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On Subgroups of Type $Z_p \times Z_p$

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1. INTRODUCTION

A few years ago, a very simple (and entirely elementary) characterization of the groups $S_p(2n, 2)$ was given in [4]. The following theorem on involutions was an immediate corollary of this characterization.

THEOREM 1 [4]. *Let K be a class of involutions in a finite group G and suppose the following*:

(i) *If t_1 and t_2 are two involutions in K which commute, then $t_1t_2 \in K$, and any involution s in $K - \{t_1, t_2, t_1t_2\}$ commutes with at least one of the involutions in $\langle t_1, t_2\rangle$.*

(ii) *At least two involutions in K commute.*

Then all involutions in K commute with one another and generate a normal abelian 2-subgroup of G.

Since no reference to Sp$(2n, 2)$ appears anywhere in the statement of the of the theorem, J. Alperin wondered whether a more natural direct proof could be found. Indeed he found a short elementary proof (utilizing nothing worse than Baer's crirerion that an involution belong to $O_2(G)$) of a more general result, namely:

THEOREM 2 (Alperin [1]). *Suppose A is a fours-group and is a subgroup of the finite group G. Let K be the set of conjugates in G of all involutions in A and suppose, for every involution t in K, $C(t) \cap A$ is nontrivial. Then $A \cap O_2(G)$ is also nontrivial.*

This theorem is more general in several respects. First, K is not necessarily one conjugacy class of G. Second, the commuting condition is entirely relative to a single fours-group A, rather than all fours-groups generated by commuting pairs in K.

For some time the author has been wondering if there were versions of these theorems for odd primes. Finally, because of a recent geometric theorem by F. Beukenhout and the author, we can prove the following.

THEOREM 3. *Let K be a union of conjugacy classes of cyclic groups of prime order p in a finite group G. We assume at least two members of K commute and that if X and Y are two commuting subgroups in K, then*

(a) *every subgroup of order p in $\langle X, Y\rangle$ belongs to K, and*

(b) *for each subgroup W in K, $C(W) \cap \langle X, Y\rangle$ is nontrivial.*

Then K contains a subgroup V commuting with all other members of K. In particular, $V \leqslant O_p(G)$, and G is not simple.

Note that when we say that two members of K commute (as subgroups), they generate a $Z_p \times Z_p$ and so centralize one another.

Remark. With only a very minor modification of the proof, the above theorem holds if (a) and (b) are replaced respectively by (a'). *At least three subgroups of order p in $\langle X, Y\rangle$ belong to K, and* (b') *for each subgroup W in K, $C(W) \cap \langle X, Y\rangle$ contains a subgroup in K.*

In a general way, the proof of Theorem 3 given here may be viewed as an odd analogue of my earlier proof involving the Sp($2n$, 2)-characterization, and not at all analogous to Alperin's proof. The basic useful property that two involutions always generate a dihedral group seems to suggest that his proof will not generalize in any obvious way. It is thus an open question whether one can prove the following.

CONJECTURE. *Suppose $A \simeq Z_p \times Z_p$ is a subgroup of the finite group G. Let K be the set of all conjugates of cyclic subgroups of order p in G. Suppose, for each subgroup W in K that $C(W) \cap A$ is nontrivial. Then $A \cap O_p(G)$ is nontrivial. More specifically, $A \cap C_G(K)$ is nontrivial.*

2. PRELIMINARY RESULTS

A *polar space* is defined by Tits [5] as a set of points S together with distinguished subsets called *subspaces* such that

(1) a subspace together with the subspaces it contains is a d-dimensional projective space with $-1 \leqslant d \leqslant n-1$ for some integer n called the *rank of S*;

(2) the intersection of two subspaces is a subspace;

(3) given a subspace L of dimension $n-1$ and a point p in $S - L$,

there exists a *unique* subspace M containing p such that $\dim(M \cap L) = n - 2$; it contains all points of L joined to p by some subspace of dimension 1;

(4) there exists two disjoint subspaces of dimension $n - 1$.

A standard example of a polar space is the subspace $S(\pi)$ defined as follows: Let P be a Desarguesian projective space of finite dimension n. Let π be either a polarity in P or a nondegenerate quadratic form on P. Let $S(\pi)$ be the collection of *absolute points* or (respectively) the *singular points* of P with respect to π. The subspaces of $S(\pi)$ are defined to be the projective subspaces of P lying in $S(\pi)$. (For example, if $S(\pi)$ is a projective symplectic space, its polar subspaces are its *isotropic* subspaces. By allowing π to be a quadratic form as well as a polarity, the definition of $S(\pi)$ is allowed to include the singular projective points in the orthogonal geometries of characteristic 2, as well as the overlying projective symplectic geometry which contains it in this case.) Combined work of Veldkamp and Tits [6, 5] has yielded the following.

