_{n\to\infty}\frac{\mathrm{P}(n)}{G(a)^n} = K(a,b)E(a)G(a)^{x_0}. \tag{101}$$

Hence, in order to determine an exponential bound on the asymptotic decay, we are left with the computation of the constant $G(a)$. Decomposing the integral in the 0th Fourier coefficient of $\log(a)$ w.r.t. positive and negative momenta and using the fact that $\hat{s}_{-,\alpha}$ for $\alpha = L, R$ and σ_{B} are even functions in $k \in (-\pi, \pi]$, we arrive at the expressions (86) and (87) for the decay rate of the bound on the exponential decay.

The case where the EFP string starts at nonnegative sites is simpler and is treated analogously as follows.

Case 1: $x_0 \geqslant 0$.

Writing (81) as in (95), where, in this case, the second factor is absent, we can proceed as for Case 2. In particular, the determinant of the Toeplitz contribution can again be separated due to (94), (98) holds for the symbol b satisfying (97), and the constant now reads $K(a, c) = \det(1 + T^{-1}[a]H[c])$. Finally, the last factor in (101) is absent. Hence, we arrive at the assertion. $\square$

Remark 31. The study of the present problem for the anisotropic XY model, i.e. for the case where $\gamma \neq 0$ in (19) of Remark 7, is more complicated. Not only the Pfaffian structure of the correlation cannot be preserved in the present form, but also one has to cope with Toeplitz theory for operators with nonscalar symbols. We will study this set of problems for general quasifree systems elsewhere.

Acknowledgments

We would like to thank the editor and the referee for their constructive remarks. Moreover, the support provided by the German Research Foundation (DFG) is gratefully acknowledged.

Appendix A. Fermionic quasifree states

Let $\mathfrak{A}$ be the selfdual CAR algebra from Definition 1. We denote by $\mathcal{E}(\mathfrak{A})$ the set of states, i.e. the normalized positive linear functionals on the C^* algebra $\mathfrak{A}$.

Definition 32 *(Density).* The density of a state $\omega \in \mathcal{E}(\mathfrak{A})$ is defined to be the operator $S \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ with $0 \leqslant S^* = S \leqslant 1$ and $JSJ = 1 - S$ satisfying, for all $F, G \in \mathfrak{h}^{\oplus 2}$,

$$\omega\big(B^*(F)B(G)\big) = (F, SG). \tag{102}$$

An important class are the quasifree states.

Definition 33 *(Quasifree state).* A state $\omega \in \mathcal{E}(\mathfrak{A})$ is called quasifree if it vanishes on the odd polynomials in the generators and if it is a Pfaffian on the even polynomials in the generators, i.e. if, for all $F_1, \ldots, F_{2n} \in \mathfrak{h}^{\oplus 2}$ and for any $n \in \mathbb{N}$, we have

$$\omega\big(B(F_1)\ldots B(F_{2n})\big) = \mathrm{pf}(\Omega_n), \tag{103}$$

where the skew-symmetric matrix $\Omega_n \in \mathbb{C}_a^{2n\times 2n} = \{A \in \mathbb{C}^{2n\times 2n} \mid A^{\mathrm{t}} = -A\}$ is defined, for $i, j = 1, \ldots, 2n$, by

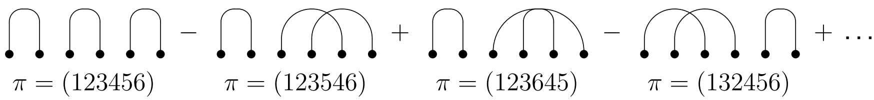

Fig. 4. Some of the pairings for $n = 3$. The total number of intersections I relates to the signature as $\operatorname{sign}(\pi) = (-1)^I$.

$$\Omega_{ij} := \begin{cases} \omega(B(F_i)B(F_j)), & \text{if } i < j, \\ 0, & \text{if } i = j, \\ -\omega(B(F_j)B(F_i)), & \text{if } i > j. \end{cases} \tag{104}$$

Here, the Pfaffian $\mathrm{pf} : \mathbb{C}_a^{2n\times 2n} \to \mathbb{C}$ is given by

$$\mathrm{pf}(A) := \sum_{\pi} \operatorname{sign}(\pi) \prod_{j=1}^{n} A_{\pi(2j-1),\pi(2j)}, \tag{105}$$

where the sum is running over all pairings of the set $\{1, 2, \dots, 2n\}$, i.e. over all the $(2n)!/(2^n n!)$ permutations π in the permutation group of $2n$ elements which satisfy $\pi(2j-1) < \pi(2j+1)$ and $\pi(2j-1) < \pi(2j)$, see Fig. 4. The set of quasifree states is denoted by $\mathcal{Q}(\mathfrak{A})$.

The following lemma has been used in Section 3.

Lemma 34 *(Pfaffian). The Pfaffian has the following properties.*

(a) *Let $X, Y \in \mathbb{C}^{2n\times 2n}$ with $Y^{\mathrm{t}} = -Y$. Then,*

$$\mathrm{pf}\big(XYX^{\mathrm{t}}\big) = \det(X)\,\mathrm{pf}(Y). \tag{106}$$

(b) *Let $X \in \mathbb{C}^{n\times n}$. Then,*

$$\mathrm{pf}\left(\begin{bmatrix} 0 & X \\ -X^{\mathrm{t}} & 0 \end{bmatrix}\right) = (-1)^{\frac{n(n-1)}{2}} \det(X). \tag{107}$$

Proof. See, for example, Stembridge [25]. □

Next, we state the properties of the NESS for the translation invariant case $\kappa = 0$, the so-called XY NESS. To this end, let $\mathfrak{f} : \mathfrak{h} \to \hat{\mathfrak{h}} := L^2(\mathbb{T})$ (with unit circle $\mathbb{T}$) be the Fourier transformation defined with the sign convention $\hat{f}(k) := (\mathfrak{f} f)(k) := \sum_{x\in\mathbb{Z}} f(x)\mathrm{e}^{\mathrm{i}kx}$. Moreover, for any $a \in \mathcal{L}(\mathfrak{h})$, we use the notation $\hat{a} := \mathfrak{f} a \mathfrak{f}^*$. We then have the following.

Theorem 35 *(XY NESS). There exists a unique quasifree NESS $\omega \in \mathcal{Q}(\mathfrak{A})$ associated with the C^*-dynamical system $(\mathfrak{A}, \tau)$ and the initial state $\omega_0 \in \mathcal{Q}(\mathfrak{A})$ whose density $S \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ has the form*

$$S = s_- \oplus s_+, \tag{108}$$

where the operators $s_\pm \in \mathcal{L}(\mathfrak{h})$ act in momentum space $\hat{\mathfrak{h}}$ as multiplication by

$$\hat{s}_\pm(k) := \frac{1}{2}\big(1 \pm \varrho_\pm(k)\big), \tag{109}$$

and the functions $\varrho_\pm : \mathbb{T} \to (-1, 1)$ *are defined by*

$$\varrho_\pm(k) := \tanh\big[\tfrac{1}{2}\big(\beta \pm \operatorname{sign}\big(\sin(k)\big)\delta\big)\cos(k)\big]. \tag{110}$$

Proof. See Aschbacher and Pillet [13]. □

Appendix B. Magnetic Hamiltonian

In this section, we summarize the spectral theory of $H_\mathrm{B} \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ needed above. To this end, we denote by $\mathrm{spec}_\mathrm{sc}(A)$, $\mathrm{spec}_\mathrm{ac}(A)$, and $\mathrm{spec}_\mathrm{pp}(A)$ the singular continuous, the absolutely continuous, and the pure point spectrum of the operator A, respectively.

Theorem 36 *(Magnetic spectrum). The magnetic Hamiltonian* $H_\mathrm{B} \in \mathcal{L}(\mathfrak{h}^{\oplus 2})$ *has the following properties.*