THEOREM (Veldkamp–Tits, [6, 5]). *If S is a finite polar space all of whose 1-dimensional subspaces have at least three points, then*

(i) $S \simeq S(\pi)$ *if rank* $S \geqslant 3$, *or*

(ii) *S is a generalized 4-gon in the sense of Tits.*

We now turn to a graph-theoretic characterization of the polar spaces. For the purposes of this paper, all graphs considered are undirected and without loops. Recently F. Beukenhout and the author considered graphs $\mathscr{G}$ satisfying the following hypothesis.

2.1. *If (x, y) is an edge in $\mathscr{G}$, there exists a (not necessarily unique) complete subgraph $C(x, y)$ containing at least three vertices and the edge (x, y) as a subgraph, such that if z is any vertex of $\mathscr{G}$ not in $C(x, y)$, then z is either joined by an edge to exactly one member of $C(x, y)$, or is joined to all of the members of $C(x, y)$.*

Although we do not require $\mathscr{G}$ to be finite, we do impose a finitness condition. If $\mathscr{G}$ is a graph satisfying 2.1, we say that a complete subgraph C of $\mathscr{G}$ is a subspace of $\mathscr{G}$ if and only if for any two points x and y of C, each $C(x, y)$ of 2.1 lies in C. The empty set is a subspace. A *flag of length d* is a chain of subspaces $K_1, K_2, \ldots, K_d$ with each K_i properly contained in K_{i+1}. Our condition is as follows.

2.2. *Every flag of l has length at most n.*

The following theorem was proved by F. Beukenhout and the author and will appear in a forthcoming publication [2].

THEOREM (Beukenhout–Shult). *Let $\mathcal{G}$ be a graph satisfying hypotheses* 2.1 *and* 2.2. *Then either*

(1) *$\mathcal{G}$ is totally disconnected,*

(2) *$\mathcal{G}$ contains a vertex arced to all other vertices of $\mathcal{G}$ or,*

(3) *the subspaces of $\mathcal{G}$ together with the vertices of $\mathcal{G}$ form a polar space.*

This theorem together with the Veldkamp–Tits theorem yields the following.

COROLLARY. *Let $\mathcal{G}$ be a finite graph satisfying hypotheses* 2.1 *and* 2.2. *Then one of the following holds:*

(i) *$\mathcal{G}$ is totally disconnected.*

(ii) *$\mathcal{G}$ contains a vertex lying on an edge with all the other vertices of $\mathcal{G}$.*

(iii) *$\mathcal{G}$ is the graph of isotropic or singular projective points of a nondegenerate projective symplectic orthogonal or unitary geometry.*

(iv) *$\mathcal{G}$ is a generalized* 4*-gon, each $C(x, y)$ is a maximal complete subgraph.* (*Note*: *Although $\mathcal{G}$ is regular it may not be strongly regular in this case, and the $C(x, y)$'s may have cardinality depending on the choice of edge (x, y).*)

Our proof of Theorem 3 rests primarily on this corollary. Most of the proof involves what to do with case (iv). However some work is required in case (iii) and to facilitate this we require two technical lemmas concerning $S(\pi)$ which we establish in this section.

LEMMA 2.1. *Let V be a finite vector space admitting a nondegenerate quadratic form π, and let S denote the set of singular vectors of V. Then either*

(i) $\langle S \rangle = V$,

(ii) $\dim V = 2$ *and* (V, π) *defines an orthogonal geometry or,*

(iii) $\dim V = 1$.

Proof. The pair (V, π) defines a symplectic, unitary or orthogonal geometry on V. We may assume $\dim V > 1$.

If (V, π) defines a symplectic geometry, $S = V^{\#}$ and so (i) holds.

Suppose (V, π) defines a unitary geometry and let W be any nondegenerate 2-dimensional subspace. We suppose $GF(q^2)$ to be the ground field. The $(q + 1)$-st power map defines an epimorphism between multiplicative groups, $GF(q^2)^* \to GF(q)^*$, and so there exist elements α and β in $GF(q^2)^*$ such that $a^{q+1} = -\beta^{q+1}$. Since $q + 1 > 2$, we can choose α and β so that $\alpha \neq \beta$. Now W contains an orthonormal basis $\{e_1, e_2\}$ and $\alpha e_1 + \beta e_2$ and $\epsilon\alpha e_1 + \beta e_2$ (where $\epsilon \neq 1$, $\epsilon^{q+1} = 1$) are two singular vectors spanning W so

$\langle (S \cap W) \rangle = W$. Since V is generated by its nondegenerate 2-dimensional subspaces, $\langle S \rangle = V$ and so (i) holds.