(a) $\mathrm{spec}_\mathrm{sc}(H_\mathrm{B}) = \emptyset$;
(b) $\mathrm{spec}_\mathrm{ac}(H_\mathrm{B}) = [-1, 1]$;
(c) $\mathrm{spec}_\mathrm{pp}(H_\mathrm{B}) = \{\pm e_\mathrm{B}\}$ *with* $e_\mathrm{B} > 1$.
The eigenvalues $\pm e_\mathrm{B}$ *are simple, and*

$$e_\mathrm{B} = \sqrt{1 + \kappa^2}. \tag{111}$$

The normalized eigenfunction of H_B *with eigenvalue* e_B *is given by* $f_\mathrm{B} \oplus 0 \in \mathfrak{h}^{\oplus 2}$*, where* f_B *is exponentially localized, i.e. for all* $x \in \mathbb{Z}$*, it has the form*

$$f_\mathrm{B}(x) := \frac{1}{n_\mathrm{B}} \mathrm{e}^{-\lambda_\mathrm{B}|x|}, \tag{112}$$

and the decay rate and the normalization constant are given by

$$\lambda_\mathrm{B} := \log(\kappa + e_\mathrm{B}), \tag{113}$$

$$n_\mathrm{B} := \sqrt{\frac{e_\mathrm{B}}{\kappa}}. \tag{114}$$

Moreover, the eigenfunction of H_B *with eigenvalue* $-e_\mathrm{B}$ *reads* $0 \oplus f_\mathrm{B} \in \mathfrak{h}^{\oplus 2}$.

Proof. Assertions (a) and (b) are proven in a more general context in Hume and Robinson [19] (which also contains the case of the truly anisotropic XY model without magnetic field, and more general perturbations). The fact that there is no eigenvalue embedded in the continuum $\mathrm{spec}_\mathrm{ac}(H_\mathrm{B}) = \mathrm{spec}_\mathrm{ac}(H) = \mathrm{spec}(H) = [-1, 1]$ also follows from [19]. Hence, in order to derive assertion (c), we compute eigenfunctions of the operator H_B by looking for solutions of the eigenvalue equation $h_\mathrm{B} f = ef$ for $e \in \mathbb{R}$ with $|e| > 1$ and not identically vanishing $f \in \mathfrak{h}$. Since such e lie in the resolvent set of h, we can write $f = -(h - e)^{-1} v f = -\kappa f(0)(h - e)^{-1}\delta_0$. By taking the scalar product of this equation with δ_x for any $x \in \mathbb{Z}$, we have

$$f(x) = -\kappa f(0)\big(\delta_x, (h-e)^{-1}\delta_0\big), \tag{115}$$

implying that $f(0) \neq 0$. Plugging $x = 0$ into (115), we get the eigenvalue equation

$$1 + \kappa\big(\delta_0, (h-e)^{-1}\delta_0\big) = 0. \tag{116}$$

Switching to the momentum space representation and using Cauchy's residue theorem, we get, for all $e \in \mathbb{R}$ with $|e| > 1$ and all $x \in \mathbb{Z}$,

$$\big(\delta_x, (h-e)^{-1}\delta_0\big) = -\operatorname{sign}(e)\frac{(e - \operatorname{sign}(e)\sqrt{e^2-1})^{|x|}}{\sqrt{e^2-1}}. \tag{117}$$

If we plug $x = 0$ into (117) and use the assumption $\kappa > 0$, we see that the equation (116) can be satisfied for $e > 1$ only. Solving (116) for this case leads to (111). Next, plugging (111) into (117), we get $f(x) = f(0)(\kappa + e_{\mathrm{B}})^{-|x|}$ from (115). Choosing $f(0) > 0$, we arrive at (112) with (113) and (114). $\square$

Remark 37. The Fourier transformation of $f_{\mathrm{B}} \in \mathfrak{h}$ is given, for all $k \in (-\pi, \pi]$, by

$$\hat{f}_{\mathrm{B}}(k) = \frac{1}{n_{\mathrm{B}}}\frac{\kappa}{e_{\mathrm{B}} - \cos(k)}. \tag{118}$$

Cauchy's residue theorem yields that, for $x \in \mathbb{Z}$ with $x \geqslant 0$, we also have

$$\mathrm{e}^{-\lambda_{\mathrm{B}}x} = \mathrm{i}e_{\mathrm{B}}\int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\frac{\mathrm{e}^{-\mathrm{i}kx}}{\sin(k) + \mathrm{i}\kappa}. \tag{119}$$

The integrand on the r.h.s. of (119) is used for the extraction of the Hankel symbol in the proof of Proposition 23.

Appendix C. Wave operators

In this section, we use the stationary approach to scattering theory in order to compute the wave operators $w_\pm(h, h_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h})$ appearing in the ac-contribution to the asymptotic correlation matrix from Lemma 18. To this end, we first need to express the resolvent of the magnetic Hamiltonian by the resolvent of the XY Hamiltonian. This is done in the following lemma.