Assume finally that (V, π) defines an orthogonal geometry. We may now assume $\dim V \geqslant 3$, since otherwise (ii) holds. The nondegenerate 2-dimensional subspaces of V are of two types, P_0 having no singular vectors and P_1, having two singular one dimensional subspaces spanning P_1. For perpendicular sums we have $P_0 \perp P_0 \simeq P_1 \perp P_1$, where the geometry isomorphisms are defined up to a scalar multiple of the quadratic form. Assume W is a nondegenerate 3-dimensional subspace. Then V is defined over a field F of odd characteristic and the quadratic form $\underset{\sim}{Q}$ can be taken (up to a scalar multiple of $\underset{\sim}{Q}$) to be defined by

$$\underset{\sim}{Q}(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 \qquad \text{or} \qquad x_1^2 + x_2^2 + gx_3^2$$

where g is a nonsquare in F. (This replacement of $\underset{\sim}{Q}$ by a nonzero scalar multiple $c\underset{\sim}{Q}$ does not affect the question whether $\langle W \cap S \rangle = W$.) In the former case $\langle (1, 0, 0), (0, 1, 0) \rangle$, $\langle (0, 1, 0), (0, 0, 1) \rangle$ are two nondegenerate 2-dimensional subspaces of type P_1 so $\langle S \cap W \rangle = W$. In the latter case there exist scalars α and β in F^* such that $\alpha^2 + \beta^2 = -g$ (for if $|F|$ is odd, every element of F is a sum of two nonzero squares). Then $\alpha \neq 0 \neq \beta$ and $s_1 = (\alpha, \beta, 1)$, $s_2 = (\alpha, -\beta, 1)$ and $s_3 = (-\alpha, \beta, 1)$ are three singular vectors whose span contains $s_1 - s_2 = (0, 2\beta, 0)$, $s_1 - s_3 = (2\alpha, 0, 0)$ and $\frac{1}{2}(s_2 + s_3) = (0, 0, 1)$, so they span W.

Now assume $\dim W = 4$. If $W \simeq P_1 \perp P_1$, $\langle S \cap W \rangle = W$. We may thus assume $W \simeq P_0 \perp P_1$. But in this case $\mathrm{Aut}(W)$ contains a normal subgroup isomorphic (projectively) to $PSL(2, q^2)$, so $\mathrm{Aut}(W)$ acts irreducibly on W. Thus $S \cap W \neq \varnothing$ implies $\langle S \cap W \rangle = W$. Thus, if W is a nondegenerate subspace of dimension 3 or 4, $\langle S \cap W \rangle = W$. If the characteristic of the field is even, $\dim V \leqslant 4$, and V is spanned by nondegenerate subspaces W of dimension 4, whence $\langle S \rangle = W$. If the characteristic is odd, $\dim V \geqslant 3$, and V is a sum of nondegenerate subspaces W of dimension 3. Thus in either case, $\langle S \rangle = V$ follows. This completes the proof of the lemma.

LEMMA 2.2. *Let V be a finite vector space admitting a nondegenerate quadratic form π and let S denote the singular vectors of V. Fix s in S. Suppose s_1 and s_2 are two vectors in S not perpendicular to s, but that* (2.1)

$$S \cap s^{\perp} \cap s_1^{\perp} = S \cap s^{\perp} \cap s_2^{\perp}.$$

Then either

(i) $s_2 \in \langle s, s_1 \rangle$,

(ii) (V, π) *defines a unitary geometry and* $\dim V \leqslant 3$,

(iii) *(V, π) defines an orthogonal geometry and* $\dim V \leqslant 3$, *or else V is of type $P_1 \perp P_0$ where P_0 and P_1 are the nondegenerate 2-dimensional orthogonal geometries having* 0 *or* 2 *singular* 1*-dimensional subspaces, respectively.*

Proof. Since s_1 is not perpendicular to s, $P = \langle s, s_1\rangle$ is a nondegenerate 2-dimensional subspace containing at least two singular 1-dimensional subspaces. We may assume (since nondegenerate symplectic spaces have even dimension) that $\dim V \geqslant 4$, since otherwise one of (ii) or (iii) holds. We may then write $V = P \perp W$ where $W = P^{\perp}$ and set $s_2 = p + w$ where $p \in P$ and $w \in W$. Then $w \neq 0$ since otherwise (i) holds. Then unless (V, π) defines an orthogonal geometry and $W \simeq P_0$, we have $\langle S \cap W\rangle = W$ by Lemma 2.1, and so there exists a vector s_3 contained in $S \cap (W - w^{\perp})$, and clearly s_3 lies in $S \cap s^{\perp} \cap s_1{}^{\perp}$ but s_3 is not in $S \cap s^{\perp} \cap s_2{}^{\perp}$ so 2.1 fails. Thus we see that (iii) holds. The proof is complete.

3. Proof of Theorem 3

Let K be the union of classes of Zp's described in the hypothesis of Theorem 3. We convert K to a graph $\mathscr{K}$ whose vertices are K and whose edges are pairs of subgroups in K which commute with one another. Given two commuting members X and Y of K, the collection $C(X, Y)$ of subgroups of order p lying in $\langle X, Y\rangle$ is a subset of K which we regard as a *subgraph* of $\mathscr{K}$. Then $C(X, Y)$ is a complete subgraph of $\mathscr{K}$ containing the arc (X, Y). We have $|C(X, Y)| = 1 + p \geqslant 3$ and from our hypothesis, for any $Z \in K - C(X, Y)$, $C_G(Z) \cap \langle X, Y\rangle$ has order p or p^2—that is, in terms of $\mathscr{K}$, Z is joined by an edge to one or all members of $C(X, Y)$. Thus the graph $\mathscr{K}$ satisfies hypothesis 2.1 and 2.2. Then by the Corollary of Section 2, (i) $\mathscr{K}$ is totally disconnected, (ii) $\mathscr{K}$ contains a vertex joined by an edge to all other vertices of $\mathscr{K}$, (iii) $\mathscr{K}$ is isomorphic to the graph of isotropic or singular projective points of a nondegenerate projective symplectic, orthogonal or unitary geometry, or (iv) $\mathscr{K}$ is a generalized 4-gon and each $C(X, Y)$ is a maximal complete subgraph of $\mathscr{K}$.

We consider these four cases individually.

(i) This case does not occur since, by the hypothesis of Theorem 3, $\mathscr{K}$ contains an edge.

(ii) In this case, there exists a group V in $\mathscr{K}$ commuting with all other subgroups in $\mathscr{K}$. Thus, writing $C(\mathscr{K})$ for the set of all subgroups of G commuting with every subgroup in $\mathscr{K}$, we see that

$$V \leqslant N = \langle \mathscr{K} \cap C(\mathscr{K})\rangle$$

and N is a normal elementary abelian p-subgroup of G. So the conclusion of our theorem holds.

(iii) In this case, there exists a projective space P and a polarity or quadratic form π such that $\mathscr{K} \simeq S(\pi)$. Let f denote the isomorphism $\mathscr{K} \to S(\pi)$, and write $x \perp y$ if x and y are elements of P and $x \in \pi(y)$. Then for groups X and Y in $\mathscr{K}$ we have $[X, Y] = 1$ if and only if $f(X) \perp f(Y)$. Each group in K acts (by conjugation) on the graph $\mathscr{K}$ and hence on $f(\mathscr{K}) \simeq S(\pi)$.

If $S(\pi)$ is the set of singular 1-dimensional subspaces of a unitary polarity π, then $S(\pi)$ satisfies 2.1 with the cliques $C(x, y)$ having $1 + q^2$ points. But if $f(X) = x$ and $f(Y) = y$ where $X, Y \in K$, $[X, Y] = 1$, we see that the family $U(X, Y)$ of subgroups of K lying in $\langle X, Y\rangle$ is the preimage of $C(x, y)$, that is

$$f(U(X, Y)) = C(x, y)$$

and contains $1 + p$ points. Since p is not q^2, the unitary case is excluded.

Suppose π is a symplectic polarity of the projective space P so $P = S(\pi)$. We may then view P as the collection of 1-dimensional subspaces of a vector space V equipped with a nondegenerate symplectic form. Since $S(\pi)$ is not totally disconnected, dim $\geqslant 4$. The group $\mathrm{Aut}(S(\pi))$ now contains two subgroups: $\rho(G)$, the induced action of $G = \langle K\rangle$ on $f(\mathscr{K}) = S(\pi)$; and the group $P\,\mathrm{Sp}(2n, q)$, the action on $S(\pi)$ induced by the symplectic group $\mathrm{Aut}(P, \pi)$.

We wish to argue first, that for each X in K, $\rho(X)$ coincides with the group of transvections of $\mathrm{Aut}(P, \pi)$ with direction $f(X) = x$. The difficulty, of course, is that we do not know that $\rho(G)$ even normalizes $\mathrm{Aut}(P, \pi)$ as subgroups of $\mathrm{Aut}(S(\pi))$.

Set $\Gamma = x^{\perp} \cap S(\pi)$ and $\Sigma = S(\pi) - \Gamma - \{x\}$. If y is a vertex in Σ, y is a 1-dimensional subspace of V not perpendicular to x, and if w is any further 1-dimensional subspace of the hyperbolic plane $\langle e, y\rangle$, then $w \in \Sigma$ and any 1-space in $x^{\perp}$ perpendicular to y is also perpendicular to w. Thus the vertices of Σ are assorted into classes $C_1, C_2, \ldots$, each class containing p vertices, such that if y and w are two elements of the same class, y and w are joined by an edge with the same set of points in Γ. Conversely, if y and w are two elements of Σ joined by edges to precisely the same set of vertices in Γ, then, putting $\langle x_0\rangle = x$, $\langle y_0\rangle = y$, $\langle w_0\rangle = w$, where x_0, y_0 and w_0 are vectors in $V^{\#}$, we have

$$V^{\#} \cap x_0^{\perp} \cap y_0^{\perp} = V^{\#} \cap x_0^{\perp} \cap w_0^{\perp},$$

where y_0 and w_0 are not perpendicular to x_0. By Lemma 2.2, this forces $w_0 \in \langle x_0, y_0\rangle$ so y and w belong to the same subset C_i of Σ. Thus the C_i are equivalence classes in Σ, for the relation of being joined by an adge to the same set of vertices in Γ.