For any operator $a \in \mathcal{L}(\mathfrak{h})$ and any $z \in \mathbb{C}$ in the resolvent set of a, we denote by $r_z(a) := (a-z)^{-1} \in \mathcal{L}(\mathfrak{h})$ the resolvent of a at the point z.

Lemma 38 *(Magnetic resolvent). Let $e \in \mathbb{R}$ and $\varepsilon > 0$. Then, at the points $e \pm \mathrm{i}\varepsilon$, the resolvent of $h_{\mathrm{B}} \in \mathcal{L}(\mathfrak{h})$ can be expressed in terms of the resolvent of $h \in \mathcal{L}(\mathfrak{h})$ as*

$$r_{e\pm\mathrm{i}\varepsilon}(h_{\mathrm{B}}) = r_{e\pm\mathrm{i}\varepsilon}(h) - \frac{\kappa}{1 + \kappa(\delta_0, r_{e\pm\mathrm{i}\varepsilon}(h)\delta_0)}\big(r_{e\mp\mathrm{i}\varepsilon}(h)\delta_0, \cdot\big)r_{e\pm\mathrm{i}\varepsilon}(h)\delta_0. \tag{120}$$

Proof. In order to simplify notation, we drop the index $e \pm i\varepsilon$ of the resolvents. With the help of the resolvent identity $r(h_{\mathrm{B}}) = r(h) - \kappa r(h) v r(h_{\mathrm{B}})$, we can write, for all $f \in \mathfrak{h}$,

$$r(h_{\mathrm{B}}) f + \kappa \big(\delta_0, r(h_{\mathrm{B}}) f\big) r(h)\delta_0 = r(h) f. \tag{121}$$

Taking the scalar product of (121) from the left with δ_0, we get

$$\big(1 + \kappa \big(\delta_0, r(h)\delta_0\big)\big)\big(\delta_0, r(h_{\mathrm{B}}) f\big) = \big(r(h)^* \delta_0, f\big). \tag{122}$$

Since, due to $(1 + \kappa(\delta_0, r(h)\delta_0))(1 - \kappa(\delta_0, r(h_{\mathrm{B}})\delta_0)) = 1$, the first factor on the l.h.s. of (122) is nonvanishing, we can solve (122) for $(\delta_0, r(h_{\mathrm{B}}) f)$. Plugging the resulting expression into (121) yields the assertion. $\square$

In the next definition, we introduce the energy space being the direct integral decomposition of the absolutely continuous subspace of the XY Hamiltonian $h \in \mathcal{L}(\mathfrak{h})$ w.r.t. which h is diagonal.

Definition 39 *(Energy space).* Let the direct integral over $\mathrm{spec}_{\mathrm{ac}}(h)$ with fiber $\mathbb{C}^2$ be denoted by

$$\tilde{\mathfrak{h}} := L^2\big([-1, 1], \mathbb{C}^2\big), \tag{123}$$

and let us call $\tilde{\mathfrak{h}}$ the energy space of h. Moreover, the mapping $\tilde{\mathfrak{f}} : \hat{\mathfrak{h}} \to \tilde{\mathfrak{h}}$ is defined, for all $\varphi \in \hat{\mathfrak{h}}$, by

$$(\tilde{\mathfrak{f}}\varphi)(e) := (2\pi)^{-1/2}\big(1 - e^2\big)^{-1/4}\big[\varphi\big(\arccos(e)\big), \varphi\big(-\arccos(e)\big)\big]. \tag{124}$$

We will use the notation $\tilde{f} := \tilde{\mathfrak{f}}\mathfrak{f} f$ for all $f \in \mathfrak{h}$, and $\tilde{a} := \tilde{\mathfrak{f}}\mathfrak{f} a \mathfrak{f}^* \tilde{\mathfrak{f}}^*$ for all $a \in \mathcal{L}(\mathfrak{h})$, where the Fourier transform $\mathfrak{f} : \mathfrak{h} \to \hat{\mathfrak{h}}$ is defined in Appendix A. Moreover, the Euclidean scalar product in $\mathbb{C}^2$ will be denoted as $\langle \cdot, \cdot \rangle$.