Let T_x be the group of transvections in $P\,\mathrm{Sp}(2n, q)$ having direction x. Then the C_i are both T_x-orbits on Σ and also $p(X)$-orbits on Σ. Thus the subgroup $H = \langle \rho(X), T_x \rangle$ of $\mathrm{Aut}(S(\pi))$ fixes $\{x\} \cup \Gamma$ pointwise, and stabilizes each C_i.

There are three facts concerning the subgraph Σ which we must establish:

(3.1) *The C_i are totally disconnected.*

(3.2) *If $i \neq j$ each vertex of C_i is joined by an edge to exactly one vertex of C_i so that edges between C_i and C_j define a* 1–1 *correspondence between C_i and C_j.*

(3.3) *The subgraph Σ is connected.*

3.1 is clear since the C_i are the 1-dimensional subspaces distinct from x in a hyperbolic plane containing x.

Set $x = \langle x_0 \rangle$. If $\langle y_0 \rangle$ is an element of C_i, chosen so that $x_0 \cdot y_0 = 1$, we may set $P = \langle x_0, y_0 \rangle$ and write $V = P \perp W$ where $W = P^{\perp}$. Then for some $w \in W^{\neq}$, the elements of C_j are

$$\langle y + w \rangle, \langle y_0 + w + x_0 \rangle, \ldots, \langle y_0 + w + (p-1)x_0 \rangle.$$

Then $y_0 + \alpha x_0$ is perpendicular to $y_0 + w + \beta x_0$ if and only if $\beta = -\alpha$; thus $\langle y_0 + \alpha x_0 \rangle$, a typical element of C_i, is perpendicular to one and only one element of C_j. This proves 3.2.

To prove 3.3 we first show that if $i \neq j$, any element of C_i has distance 2 from any element of C_j. Since $|C_i| \geqslant 2$, it then follows from 3.2 that each vertex of C_i has distance at most 3 from any other vertex of C_i, and this proves 3.3. As in the previous paragraph we may define x_0, y_0, P, W and w. Let the vertex in C_i be $a = \langle y_0 \rangle$, without loss of generality, and the vertex in C_j be $b = \langle y_0 + w + \alpha x_0 \rangle$. Since $\dim V \geqslant 4$, W is nondegenerate and so there exists a vector w_2 in W such that $w_2 \cdot w = -\alpha$. Then set $C = \langle y_0 + w_2 \rangle$. Then c is a vertex in Σ and is perpendicular to $a = \langle y_0 \rangle$. But

$$(y_0 + w_2) \cdot (y_0 + w + \alpha x_0) = -\alpha + \alpha = 0$$

so c is perpendicular to b. Thus c is an element of Σ lying on an edge with a and b and so the proof of 3.3 is complete.

Suppose, now, that on some C_i, say C_1, the groups $\rho(X)$ and T_x do not induce the same group of permutations. Then H, restricted to C_1, is not solvable (for the solvable groups of degree p have a unique p-Sylow subgroup). By a well known theorem of Burnside, H is doubly transitive on C_1. Let H_β be the subgroup of H stabilizing a vertex β in C_1. Then H_β transitively permutes $C_1 - \{\beta\}$. Consider $j > 1$. H_β acts on C_j, and fixes the unique vertex of C_j on an edge with β. If H_β fixed vertex α in C_j, then H_β also fixes the unique vertex in C_1 on an edge with α (by 3.2). This is necessarily β since H_β is transitive on $C_1 - \{\beta\}$. Thus H_β fixes only the vertex in C_j on an edge with β. We thus see

(3.4) *H_β fixes exactly one vertex in each C_i. The subgraph Σ_0 of vertices of Σ fixed by H_β is a complete subgraph of Σ_0 and contains one vertex from each C_i.*

It follows from 3.2 that Σ_0 has the same valence as Σ. Since by 3.4 Σ_0 is complete, no edge contains a vertex of Σ_0 and $\Sigma - \Sigma_0$. Since $\Sigma - \Sigma_0$ is nonempty (for $p = |C_i| > |C_i \cap \Sigma_0| = 1$), this means that Σ is not connected. But this contradicts 3.3.