We then have the following lemma.

Lemma 40 *(Diagonalization). The mapping $\tilde{\mathfrak{f}} \in \mathcal{L}(\hat{\mathfrak{h}}, \tilde{\mathfrak{h}})$ is unitary, and the XY Hamiltonian $h \in \mathcal{L}(\mathfrak{h})$ acts, on any $\eta \in \tilde{\mathfrak{h}}$, as the multiplication by the energy variable e,*

$$(\tilde{h}\eta)(e) = e\eta(e). \tag{125}$$

Proof. A simple computation shows that $\tilde{\mathfrak{f}}$ is a surjective isometry with $\tilde{\mathfrak{f}}^{-1} = \tilde{\mathfrak{f}}^* : \tilde{\mathfrak{h}} \to \hat{\mathfrak{h}}$ acting on all $\eta =: [\eta_1, \eta_2] \in \tilde{\mathfrak{h}}$ as

$$\big(\tilde{\mathfrak{f}}^*\eta\big)(k) = (2\pi)^{1/2}\big(1 - \cos^2(k)\big)^{1/4}\big[\chi_{[0,\pi]}(k)\, \eta_1\big(\cos(k)\big) + \chi_{[-\pi,0]}(k)\eta_2\big(\cos(k)\big)\big]. \tag{126}$$

Equality (125) then follows immediately. $\square$

We introduce the following abbreviations.

Definition 41 *(Boundary values).* Let $e \in \mathbb{R}$ and $\varepsilon > 0$. For all $f, g \in \mathfrak{h}$, we define

$$\varrho_{f,g}(e \pm \mathrm{i}\varepsilon) := \big(f, r_{e\pm\mathrm{i}\varepsilon}(h)g\big), \tag{127}$$

$$\gamma_{f,g}(e,\varepsilon) := \frac{1}{2\pi\mathrm{i}}\big(\varrho_{f,g}(e+\mathrm{i}\varepsilon) - \varrho_{f,g}(e-\mathrm{i}\varepsilon)\big). \tag{128}$$

Moreover, if the limits exist, we write

$$\varrho_{f,g}(e \pm \mathrm{i}0) := \lim_{\varepsilon\to 0^+} \varrho_{f,g}(e \pm \mathrm{i}\varepsilon), \tag{129}$$

$$\gamma_{f,g}(e) := \lim_{\varepsilon\to 0^+} \gamma_{f,g}(e,\varepsilon). \tag{130}$$

The wave operators then have the following form.

Proposition 42 *(Wave operators). In energy space $\tilde{\mathfrak{h}}$, the action of the wave operators $w_\pm(h, h_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h})$ on any $f \in \mathfrak{h}$ has the form*

$$\tilde{w}_\pm(h, h_{\mathrm{B}})\tilde{f}(e) = \tilde{f}(e) - \frac{\kappa\varrho_{\delta_0,f}(e \pm \mathrm{i}0)}{1 + \kappa\varrho_{\delta_0,\delta_0}(e \pm \mathrm{i}0)}\tilde{\delta}_0(e). \tag{131}$$

Proof. In order to compute the wave operators $w_\pm(h, h_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h})$ with the help of the stationary scheme in scattering theory (see, for example, Yafaev [27]), we write them in the weak abelian form

$$w_\pm(h, h_{\mathrm{B}}) = \operatorname*{w-lim}_{\varepsilon\to 0^+} 2\varepsilon \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-2\varepsilon t} 1_{\mathrm{ac}}(h)\mathrm{e}^{\pm\mathrm{i}th}\mathrm{e}^{\mp\mathrm{i}th_{\mathrm{B}}} 1_{\mathrm{ac}}(h_{\mathrm{B}}). \tag{132}$$

Applying Parseval's identity to (132) and using that $r_{e\pm\mathrm{i}\varepsilon}(h) = \pm\mathrm{i}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mp\mathrm{i}t(h-(e\pm\mathrm{i}\varepsilon))}$, we can write, for all $f, g \in \mathfrak{h}$,

$$\big(f, w_\pm(h, h_{\mathrm{B}})g\big) = \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi} \int_{-\infty}^{\infty} \mathrm{d}e \big(r_{e\pm\mathrm{i}\varepsilon} 1_{\mathrm{ac}}(h) f, r_{e\pm\mathrm{i}\varepsilon}(h_{\mathrm{B}}) 1_{\mathrm{ac}}(h_{\mathrm{B}})g\big). \tag{133}$$