Thus we are forced to conclude that $\rho(X)$ and T_x induce the same permutation group on each C_i in Σ. Then the commutator $[\rho(X), T_x]$ fixes pointwise each C_i as well as $\{x\} \cup \Gamma$ and so is the identity of $\mathrm{Aut}(S(\pi))$. Thus $H = \langle \rho(X), T_x \rangle \simeq Z_p$ or $Z_p \times Z_p$. In the latter case there is a kernel U_i of the action of H on each C_i. But by 3.2 $U_i = U_j$ for all j so $U_i = 1 \in \mathrm{Aut}(S(\pi))$. Thus $\rho(X) = T_x$ as subgroups of $\mathrm{Aut}(S(\pi))$.

(3.5) *$\rho(X)$ induces on $S(\pi)$ the same group of automorphisms as induced by the transvections with direction $x = f(X)$.*

Now suppose X and Y are distinct mutually commuting subgroups in K, let Z be a subgroup of $\langle X, Y \rangle$ distinct from X and Y. Then by 3.5 $\rho(Z)$ induces on $S(\pi)$ a group of automorphisms identical with that induced by the group of transveections T_z having direction $z = f(Z)$. But Z is generated by $z_0 = z_0 y_0$ where x_0 and y_0 generate X and Y, respectively. Thus $\rho(z_0)$ corresponds to the action on $S(\pi)$ of the product of two commuting transvections (induced by $\rho(x_0)$ and $\rho(y_0)$) having directions x and y. But this is impossible since the product of two commuting transvections with directions x and y has only $x^\perp \cap y^\perp \cap S(\pi)$ as its fixed point set, and this has smaller cardinality than $z^\perp \cap S(\pi)$. This contradiction disposes of the case that π is a symplectic polarity.

Finally, we may suppose π is a nondegenerate quadratic form on a vector space V, inducing an orthogonal geometry. Fix $X, Y \in K$ such that $[X, Y] \neq 1$, put $f(X) = x$, $f(Y) = y$ and set $Y_1 = Y^{x_0}$ where $\langle x_0 \rangle = X$. Then $Y_1 \neq Y$ and both Y and Y_1 fail to commute with X. Writing $S(\pi) = x \cup \Gamma \cup \Sigma$, where $\Gamma = x^\perp \cap S(\pi)$ and $\Sigma = S(\pi) - (\{x\} \cup \Gamma)$, we see that y and y_1 are joined by edges to the same set of vertices in Γ. Thus if we write $P = \langle x, y \rangle$, $V = P \perp W$ where $W = P^\perp$, y_1 is generated by a vector $p + w$ where $p \in P$ and $w \in W$. Then $w \neq 0$ since otherwise y_1 is a singular line in P and this would be impossible since P, being a nondegenerate orthogonal 2-dimensional space already containing singular points x and y cannot possess the third singular point y_1. Since y_1 is arced to the same vertices in $\Gamma = S(\pi) \cap x^{\perp}$ as y is, we see that the singular vectors of W_0 are all perpendicular to w. By Lemma 2.1, this is possible only if W contains no singular vectors at all. In this case V is orthogonal of type $P_1 \perp P_0$. But then $S(\pi)$ is totally disconnected, a case already excluded.

(iv) We are assuming here that $\mathscr{K}$ is a generalized 4-gon and that each $C(X, Y)$ is a maximal clique in $\mathscr{K}$.

Fix vertex X in $\mathscr{K}$ and let Γ denote those members of $\mathscr{K} - \{X\}$ which commute with X and set $\Sigma = \mathscr{K} - (\Gamma \cup \{X\})$. This gives us a decomposition, $\mathscr{K} = \{X\} + \Gamma + \Sigma$. The subgraph Γ is a union of $C(X, Y) - \{\{X\}$ as Y ranges over Γ. Since the cliques $C(X, Y)$ are maximal complete subgraphs the existence of an edge between $C(X, Y_1) - \{X\}$ and $C(X, Y_2) - \{X\}$ implies $C(X, Y_1) = C(X, Y_2)$. Thus

(3.6) *Γ is the disjoint union of f complete subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_f$.*

Also,

(3.7) *Each vertex in Σ is arced to exactly one vertex in each Γ_i, $i = 1, \ldots, f$.*

Since case (ii) is excluded, Σ is nonempty. Fix a vertex Z in Σ and let $Y_i \in \Gamma$ be the unique vertices in each Γ_i lying on an edge with Z, $i = 1, 2, \ldots, f$. Now Y_1, being a group of order p acting on $\mathscr{K}$ by conjugation centralizes X and hence permutes the connected components $\Gamma_1, \ldots, \Gamma_f$ as wholes. Suppose Y_1 stabilizes Γ_j for some $j > 1$. Then, since Y_1 commutes with Z, Y_1 fixes the unique vertex Y_j in Γ_j lying on an edge with Z. Thus Y_1 commutes with Y_j, forcing an edge between Γ_1 and Γ_j. This contradicts 3.6. Thus Y_1 stabilizes Γ_1 and permutes $\{\Gamma_2, \ldots, \Gamma_f\}$ in cycles of length p. Similarly, Y_i stabilizes Γ_i and permutes the remaining $\Gamma_j (j \neq i)$ in cycles of length p. It follows that the group $\langle Y_1, \ldots, Y_f \rangle$ transitively permutes the Γ_i. Since $\mathscr{K}$ is connected, the line graph defined by the cliques is also connected. It follows that

(3.8) Aut($\mathscr{K}$) *is transitive on cliques and* $f = 1 \bmod p$.