Moreover, if the limits $\varepsilon \to 0^+$ of $(r_{e\pm\mathrm{i}\varepsilon}(h)f, r_{e\pm\mathrm{i}\varepsilon}(h_{\mathrm{B}})g)$ exist for all $f, g \in \mathfrak{h}$ and almost all $e \in \mathbb{R}$ (the set of full measure depending on f and g), we get

$$\big(f, w_\pm(h, h_{\mathrm{B}})g\big) = \int_{-1}^{1} \mathrm{d}e \lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\pi}\big(r_{e\pm\mathrm{i}\varepsilon}(h) f, r_{e\pm\mathrm{i}\varepsilon}(h_{\mathrm{B}})g\big) \tag{134}$$

because $1_{\mathrm{ac}}(h) = 1$ and $\mathrm{spec}(h) = [-1, 1]$. In order to compute the limit in (134), we express the resolvents $r_{e\pm\mathrm{i}\varepsilon}(h)$ of the magnetic Hamiltonian in terms of the resolvents $r_{e\pm\mathrm{i}\varepsilon}(h)$ of the XY Hamiltonian. Plugging (120) from Lemma 38 into the scalar product on the r.h.s. of (134) and using (127) and (128) from Definition 41, we have

$$\frac{\varepsilon}{\pi}\big(r_{e\pm\mathrm{i}\varepsilon}(h) f, r_{e\pm\mathrm{i}\varepsilon}(h_{\mathrm{B}})g\big) = \gamma_{f,g}(e,\varepsilon) - \frac{\kappa}{1 + \kappa\varrho_{\delta_0,\delta_0}(e \pm \mathrm{i}\varepsilon)}\gamma_{f,\delta_0}(e,\varepsilon)\varrho_{\delta_0,g}(e \pm \mathrm{i}\varepsilon). \tag{135}$$

Here, we made use of the fact that, due to the resolvent identity, we have the equality $\gamma_{f,g}(e,\varepsilon) = (f, \frac{\varepsilon}{\pi} r_{e\pm i\varepsilon}(h) r_{e\mp i\varepsilon}(h) g)$. Now, we know that, for any $f, g \in \mathfrak{h}$ and almost all $e \in [-1,1]$, the following limits exist,

$$\varrho_{f,g}(e \pm \mathrm{i}0) = \pm\pi\mathrm{i}\frac{\mathrm{d}(f, \rho(e)g)}{\mathrm{d}e} + \mathrm{p.v.}\int_{-1}^{1} \mathrm{d}e' \frac{1}{e'-e}\frac{\mathrm{d}(f, \rho(e')g)}{\mathrm{d}e'}, \tag{136}$$

where the p.v.-integral denotes Cauchy's principle value, the mapping $\rho : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathfrak{h})$ with $\mathcal{B}(\mathbb{R})$ the Borel sets on $\mathbb{R}$ is the projection-valued spectral measure of the XY Hamiltonian h, and we used that

$$\mathrm{d}\big(f, \rho(e)g\big) = \chi_{[-1,1]}(e)\frac{\mathrm{d}(f, \rho(e)g)}{\mathrm{d}e}\,\mathrm{d}e. \tag{137}$$

Moreover, we get from (128) and (136),

$$\gamma_{f,g}(e) = \frac{\mathrm{d}(f, \rho(e)g)}{\mathrm{d}e}. \tag{138}$$

Therefore, plugging (135), (136), and (138) into (134), we can write

$$\big(f, w_\pm(h, h_{\mathrm{B}})g\big) = (f, g) - \kappa \int_{-1}^{1} \mathrm{d}e \frac{\gamma_{f,\delta_0}(e)\varrho_{\delta_0,g}(e \pm \mathrm{i}0)}{1 + \kappa\varrho_{\delta_0,\delta_0}(e \pm \mathrm{i}0)}, \tag{139}$$

where, in the first term on the r.h.s., we used $\int_{-1}^{1} \mathrm{d}e\gamma_{f,g}(e) = (f, 1_{\mathrm{ac}}(h)g) = (f, g)$. In order to write the derivatives in (138) entering (139) more explicitly, we switch to the energy space representation from Definition 39. Using the diagonalization (125), we have, for all $f, g \in \mathfrak{h}$, that