It is not clear yet that Aut($\mathscr{K}$) is transitive on vertices. (Otherwise it would then follow that Aut($\mathscr{K}$) would be transitive on "flags", that is, incident vertex-clique pairs.)

(3.9) *For any vertex W in $\mathscr{K}$, the subgraph of all vertices in $\mathscr{K}$ joined by an edge to W is also the disjoint union of f complete subgraphs. In particular $\mathscr{K}$ is regular.*

For any vertex W in $\mathscr{K}$ let $A(W)$ be the set of vertices lying on an edge joined to W. We have already seen that $A(W)$ is a union of complete graphs, corresponding to the cliques containing W. We first show that if W is in Σ, the number of complete subgraphs in $A(W)$ is f. First, if C is a complete subgraph of $A(W)$, $C \cup \{W\}$ is a clique of Hypothesis 2.1 and so X lies on an edge joined to exactly one of its points R in C. Then $R \in \Gamma_i$ for some i. If C' is a second complete subgraph of $A(W)$, similarly $| C' \cap \Gamma | = 1$, but $C' \cap \Gamma_i$ is empty since otherwise a vertex in C and a vertex in C' would form an edge. Thus each complete subgraph in $A(W)$ meets Γ at a distinct Γ_i, at so by 3.7, the number of such connected components of $A(W)$ is exactly f.

Now let $Y \in \Gamma$. We claim there exists a vertex W in Σ not on an edge with Y. Suppose otherwise. Then $\Sigma \subseteq A(Y)$, the set of vertices arced to Y. Fix Z in Σ, and let $y_i = A(Z) \cap \Gamma$. Without loss of generality assume $y = y_1$. Then if $f > 1$, we can choose $Y_i \neq Y$ and consider the clique $C(Y_i, Z)$. Since this contains at least three vertices, $\Sigma \cap C(Y_i, Z)$ consists of at least two points arced to Y. By hypothesis, Y is arced every vertex of $C(Y_i, Z)$ including Y_i. But $Y = Y_1$ is not arced to Y_j by choice of j. Thus we may assume $f = 1$. In this case our vertex Y is arced to every further vertex of $\mathscr{K}$. But this was case (ii), already excluded. Thus we must assume there exists a vertex W in Σ not arced to Y. From the previous paragraph $[X, W] \neq 1$ implied $A(W)$ has f components, and similarly $[W, Y] \neq 1$, implies, in turn, that $A(Y)$ has exactly f components. From our general choice of $Y \in \Gamma$ we see now that 3.9 holds. Regularity follows, since by 3.8 all cliques have the same cardinality and so $| A(Z)| = fp$ for every Z in $\mathscr{K}$.

(3.10) *The relation "$\sim$" defined on Σ by the rule $W \sim V$ if and only if $A(W) \cap \Gamma = A(V) \cap \Gamma$ is an equivalence relation on Σ. The equivalence classes $\Sigma_1, \ldots, \Sigma_m$, each have cardinality a multiple of p.*

That "$\sim$" is an equivalence relation is obvious. Since X acts on the elements of each Σ_i in orbits of length p, the second statement follows.

(3.11) *In the subgraph Σ each vertex is arced to $f(p-1)$ others. Two vertices lie in a common Σ_i if and only if their distance* (in Σ) *from one another is at least* 3.

The first statement follows from 3.7 and 3.9. The situation of 3.7 now holds generally because of 3.9, that is, any pair of vertices *not* forming an edge are joined to exactly f vertices in common. Thus if V, W in Σ, and lie in a common Σ_i, then $A(V) \cap A(W)$ contains f points in Γ not arced to one another. If $[V, W] = 1$, then $A(V) \cap A(W) = C(V, W) - \{V, W\}$, a complete subgraph of $p - 1$ points. Thus $[V, W] \neq 1$. Then $A(V) \cap A(W) \cap \Gamma = A(V) \cap A(W)$, since all f vertices of the right side lie in the left. Thus $A(V) \cap A(W) \cap \Gamma$ so V and W have distance 3 in Σ. Conversely if V and W are distance 3 in Σ,

$A(V) \cap A(W) \subseteq \Gamma$ so $A(V) \cap \Gamma = A(W) \cap \Gamma$, and so V and Σ lie in the same Σ_i.

$$| \Sigma | = (f-1) p^2. \tag{3.12}$$

We simply count edges with one end point in Γ and one in Σ.

$$| \Sigma_i | = p, \qquad i = 1,..., m. \tag{3.13}$$

For $W, V \in \Sigma_i$, $(\{W\} \cup (A(W) \cap \Sigma)) \cap (\{V\} \cup (A(V) \cap \Sigma))$ is empty. Thus

$$\bigcup_{V \in \Sigma_i} ((A(V) \cap \Sigma) \cup \{V\})$$

has cardinality $| \Sigma_i | (1 + f(p-1))$ and so by (3.12) we obtain

$$| \Sigma_i | = \frac{(f-1)p^2}{1 + f(p-1)}. \tag{3.14}$$

By 3.8 $f = kp + 1$ for some positive integer k. Substituting yields

$$| \Sigma_i | = \frac{kp^2}{1 + kp - k} < \frac{kp^2}{kp - \frac{1}{2}(kp)} = 2p, \tag{3.15}$$

where the inequality comes from $k - 1 < \frac{1}{2}kp$ in the denominator. Now $| \Sigma_i |$ is an integer multiple of p (by 3.10 less than $2p$. It follows that $| \Sigma_i | = p$ for all i.

Now it is easy to verify

(3.15) *If* $Y \in \Sigma_i$, *then* $\langle X, Y \rangle$ *induces a group of permutations on* $\{X\} \cup \Sigma_i$, *isomorphic to* $PSL(2, p)$ *acting on* $1 + p$ *letters.*

First we must show $\langle X, Y \rangle$ stabilizes $\{X\} \cup \Sigma_i$. But $\{X\} \cup \Sigma_i - \{Y\}$ is a set of vertices V such that $A(V) \cap A(Y)$ is the same set of vertices for all V in the set, and each V does not centralize Y. Thus these V's make up an equivalence class of $\mathscr{K} - A(Y)$ for the relation of being arced to the same subset of $A(Y)$, by 3.10 with Y in the role of X. Thus Y stabilizes $\{X\} \cup \Sigma_i$, and so $H = \langle X, Y \rangle$ stabilizes it, and is doubly transitive on it. But X is a normal subgroup $H \cap N(X)$, regular on the remaining p letters, Σ_i. Thus the group of permutations induced on $\{X\} \cup \Sigma_i$ is a split (B, N)-pair of rank one. We may now apply the result of Hering, Kantor, and Seitz [3]; this together with the fact that H acts on $1 + p$ letters, where p is a prime, and is generated by p-elements, yields (3.15).

(3.16) *Hypothesis* (a) *of Theorem* 3 *forces a contradiction.*

From (3.15) we see that any two members of $\mathscr{K}$ which do not commute generate a group H whose action on the H-orbit containing either of these elements is that of $PSL(2, p)$ on its $1 + p$ p-Sylow subgroups. As before $H = \langle X, Y \rangle$ is 2-transitive on $\{X\} \cup \Sigma_i = \{X, Y_1, ..., Y_p\}$. There exists a vertex V in Γ commuting with every subgroup of $\{X\} \cup \Sigma_i$. Put $X = \langle x \rangle$, $Y_i = \langle y_i \rangle$ $i = 1, ..., p$, and $V = \langle v \rangle$. By Hypothesis (a) of Theorem 3 [or with Hypothesis (a') a suitable generator v of V can be found such that] $Y_1' = \langle vy_1 \rangle$ is an element of K. Without loss of generality choose the y_i so that $y_1{}^X = \{y_1, ..., y_p\}$. Then Y_1' does not commute with X and if $Y_i' = \langle vy_i \rangle$ we have $(Y_1')^X = \{Y_1', ..., Y_p'\} = \Sigma_j$ for some $j \neq i$. Since H is 2-transitive on $\{X\} \cup \Sigma_i$, we can choose the indices i so that $X = Y_p^{y_1} = y_1^{-1} Y_p y_1$, and put $y_1^{-1} y_p y = x^b$. Then

$$(vy_1)^{-1}(vy_p)(vy_1) = vx^a$$

generates a subgroup lying in the Y_1'-orbit of Y_p. But since $H_j = \langle X, Y_1' \rangle$ is 2-transitive on $\{X\} \cup \Sigma_j$, its subgroup $\langle Y_1', Y_p \rangle$ is also 2-transitive on it, and so the subgroup $\langle vx^a \rangle$ is one of $\{X, Y_1', ..., Y_p'\}$. This is impossible since none of these $1 + p$ subgroups commutes with any other, yet $\langle vx^a \rangle$ centralizes X and yet is distinct from it. This contradiction completes the proof of Theorem 3.

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