$$\frac{\mathrm{d}(f, \rho(e)g)}{\mathrm{d}e} = \big\langle \tilde{f}(e), \tilde{g}(e)\big\rangle, \tag{140}$$

where we recall from Definition 39 that $\langle\cdot,\cdot\rangle$ denotes the scalar product in the fiber $\mathbb{C}^2$ of the direct integral $\tilde{\mathfrak{h}} = L^2([-1,1], \mathbb{C}^2)$, and $\tilde{f} = \tilde{\mathfrak{f}}\mathfrak{f}f$ for all $f \in \mathfrak{h}$. Hence, plugging (138) and (140) into (139), we arrive at the assertion. □

Finally, since the wave operators $w_\pm(h, h_{\mathrm{B}}) \in \mathcal{L}(\mathfrak{h})$ appearing in the ac-contribution to the asymptotic correlation matrix act on completely localized wave functions $\delta_x \in \mathfrak{h}$ with $x \in \mathbb{Z}$, we compute the terms $\varrho_{\delta_0,\delta_x}(e \pm \mathrm{i}0)$ on the r.h.s. of (131) in Proposition 42.

Lemma 43 *(Boundary values). Let $x \in \mathbb{Z}$ and $e \in (-1,1)$. Then, we have*

$$\varrho_{\delta_0,\delta_x}(e \pm \mathrm{i}0) = \pm\mathrm{i}\frac{(e \mp \mathrm{i}\sqrt{1-e^2})^{|x|}}{\sqrt{1-e^2}}. \tag{141}$$

Proof. Let $x \in \mathbb{Z}$ with $x \geqslant 0$, $e \in (-1, 1)$, and $\varepsilon > 0$ sufficiently small. Writing $\varrho_{\delta_0,\delta_x}(e - \mathrm{i}\varepsilon)$ in the momentum space representation, using Cauchy's residue theorem, and taking the limit $\varepsilon \to 0^+$, we get the expression (141) for $\varrho_{\delta_0,\delta_x}(e - \mathrm{i}0)$. Moreover, using (127), the translation and parity invariance of h, i.e. $[h, u] = 0$ and $[h, \theta] = 0$, respectively, where $\theta : \mathfrak{h} \to \mathfrak{h}$ is defined, for all $f \in \mathfrak{h}$, by $(\theta f)(x) := f(-x)$, we have, for all $x \in \mathbb{Z}$ with $x \geqslant 0$, that

$$\varrho_{\delta_0,\delta_x}(e + \mathrm{i}\varepsilon) = \overline{\varrho_{\delta_0,\delta_{-x}}(e - \mathrm{i}\varepsilon)}, \tag{142}$$

$$\varrho_{\delta_0,\delta_{-x}}(e - \mathrm{i}\varepsilon) = \varrho_{\delta_0,\delta_x}(e - \mathrm{i}\varepsilon). \tag{143}$$

This yields the assertion. $\square$

Appendix D. Asymptotic correlation matrix

In this section, we compute the nonvanishing entries of the total asymptotic correlation matrix used above.

Lemma 44 *(Structure). Let $x_0 \geqslant 0$ and $i, j \geqslant 1$, or $x_0 < 0$ and $i, j \geqslant 1 - x_0$. Then, the entries of the asymptotic correlation matrix have the structure*

$$\Omega^{\mathrm{aa},-}_{2i-1\,2j} + \Omega^{\mathrm{pp},-}_{2i-1\,2j} = (\mathrm{e}_{i-j}, \hat{s}_-\mathrm{e}_0) + (\mathrm{e}_{i-j}, a_-\mathrm{e}_0) + (\mathrm{e}_{i+j}, b_-\mathrm{e}_0), \quad \text{if } i \leqslant j, \tag{144}$$

$$\Omega^{\mathrm{aa},+}_{2j\,2i-1} + \Omega^{\mathrm{pp},+}_{2j\,2i-1} = (\mathrm{e}_{j-i}, \hat{s}_+\mathrm{e}_0) + (\mathrm{e}_{i-j}, a_+\mathrm{e}_0) + (\mathrm{e}_{i+j}, b_+\mathrm{e}_0), \quad \text{if } i > j, \tag{145}$$

where the functions $a_\pm, b_\pm \in L^\infty(\mathbb{T})$ are defined by

$$a_\pm(k) := \kappa^2 \chi_{[0,\pi]}(k) \frac{\hat{s}_{\pm,R}(k) - \hat{s}_{\pm,L}(k)}{\sin^2(k) + \kappa^2}, \tag{146}$$

$$b_\pm(k) = (-\mathrm{i}\kappa) \frac{\hat{s}_{\pm,R}(k)}{\sin(k) + \mathrm{i}\kappa} \mathrm{e}^{-2\mathrm{i}k(x_0-1)} + \frac{\kappa}{n_{\mathrm{B}}^2} (f_{\mathrm{B}}, s_{0,\pm} f_{\mathrm{B}}) \frac{\mathrm{e}^{-2\lambda_{\mathrm{B}}(x_0-1)}}{e_{\mathrm{B}} - \cos(k)}. \tag{147}$$

Proof. For $i \leqslant j$, plugging the wave operator (59) into ac-contribution (54) yields

$$\begin{aligned}
\Omega^{\mathrm{aa},-}_{2i-1\,2j} &= (\delta_{i+x_0-1}, s_- \delta_{j+x_0-1}) \\
&\quad + \mathrm{i}\kappa \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \hat{s}_-(k) \frac{\mathrm{e}^{-\mathrm{i}(k(i+x_0-1)-|k||j+x_0-1|)}}{\sin(|k|) - \mathrm{i}\kappa} \\
&\quad - \mathrm{i}\kappa \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \hat{s}_-(k) \frac{\mathrm{e}^{-\mathrm{i}(|k||i+x_0-1|-k(j+x_0-1))}}{\sin(|k|) + \mathrm{i}\kappa} \\
&\quad + \kappa^2 \int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi} \hat{s}_-(k) \frac{\mathrm{e}^{-\mathrm{i}|k|(|i+x_0-1|-|j+x_0-1|)}}{\sin^2(k) + \kappa^2}.
\end{aligned} \tag{148}$$

Moreover, using Lemma 22, we have

$$\Omega^{\mathrm{pp},-}_{2i-1\,2j} = \frac{1}{n_{\mathrm{B}}^2}(f_{\mathrm{B}}, s_{0,-} f_{\mathrm{B}}) \mathrm{e}^{-\lambda_{\mathrm{B}}(|i+x_0-1|+|j+x_0-1|)}. \tag{149}$$

The expressions for $i > j$ are analogous. Then, if $x_0 \geqslant 0$ and $i, j \geqslant 1$, or $x_0 < 0$ and $i, j \geqslant 1 - x_0$, we use the translation invariance of $s_\pm$, resolve the absolute values, and decompose the integrals w.r.t. the sign of the momentum in order to get rid of the sign function in the density $\hat{s}_\pm$. This leads to (144) and (145). □

Appendix E. Toeplitz symbol regularity

The following proposition is used in Theorem 26.

Proposition 45 *(Regularity). The Toeplitz symbol $a \in L^\infty(\mathbb{T})$ of Proposition* 23 *has the following properties.*

(a) $a \in C^1(\mathbb{T}) \cap PC^\infty(\mathbb{T})$;
(b) *The left and right derivatives $D_\pm a'(k)$ exist for all $k \in (-\pi, \pi]$, but, for $k_+ := 0$ and $k_- := \pi$, we have*

$$D_+ a'(k_\pm) - D_- a'(k_\pm) = \pm \frac{1}{\kappa^2} \frac{\sinh[\frac{1}{2}(\beta_R - \beta_L)]}{\cosh[\frac{1}{2}\beta_R]\cosh[\frac{1}{2}\beta_L]}. \tag{150}$$

Proof. From the very form of the symbol given in (65)–(67), we get $a \in PC^\infty(\mathbb{T})$ with jumps at $k_\pm$, and since $\varphi_B \in C(\mathbb{T})$ for nonvanishing coupling, we also have $a \in C(\mathbb{T})$. Moreover, the one-sided limits yield $a'(k_\pm + 0) = a'(k_\pm - 0) = 0$ which is assertion (a), and analogously for assertion (b). □

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